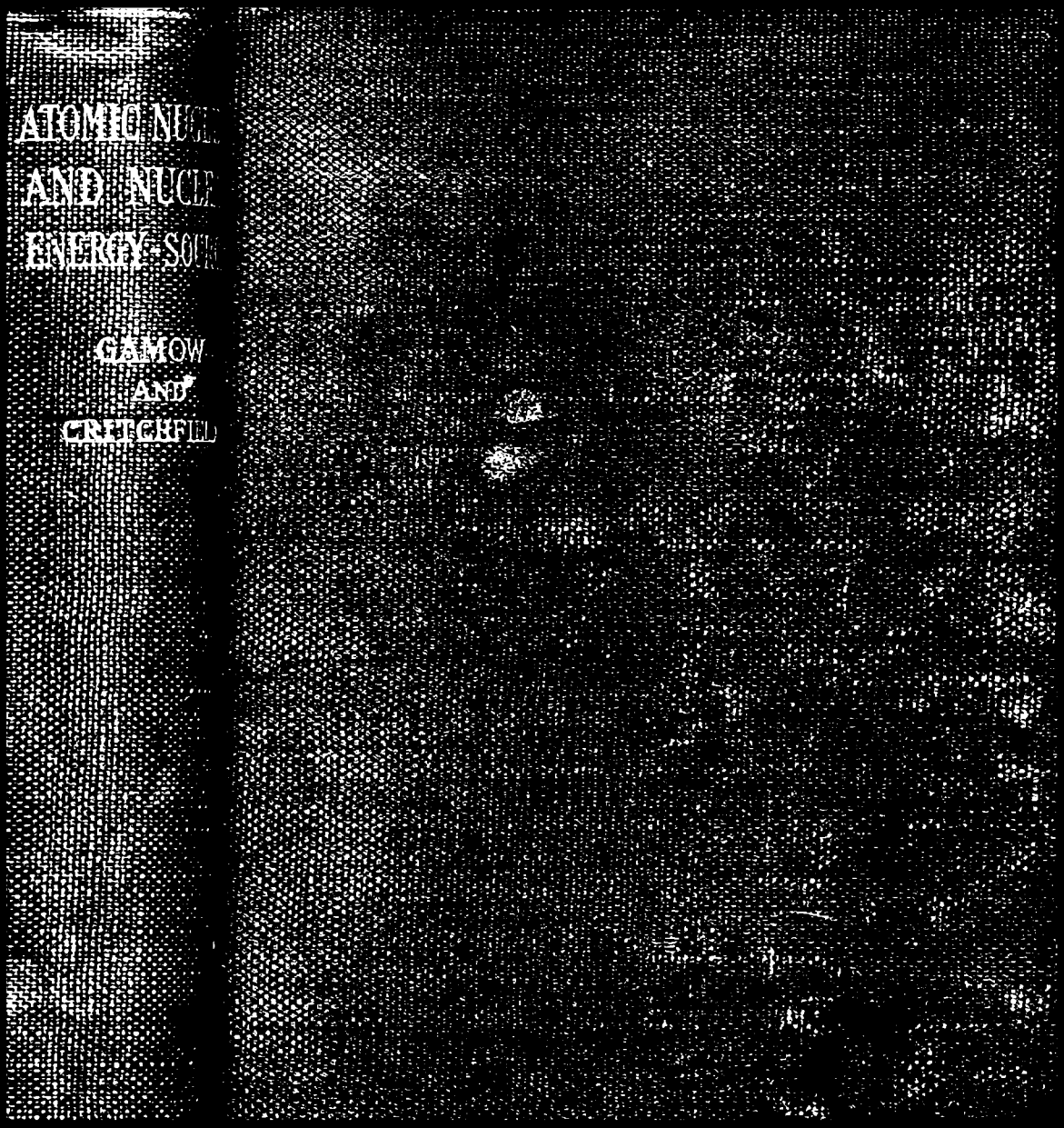
ATOMIC NUCL
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THE

INTERNATIONAL SERIES

OF

MONOGRAPHS ON PHYSICS

GENERAL EDITORS

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N. F. MOTT, E. C. BULLARD

THE INTERNATIONAL SERIES OF MONOGRAPHS ON PHYSICS

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THEORY OF ATOMIC NUCLEUS AND NUCLEAR ENERGY-SOURCES

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Being the third edition of

STRUCTURE OF ATOMIC NUCLEUS AND NUCLEAR TRANSFORMATIONS

INV. 69—NIST OR

OXFORD
AT THE CLARENDON PRESS
1949

Oxford University Press, Amen House, London E.C. 4

GLASGOW NEW YORK TORONTO MELBOURNE WELLINGTON

BOMBAY CALCUTTA MADRAS CAPE TOWN

Geoffrey Cumberlege, Publisher to the University

PRINTED IN GREAT BRITAIN

PREFACE

'It is my personal conviction that if we knew more about the nucleus, we should find it much simpler than we suppose. I am always a believer in simplicity, being a simple fellow myself.'

LORD RUTHERFORD (Göttingen Lecture—Dec. 14, 1931)

DURING the decade that has passed since the appearance of the second edition of this book nuclear physics has grown to such a size that it is quite impossible to write a comprehensive treatise on the subject except in the form of a multi-volume handbook. The difficulties of collecting all available material in a single volume have been augmented by the discovery of practical methods for the large-scale liberation of nuclear energy which led to a deluge of mostly semi-technical studies, many of which are, however, quite important also from the purely scientific point of view. Under these circumstances it has become again necessary to rewrite the book practically anew, as well as to increase its volume. Taking into consideration the limits of human endurance, this could be done only by doubling the authorship; thus G. Gamow of the first two editions becomes G. Gamow and C. Critchfield of the present one.

Despite the enormous increase of the factual knowledge concerning the properties of atomic nuclei, comparatively little progress has been made in actually understanding the basic properties of nuclear structure; in fact, it seems that during these ten years we have even unlearned a little about the most fundamental question of the nature of nuclear forces. Thus, there arose the question of whether it would not be better to delay the book until the fundamental principles of nuclear physics are finally settled. The authors have decided, however, to go ahead with the book on the basis of the superstition that whenever a new edition of this book nears publication a new major discovery is made in nuclear physics (discovery of neutrons after the first edition; theory of the 'compound nucleus' after the second edition), making the book out of date. If this happens again, the contribution to the progress of science will certainly justify the short 'up-to-date-ness' of the edition.

As to the actual composition of the book, the following remarks can be made. It was attempted to write the book in an inductive way, starting from the properties of elementary particles from which the

nuclei are made and working up to more and more complex nuclear properties. A considerable space has been devoted to thermonuclear reactions and their application to cosmological problems (Chapter X), since, at the present time there is no book on astrophysics in which this class of problem receives an adequate treatment. On the other hand, comparatively little is said about the large-scale energy liberation of nuclear energy by means of chain reactions in fissionable substances, since it is felt that this class of problem belongs more to what can be called 'nuclear technology' rather than to pure science. Consequently Chapter XI, which treats the problem of nuclear chain reactions, is limited to purely schematic descriptions of the processes involved. New important findings, which appeared during the printing of this book, are included in the appendixes at the end of the volume.

It may be added in conclusion that the authors certainly hope that the fourth edition of the book will represent the final version of the theory of the atomic nucleus.

G. G.
C. L. C.

September 1947
LOS ALAMOS.
NEW MEXICO.

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I

GENERAL PROPERTIES OF THE ATOMIC NUCLEUS

1. Atoms, nuclei, nucleons

In its incessant effort to become adjusted to the immense complexity of the phenomena of nature, the human mind strives to analyse these phenomena and to reduce them to the smallest possible number of elementary notions and laws from which the entire picture of the physical world can be derived by the method of purely logical deduction. This desire to reduce the observed complexity of nature to the logical simplicity of basic postulates represents the main driving force in the development of science, and the degree of simplification thus achieved (as characterized by the number of elementary notions and laws necessary for complete deduction) can be used as a measure of the progress of science towards its final goal.

It goes without saying that this search for basic elementary postulates results sometimes in over-simplification of the theoretical picture thus obtained, and that the notions that at a certain epoch of scientific development were considered as being elementary often turned out to be quite complex when some previously neglected or later discovered empirical facts were taken into consideration. However, studying the history of science for the few thousand years of its existence, one can hardly escape the conviction that, in spite of all these ups and downs, the reduction of the complexity of nature to a comparatively small number of basic postulates represents a converging process.

At the dawn of the scientific study of matter, the desire for basic simplicity underlying the apparent complexity of different forms of matter led the philosophers of ancient Greece to the recognition of four basic elements from which the world was supposed to be made; these are stone, water, air, and fire. According to the ideas first formulated by Democritus, these four elementary substances were made up of a large number of essentially indivisible particles, *the atoms* which, being mixed in different proportions, were deemed responsible for the formation of all the variety of known substances. Thus the soil was considered as being a mixture of stone- and water-atoms, wood as a mixture of the atoms of stone, water, and fire, whereas various metals were looked upon as stone- and fire-atoms mixed in different proportions (more fire in gold, less in iron).

Further development of science extended and amplified these ancient views without, however, changing essentially the fundamental idea. The three 'material' elements, stone, water, and air, were replaced by ninety-two chemical elements, whereas the atoms of fire evolved into the notion of radiant energy, and later into the modern idea of light quanta. This transition from only a few atomic species, as proposed by the ancient philosophers, to a much larger number of distinct elements which are required to establish the science of chemistry represents a typical 'forced retreat' on our way towards the reduction of the complexity of nature.

However, this largely increased number of distinct elements from which matter was supposed to be built led immediately to attempts at a new reduction; and early in the nineteenth century the French chemist Prout formulated an hypothesis according to which the different chemical elements represent just *different degrees of 'condensation' of a single basic element, viz. hydrogen.* Prout based his views on contemporary estimates of atomic weights which seemed to indicate that the weights of different atomic species relative to hydrogen were always represented by integers. This far-seeing hypothesis was 'disproved', however, and thrown into oblivion by more precise chemical measurements which showed that 'Prout's integer rule' held only in first approximation, and that in some cases (as in chlorine with chemical atomic weight 35·5) it did not hold at all.

The difficulties standing in the way of Prout's original hypothesis were removed almost a full century later by the discovery of *isotopes*, and by recognition of the fact that the so-called chemical atomic weights represent, in general, only the weighted means of the weights of various isotopes of the element. Furthermore, it became clear that the small deviations from Prout's integer rule (mass-defects) which are observed even in the cases of single isotopes must be interpreted on the basis of the relativistic mass-energy relation as due to the internal energy that binds the atoms of hydrogen into the more complex atoms of various heavier elements. Thus the very same objections that caused the early downfall of Prout's original hypothesis of condensation became instrumental in the ultimate interpretation of the structure of the atom.

Already at an early stage in atomic studies it had become clear that atoms must be considered as some kind of complex mechanical

systems built of positively and negatively charged parts; and the discovery of the *electron* by J. J. Thomson led him to propose an atomic model in which the positively charged material, representing the major part of the atomic mass, was assumed to be distributed more or less uniformly throughout the body of the atom, whereas the light, negative electrons were scattered through this material as the seeds in a water-melon. According to this picture, the condensation of several hydrogen atoms into more complex atomic species would be visualized as the fusion of the spherical positive charges of the original atoms into a single, larger, spherical charge and followed by a redistribution of the electrons within the new domain.

This Thomson atomic model, which for a while dominated the theory of atomic structure, was, however, disproved in 1911 by the classic experiments of E. Rutherford, who studied the scattering of beams of α-particles passing through thin foils of different materials. It was found, in fact, that the large deflexions suffered by individual α-particles in their collisions with the atoms of the scattering material could not possibly be explained under the assumption of a uniform distribution of the positive charge (and its mass) through the entire body of the atom; and that in order to obtain electric forces that are strong enough to account for the observed deflexions one must assume that the positive charge of the atom, as well as most of its mass, is concentrated within a very small region around the atomic centre. This heavy, positively charged part of the atom was given the name of *atomic nucleus*, and the old Thomson model gave way to the new Rutherford model of the atom in which, instead of moving inside a spherical positive charge, atomic electrons move around the positively charged centre under the influence of the inverse square Coulomb attraction. Rutherford's experiments on α-scattering in different materials also permitted a direct estimate of the electric charge on the corresponding nuclei or, what is the same thing, the number of electrons composing the outer atomic envelopes, and led to the important result that the number of atomic electrons coincides with the ordinal number (*atomic number*) of the element in question in the natural sequence of elements arranged in order of increasing atomic weights.

In the light of Rutherford's atomic model one would be inclined to reformulate Prout's original hypothesis by applying it directly to the *nuclei* of different atomic species, and by saying that the atomic

nuclei of various elements are formed by the 'condensation' of hydrogen-nuclei, or *protons* as they were called because of their fundamental importance in the problems of the structure of matter. Such a formulation would be, however, only partially true since, as it is easy to see, building a composite atom from a number of hydrogen-atoms requires a redistribution of the electric charge between the nucleus and the outer envelope. Thus, for example, considering the atom of the principal isotope of oxygen (with atomic weight 16) as constructed from sixteen hydrogen-atoms by direct fusion of their nuclei, with all outer electrons retained on the outside, we should arrive at an element with atomic number 16 instead of 8. Thus, in order to retain the principal feature of Prout's elementary hypothesis, we must assume that in the 'condensation' process leading to the formation of an oxygen atom from sixteen hydrogens, only eight of the original electrons remain on the outside. The other eight are, so to speak, 'assimilated' by the nucleus, the positive charge on which is reduced accordingly.

We could imagine the electrons that are assimilated by the nucleus either as retaining their individuality as components of the nuclear structure or as being 'swallowed' by the corresponding number of nuclear protons which thus lose their positive charge and turn into neutral particles. The serious theoretical difficulties which arose in early attempts to consider individual electrons as permanent members of the nuclear structure, as well as the empirical fact that neutral protons, known as *neutrons*, are ejected from nuclei with the same, or often greater, ease as are ordinary protons in experiments on atomic bombardment, force us to accept the second possibility. Assuming the neutron-proton hypothesis for the constitution of nuclei, we find that the number of protons in a given nucleus is given directly by its atomic number Z, whereas the number of neutrons must be computed as the difference between the total number of constituent particles (*nucleons*) and the number of protons. It will be noticed at this point that the number of nucleons forming a given nucleus and known as its *mass-number* A must not be expected to coincide necessarily with the integer nearest the exact atomic weight, since the value of the mass-defect of a composite nucleus may exceed the mass of one nucleon. In fact, following the values of mass-defects we find that they become larger than one m.u. for all elements heavier than Sn. Thus, expressing relative atomic weights on the basis of hydrogen as

exactly unity (instead of oxygen exactly 16),† we should not get a clear-cut 'integer-rule' and the correct values of the mass-number A could be obtained only by studying the *continuity* of the variation of atomic masses throughout the whole system of the elements. However, referring atomic weights to oxygen we avoid this unnecessary complication since in this case the deviations from integers happen never to exceed one-half. Nearly integer atomic weights would be obtained as well if we used C^{12} as the standard with the value exactly 12, or if we used He^4 as exactly 4. As we shall see later, this result is due to the fact that the major part of the nuclear binding energy is contained in the so-called α-shells (a combination of two constituent protons and two constituent neutrons), and that the nuclei, He^4, C^{12}, O^{16}, etc., represent saturated systems of such shells.

It must be emphasized also from the very beginning that, in considering the nuclei of different elements as built of protons and neutrons, we do not consider neutrons as some composite systems formed by two 'really' elementary particles, e.g. a proton and an electron, or, conversely, consider protons as composite systems of elementary neutrons and positive electrons. It is, in fact, much more rational to assume in this matter a *symmetrical point of view* first formulated by W. Heisenberg and *to consider the proton and the neutron as two possible electrical states of a single heavy elementary particle* which was given the name of *nucleon*.‡ The transition of a nucleon from one of its states into another is accompanied by the emission of the corresponding charge-difference in the form of a free electron (positive or negative). The reverse of this process corresponds to the change of the electric state of the nucleon caused by the absorption of an electron from the outside. Processes of the type just described are usually known as β-transformations and represent the means by which nuclei, in which either the number of neutrons or the number of protons is relatively too large, adjust their electric charge to the value corresponding to the most stable state (least total mass for the given number of nucleons).

† In this case, for example, the atomic weight of the heavier U-isotope would be $\frac{238{\cdot}07}{1{\cdot}00893} = 235{\cdot}97$, giving $A = 236$ instead of 238.

‡ It may be pointed out here that it is consistent with this view of the heavy elementary particle that there exist other states of the nucleon, such as the state with a negative electrical charge (the hypothetical *negative proton*) or states with multiple charges of either sign. However, such particles have not been detected hitherto.

The detailed study of energy-liberation in the various β-transformations revealed an unexpected and alarming discrepancy in the total energy-balance connected with these processes: it seemed that a part of the energy (as calculated from the difference in masses of the isotopes, cf. § 5) is always mysteriously lost whenever an electron is emitted from a nucleus. In order to explain this discrepancy it was necessary to assume that, in each process of this kind, a hitherto unknown energy-carrier, escaping all known means of observation, is emitted from the nucleus simultaneously with the electron. This new hypothetical particle, which had to be considered as carrying no electric charge and as possessing a much smaller mass (if any) than the electron mass, was given the somewhat romantic name *neutrino*† (denoted by ν). It was found necessary to ascribe to this particle the ability to carry mechanical momentum, and spin, so that it became a full-fledged member in the society of elementary particles of physics. Taking into account the existence of neutrinos, we can write various possible transformations of the nucleons in the forms:

$$
\begin{aligned}
n &\rightarrow p+e^-+\nu \\
p &\rightarrow n+e^++\nu \\
n+e^+ &\rightarrow p+\nu \\
p+e^- &\rightarrow n+\nu \\
n+\nu &\rightarrow p+e^- \\
p+\nu &\rightarrow n+e^+.
\end{aligned}
$$

Some of these transformations, as for example the first one, are exothermic and can take place spontaneously, whereas the others require a supply of external energy in excess of a certain threshold value. We shall return to the problem of neutrinos in the chapters discussing the nature of nuclear forces (Chap. III) and the process of β-transformations (Chap. V).

We cannot finish this introductory survey of the elementary particles without mentioning the newly discovered brand known as *mesons*.‡ These new particles, which were first introduced by Yukawa for the purpose of explaining nuclear forces, are supposed to possess

† As originally introduced in 1930 by W. Pauli, it was known by the name of 'neutron'. When two years later J. Chadwick used the same name for the chargeless protons ejected from beryllium by α-particles, the controversy of names was settled by E. Fermi, who said that Pauli's particle 'is not a neutron but a neutrino' (*neutrino* and *neutretto* being the corresponding diminutives in the Italian language).

‡ Also sometimes called 'mesotrons' in violation of Greek syntax.

a mass (or masses) intermediate between that of the proton and that of the electron, and can carry either a positive or a negative charge, or be neutral. Whereas the role of these particles in the theory of nuclear force is surrounded at the present moment by a rather dense fog (cf. Chap. III), there is no doubt that they (at least the charged ones) do exist in nature in the free state. The study of cosmic rays indicates that the positive and negative mesons are being produced (by a process which is probably similar to the ordinary electron-pair production) in the upper layers of the terrestrial atmosphere by the fast primary particles (protons ?) which come from interstellar space. In fact, about 75 per cent. of the total ionization produced by cosmic rays at sea-level is due to the action of these charged mesons. The mass of these particles was found to be about 200 electron masses, and we also have convincing evidence that they are intrinsically unstable, breaking up into an electron (positive or negative) and a neutrino with a mean lifetime of about 2 microseconds. Very little evidence as yet has been found for the existence of neutral mesons. If such existed it could be considered as an energy-rich system of positive and negative electron.† (Compare Appendix I.)

Although electrons, neutrinos, and various types of mesons undoubtedly play important roles in both the transformations and the attractive forces that occur between nucleons, thus securing the stability and even the existence of the composite atomic nuclei, they should not be considered as full-fledged constituent parts of the nuclear structure, and to a certain extent fall outside the scope of nuclear physics proper. The situation is somewhat similar to that which existed in the kinetic theory of gases where various properties of the gas as a whole could be obtained by considering the molecules as rigid spheres of a certain diameter and interacting with each other according to certain empirically established laws. In fact, it is possible to develop the theory of the fundamental nuclear properties and the various nuclear reactions essentially on the basis of empirically determined interaction-laws between the nucleons from which the composite nuclei are built. Here lies a convenient, even though not very sharply defined, boundary between *nuclear physics*

† Since the neutrino, like electron and positron, is supposed to have spin $\frac{1}{2}$, the mesons just described would have integral spin-values, i.e. either 0 or unity. The spin of the cosmic-ray meson has not been determined experimentally. If it should turn out to be $\frac{1}{2}$, the decaying meson would produce a light quantum simultaneously with the electron, and there would be no connexion with β-decay.

proper, and the next, as yet rather unexplored, division of the science of matter which can be called tentatively *the physics of elementary particles*.

2. Nuclear radii

We shall start our study of atomic nuclei by describing first their external characteristics, i.e. the properties that can be investigated without disturbing the internal nuclear structure. This includes the study of nuclear radii, their moments of momenta, their magnetic dipole, electrical quadrupole moments, and finally such information about the internal energy of nuclei that can be obtained from their measured masses.

The first information concerning the geometrical dimensions of atomic nuclei was supplied by the very same experiments on the scattering of α-particles that were responsible for the origin of the idea of the atomic nucleus itself. It was, in fact, noticed by Rutherford that in the case of large-angle scattering of particularly fast α-particles by light elements the observed angular distribution of the scattered particles deviates from the classical scattering formula:

$$I(\theta) = \left(\frac{Ze^2}{\bar{M}v^2}\right)^2 \operatorname{cosec}^4 \tfrac{1}{2}\theta, \tag{1}$$

where θ is the scattering angle in the coordinate system bound to the common centre of gravity, Z the atomic number of the scattering nucleus, and v and $\bar{M}$ the velocity and the reduced mass of the incident α-particle.† Since the above formula was derived under the assumption that the nucleus and the incident α-particle interact as two electrically charged points, and since the above stated conditions under which the 'anomalous' scattering was observed correspond to a close approach of the incident α-particle to the nucleus, it was natural to assume that the observed deviations are caused by some new kind of force acting on the particle when it comes close to the outer boundary of the nucleus. Even the earliest theoretical treatment of the 'anomalous' scattering of α-particles, based on the laws of classical mechanics,‡ led to the result that, in order to explain the

† This formula, originally derived by Rutherford on the basis of classical mechanics, is about the only classical formula that remains unaltered by wave-mechanics, a fact which used to be to Rutherford a source of great pride and satisfaction.

‡ E. S. Bieler, *Proc. Camb. Phil. Soc.* **21** (1923), 686; P. Debye and W. Hardmeier, *Phys. Zeits.* **27** (1926), 196.

observed deviations, one must assume the existence of some *short-range attractive forces* acting on the particle as it approaches the nucleus. The distance at which these attractive forces become larger than the forces of electrostatic repulsion can be assumed tentatively to characterize the actual dimensions of the nucleus, an assumption which, of course, holds better for heavier nuclei where the radius of the actual distribution of nuclear substance becomes large as compared with the range of nuclear forces. Whereas in the classical

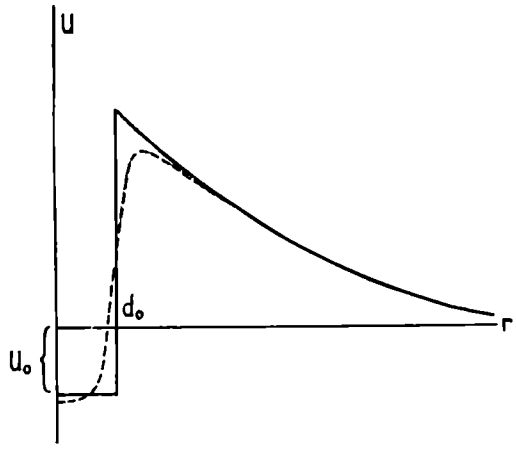

FIG. 1.

theory of 'anomalous' scattering the potential energy of the α-particle in the neighbourhood of the nucleus was represented usually in the form

$$\frac{2Ze^2}{r}-\frac{B}{r^n} \quad (n>1),$$

the wave-mechanical treatment, which is influenced not so much by the details of the force-anomalies on the outer slope of the potential barrier as by the penetration of the wave through the barrier, usually employs the simplified potential distribution shown in Fig. 1. Such a potential distribution is characterized by two numbers: the *critical distance* d_0 (which for large heavy nuclei approaches the nuclear radius r_0) and the *depth of the potential well* U_0.

Using this model for the interpretation of the observed 'anomalous' scattering of α-particles in hydrogen,† Taylor‡ came to the conclusion that the interaction between the proton and the α-particle can be

† J. Chadwick, *Proc. Roy. Soc.* A **128** (1930), 114.

‡ H. M. Taylor, ibid. A **134** (1931), 103; A **136** (1932), 605. The modern interpretation of the proton α-particle interaction involves the theory of resonance scattering (cf. Chap. VIII).

described by a potential well with $d_0 = 0{\cdot}45 \times 10^{-12}$ cm. and $U_0 = -6$ M.e.v. The method of 'anomalous' scattering of α-particles fails to give any information concerning nuclear radii of the heavier elements since, in this case, the electric repulsive forces become too large to permit the penetration of the incident particles into the region of nuclear forces. Thus, for example, Rutherford's experiments† on scattering of α-particles in uranium, in which no deviations from predictions on an inverse square law of force were observed, could only lead to the conclusion that the radius of the uranium-nucleus must be smaller than 3×10^{-12} cm., which represents the distance at which the Coulomb energy of the α-particle-nuclear system becomes equal to the kinetic energy of bombardment (classical distance of closest approach).

The first indications concerning nuclear radii of the heavier elements were supplied by the quantum theory of α-decay which considers this process as the wave-mechanical penetration of nuclear α-particles through the potential barrier formed by the short-range attractive forces on the inner side and the Coulomb repulsion on the outside. Assuming the simplified potential distribution, where in the case of heavy nuclei we can put $d_0 = r_0$ with sufficient accuracy, one obtains for the theoretical value of the decay constant λ the expression‡

$$\log_{10}\lambda = 21{\cdot}67 - 1{\cdot}19 \cdot 10^9 \frac{Z-2}{\bar{v}} + 4{\cdot}08 \cdot 10^6 \sqrt{\{(Z-2)r_0\}}, \qquad (2)$$

where $\bar{v}$ is the reduced velocity of the emitted α-particle. Using the measured values of λ and $\bar{v}$ for various radioactive elements, one can calculate the values of the nuclear radii r_0. One finds thus the values ranging from $0{\cdot}7 \times 10^{-12}$ for polonium (lightest α-emitter) to $0{\cdot}9 \times 10^{-12}$ cm. for uranium 238 (heaviest natural α-emitter).

The most general method of measuring nuclear radii is to measure the reduction in the intensity of a beam of fast neutrons caused by passing through several centimetres of the material being studied. Each nucleus in the material presents its geometrical cross-section (or rather πd_0^2) as an obstacle in the path of a very fast neutron. If the neutron hits such an obstacle its energy is degraded, or converted, so that it is effectively absorbed from the beam. Even if the neutron is not absorbed, however, it may be deflected from its original trajec-

† E. Rutherford, *Phil. Mag.* Ser. 7, **4** (1928), 580.
‡ Compare Chapter VI.

tory by *diffraction* caused by the obstacle. Each of these processes presents a cross-section πd_0^2 so that the total cross-section for removal from the beam is $2\pi d_0^2$ per nucleus.†

In Fig. 2 we give the results of such measurements carried out by Grahame and Seaborg‡ with neutrons of 13·7 M.e.v. (empty circles) and by Sherr§ with neutrons of 25·4 M.e.v. (filled circles). In the same graph we give the values of nuclear radii of the radioactive elements as calculated from the known decay constants (asterisk). The values of d_0 are plotted against $\sqrt[3]{A}$ (A = atomic weight) and they fall on a straight line described by the equation.

$$d_0 = [1{\cdot}7 + 1{\cdot}22\sqrt[3]{A}]10^{-13} \text{ cm.} \quad (3)$$

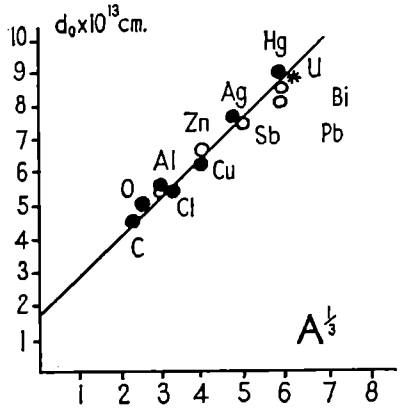

FIG. 2.

The constant term, $1{\cdot}7 \times 10^{-13}$ cm., in the above equation can be interpreted as the range of nuclear forces, whereas the second term, which depends upon the atomic weight, must represent the 'actual' radius of the nucleus. The fact that the nuclear radius is proportional to the cube root of the nuclear mass indicates that *the mean nuclear density remains constant throughout the whole periodic system of elements.* This result suggests that, in contrast to the situation existing in the atomic model, *the material forming atomic nuclei must be considered as rather similar to ordinary liquids* where the individual particles are packed closely together maintaining a constant density independent of the size of the sample. We may thus speak of the *universal nuclear fluid* which forms small droplets of various sizes and mostly spherical in shape. Using the observed nuclear radii, we find for the density of the nuclear fluid the value $2{\cdot}2 \times 10^{14}$ gm./cm.^{3}

3. Nuclear spin and magnetic moments

The suggestion that certain atomic nuclei must be regarded as possessing mechanical angular momentum (*spin*) as well as a mag-

† Compare Chapter VIII. Since the diffraction is mostly in the forward direction not all of the diffracted (scattered) neutrons are removed in an actual experimental set-up and a correction must be applied.

‡ D. C. Grahame and G. T. Seaborg, *Phys. Rev.* **53** (1938), 795.

§ R. Sherr, ibid. **68** (1945), 240.

netic dipole moment was first introduced by Pauli in connexion with the explanation of the so-called hyperfine structure of spectral lines. The simplest way of estimating the nuclear spin lies in the study of the relative intensities in the band-spectra of the element in question. For a homonuclear diatomic molecule the wave-function describing a given quantum state may be either symmetrical or antisymmetrical with respect to the spins of the two nuclei. Since the nuclei are indistinguishable (e.g. the same isotope, if there are several) they must obey either Einstein–Bose or Fermi–Dirac statistics. Thus the symmetrical spin states are allowed only if the rotational quantum number is even for Einstein–Bose statistics or odd for Fermi-Dirac statistics, and the antisymmetrical spin states are allowed in the alternate possibilities for the orbital quantum number. Given two nuclei of spin i in units of $\hbar$ the number of states that are symmetrical in the spins is larger than the number that is antisymmetrical, and simple considerations show that the ratio of these numbers is

$$\frac{W_{\text{sym}}}{W_{\text{anti}}} = \frac{i+1}{i}. \tag{4}$$

Thus, in a gaseous state in which many rotational levels are excited the number of molecules with symmetrical spin-states and even (odd) rotational quantum numbers, divided by the number of molecules with antisymmetrical spin-states and hence odd (even) rotational quantum numbers, will be this same ratio. Consequently, since transitions involving a change of molecular symmetry type do not occur in any appreciable amount, the corresponding electronic emission lines in the spectra of such molecules must show this ratio of intensities. Since the lines in a band-spectrum belong to consecutive values of the rotational quantum number, this means that the intensities of the lines should alternate, the ratio of neighbouring lines being given by the relation (4). Thus the study of intensity ratios in the band-spectrum of hydrogen leads to the important result that the proton has the spin $i = \frac{1}{2}$. Since, however, this measurement cannot be carried out with very high accuracy, the method can be applied satisfactorily only when the nuclear spin is small. Also, in the case of heavy elements the separation of individual lines becomes so small that the structure of the band fuses into a continuum making any such measurements impossible.

A more satisfactory method of measuring nuclear spins as well as

magnetic moments is to study the *hyperfine structure* of spectral lines for which these notions were originally introduced. This structure is due to a splitting of the atomic energy-levels caused by the magnetic interaction between the nuclear moment and the field created by the orbital electrons. The very small separation of the levels found experimentally in the normal hyperfine-structure pattern must be ascribed to the smallness of the nuclear magnetic moment, as compared with Bohr's electron-magneton: $\mu_e = e\hbar/2mc$. Nuclear magnetic moments are usually expressed in terms of the *nuclear magneton* $\mu_{\text{nuc}} = e\hbar/2Mc$ (M being the proton mass) which is 1,840 times smaller than the electron-magneton. (However, as we shall see later, μ_{nuc} is not equal to the magnetic moment of a proton which is about 2·8 times larger.)

The value of the nuclear spin can be derived directly from the number of hyperfine components of the spectral term provided the angular momentum in the electron system is large enough and is known. Let j be the total angular momentum of the atomic electron configuration (resultant of orbital momentum and electron spin) and i the angular momentum of the nucleus. Then the resultant momentum f will have different possible values corresponding to the various relative orientations of the vectors j and i. Applying the rules of addition of quantum vectors we see that f may take the values:

$$\begin{aligned} &f_0 = j+i, \quad f_1 = j+i-1, \quad \ldots, \quad f_{2i} = j-i \quad (j \geqslant i), \\ &f_0 = i+j, \quad f_1 = i+j-1, \quad \ldots, \quad f_{2j} = i-j \quad (i \geqslant j), \end{aligned} \tag{5}$$

so that, on account of the nuclear spin, each electronic energy level will split into $2i+1$ or $2j+1$ sub-levels according to whether $i \leqslant j$ or $i \geqslant j$. The relative separation of these sub-levels may be calculated also if it be assumed that the additional energy due to the magnetic interaction of nuclear and electronic moments is proportional to the scalar product of the two momenta, i.e. proportional to $\cos(i,j)$:

$$(\mathbf{i},\mathbf{j}) = \frac{f(f+1)-i(i+1)-j(j+1)}{2}.$$

Inserting the possible values of f as given in eq. (5) we find that the relative energy-differences between neighbouring hyperfine levels are given by the ratios:

$$\begin{aligned} &f_0 : f_1 : f_2 : \ldots : f_{2i-1} \quad (j \geqslant i), \\ &f_0 : f_1 : f_2 : \ldots : f_{2j-1} \quad (i \geqslant j), \end{aligned} \tag{6}$$

f_r being defined by the equations (5). This is the so-called interval rule, first proposed by Landé.

As an example of the application of this method we give here an analysis of the hyperfine structure in the Pr spectrum, for which the energy-level $^5K_7$, corresponding to $j = 7$, is known to be split into six components with relative energy differences, 19·0, 16·7, 14·4, 12·7, 10·4. As the number of sub-levels is less than $2j+1$ $(= 15)$ we conclude that in this case $i < j$, and the relation $2i+1 = 6$ gives $i = 5/2$ directly. For these values of i and j the relative separations predicted by (6) become 19:17:15:13:11 in good agreement with the observed values. Historically, the first determinations of the hyperfine splitting of atomic levels were made as deductions from the optical, interferometric spectra. Recent advances in the high radio-frequency techniques, however, make it possible to induce and detect transitions from one member of a hyperfine multiplet to another member of the same multiplet, thereby determining the *radio-frequency spectrum* of the lowest states of the atom directly (the wave-lengths of the radiation range from 2·5 cm. to 200 cm.). The method of detection used in this work† is closely related to the molecular beam magnetic resonance method of measuring magnetic moments which is described below and relies on the transitions being made in an external magnetic field. In this way the Zeeman effect and Paschen–Back effect are observed for the radio-frequency spectrum. The method is one of extremely high accuracy and resolution.

The value of the magnetic moment of the nucleus can be estimated from the absolute values of the hyperfine structure separation. For this, however, one must know the electron current density near the nucleus and this is difficult to evaluate. Hence the values so obtained are inaccurate and this method has been superseded by more direct measurements of the magnetic moments.

A direct measurement of the magnetic moment of the proton was first effected by Stern and his associates‡ by deflecting a beam of molecular hydrogen with an inhomogeneous magnetic field. The accuracy obtainable with the molecular beam method has been greatly enhanced by the resonance technique developed by Rabi and his co-workers.§

† P. Kusch, S. Millman, I. I. Rabi, *Phys. Rev.* **57** (1940), 765.

‡ O. Stern and R. Frisch, *Zs. f. Phys.* **85** (1933), 4; O. Stern and I. Esterman, ibid. **86** (1933), 132.

§ I. I. Rabi, J. Kellogg, J. Zacharias, *Phys. Rev.* **46** (1934), 157; I. I. Rabi, S. Millman, P. Kusch, J. R. Zacharias, ibid. **55** (1939), 526.

The method is to pass a beam of neutral molecules having no electronic magnetic moment through three magnetic fields. The first is a long, inhomogeneous field that accelerates the molecules in a direction perpendicular to the beam by reason of the nuclear magnetic moments. This causes the beam to fan out a little because various molecules have their nuclear spins and nuclear rotations lined up differently relative to the field (see Fig. 3). The third field is of the same type and accelerates the molecules in the direction opposite to that in the first field so that the third field refocuses the

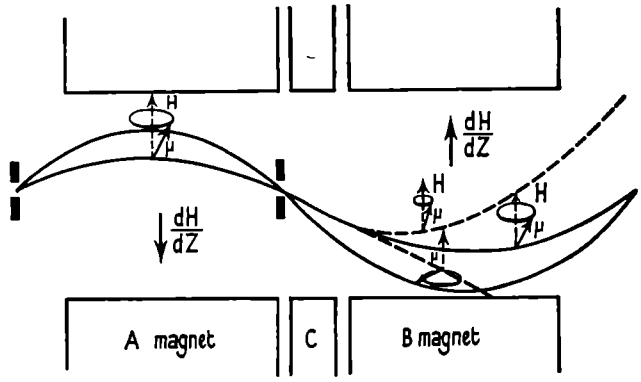

FIG. 3.

beam if all molecules retain the orientation they had in the first acceleration. Between these magnets is placed the second, homogeneous field. In this second field each molecular spin simply precesses with the Larmor frequency characteristic of the $g(i)$ of the molecular system, where

$$g(i) = \frac{\text{magnetic moment}}{\text{angular momentum}},$$

and of the strength of the homogeneous magnetic field H, viz.

$$\omega_L = Hg(i).$$

By inserting a small loop of wire into the second field and sending an oscillating current through so as to generate a high-frequency magnetic field at right angles to the homogeneous field the molecules can be made to change from one orientation to another. Since the amount of energy received from the high frequency is very small in one cycle of the Larmor precession, the effect is not noticeable unless the high frequency is in almost exact resonance with the Larmor

frequency. In resonance, however, many molecules pick up enough energy from the perturbing field to change their orientation. These molecules are then removed from the path on which they would have been focused by the third field and are, consequently, lost to the detector. By correlating dips in the intensity of the beam with the radio-frequency or, more practically, with the strength of the homogeneous field, the ratio of magnetic to mechanical moments can be determined. Relative comparisons between nuclear moments are limited in accuracy only by the time spent in the homogeneous field and are highly precise. Absolute measurements are limited by absolute measurements of the strength of the field. It is an adaptation of this method that was applied to the measurement of the radio-frequency spectrum of atoms, alluded to above.

Quite recently an entirely new method for measuring nuclear magnetic moments has been worked out by Bloch, Hansen, and Packard† and is based on the phenomenon that they call *nuclear magnetic induction*. When a sample of some material is placed in a strong magnetic field, and is subjected in addition to a weaker radio-frequency field directed at right angles to the strong field, the total magnetic polarization will be subject to a forced precession about the direction of the constant field, thus giving rise to a measurable polarization in the direction perpendicular to both fields. The amplitude of this component is expected to show the phenomenon of resonance when the Larmor frequency of the nuclear magnet coincides with the applied radio-frequency. This opens a simple way for determining nuclear magnetic moments. This new method has the advantage of permitting the determination of nuclear magnetic moments in the ordinary (solid or liquid) state of matter. In Table I we give the values of nuclear spins and magnetic moments obtained by the various methods.

The table does not contain nuclei for which both Z and A are even numbers (even number of protons and even number of neutrons). Although most of the nuclei in the universe are of this type, and the spins have been determined in many cases, the spins are always found to be zero. They can have no moments of any kind, therefore. Nuclei having spin 1/2 cannot have a quadrupole (or higher) moment and this fact is indicated by the dashes in the

† F. Bloch, *Phys. Rev.* **70** (1946), 460; F. Bloch, W. W. Hansen, and M. Packard, ibid. **70** (1946), 474.

TABLE I

Nuclear spins and moments

Nucleus	*Spin* ($\hbar$)	*Magnetic moment* (μ_{nuc})	*Electric moment* (10^{-24} *cm.*^{2})	*Nucleus*	*Spin* ($\hbar$)	*Magnetic moment* (μ_{nuc})	*Electric moment* (10^{-24} *cm.*^{2})
Neutron	1/2	−1·9103	—	Ag^{109}	1/2	−0·19	—
H^1	1/2	2·7896	—	Cd^{111}	1/2	−0·65	—
H^2	1	0·85647	·0·00273	Cd^{113}	1/2	−0·65	—
H^3	1/2	2·9754	—	In^{113}	9/2	5·48	..
Li^6	1	0·8213	..	In^{115}	9/2	5·49	0·8
Li^7	3/2	3·2532	..	Sn^{117}	1/2	−0·89	—
Be^9	3/2	−1·176	..	Sn^{119}	1/2	−0·89	—
B^{10}	(1)	(0·597)	..	Sb^{121}	5/2	3·7	..
B^{11}	3/2	2·682	..	Sb^{123}	7/2	2·8	..
C^{13}	1/2	0·701	—	I^{127}	5/2	2·8	..
N^{14}	1	0·4023	..	Xe^{129}	1/2	−0·8	—
N^{15}	1/2	0·280	—	Xe^{131}	3/2	0·7	0·7
F^{19}	1/2	2·6248	—	Cs^{133}	7/2	2·572	..
Na^{23}	3/2	2·2148	..	Ba^{135}	3/2	0·836	..
Al^{27}	5/2	3·628	..	Ba^{137}	3/2	0·935	..
P^{31}	1/2	..	—	La^{139}	7/2	2·8	..
Cl^{35}	5/2	1·368	..	Pr^{141}	5/2	..	..
Cl^{37}	5/2	1·136	..	Eu^{151}	5/2	3·4	1·2
K^{39}	3/2	0·391	..	Eu^{153}	5/2	1·5	2·5
K^{40}	4	−1·290	..	Tb^{159}	3/2	..	..
K^{41}	3/2	0·215	..	Ho^{165}	7/2	..	..
Sc^{45}	7/2	4·6	..	Tu^{169}	1/2	..	—
V^{51}	7/2	..	..	Yb^{171}	1/2	0·45	—
Mn^{55}	5/2	3·0	..	Yb^{173}	5/2	−0·65	3·9
Co^{59}	7/2	2–3	..	Lu^{175}	7/2	2·6	5·9
Cu^{63}	3/2	2·43	−0·1	Lu^{176}	7	3·8	6
Cu^{65}	3/2	2·54	−0·1	Ta^{181}	7/2	..	..
Zn^{67}	5/2	0·9	..	W^{183}	1/2	..	—
Ga^{69}	3/2	2·11	1	Re^{185}	5/2	3·3	2·8
Ga^{71}	3/2	2·69	0·5	Re^{187}	5/2	3·3	2·6
As^{75}	3/2	1·5	0·3	Os^{189}	1/2 or 3/2	..	..
Se^{77}	1/2	..	—				
Br^{79}	3/2	2·6	..	Ir^{191}	1/2	..	—
Br^{81}	3/2	2·6	..	Ir^{193}	3/2	..	..
Kr^{83}	9/2	−1·0	0·15	Pt^{195}	1/2	0·6	—
Rb^{85}	5/2	1·345	..	Au^{197}	3/2	0·23	..
Rb^{87}	3/2	2·741	..	Hg^{199}	1/2	0·55	—
Sr^{87}	9/2	−1·1	..	Hg^{201}	3/2	—0·62	0·5
Cb^{93}	9/2	3·7	..	Tl^{203}	1/2	1·45	—
Mo^{95}	1/2	..	—	Tl^{205}	1/2	1·45	—
Mo^{97}	1/2	..	—	Pb^{207}	1/2	0·60	—
Pd^{105}	3/2	small	..	Bi^{209}	9/2	3·60	−0·4
Ag^{107}	1/2	−0·10	—	Pa^{231}	3/2	..	..

table. It will be noted also that very few of the measured spins are integer. The nuclei having integer spins have an odd number of protons and an odd number of neutrons, and, in general, such

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systems are not stable. These points are discussed more fully in Chapter IV.

4. Nuclear shape and quadrupole moments

Further study of the hyperfine structure of spectral lines, as pursued principally by Schüler and his collaborators, has produced two new types of information regarding nuclei in addition to information about spins and magnetic moments. The first of these is the so-called isotope displacement of spectral lines which was first observed in the spectrum of thallium† and has since been shown to be a phenomenon of fairly frequent occurrence among the heavier elements. It consists in a rather small displacement separating the lines due to the different isotopes of the element under investigation. Obviously the various components of this isotopic hyperfine structure have intensities proportional to the relative abundances of the corresponding nuclear species in the mixed element. They may or may not exhibit magnetic hyperfine structure also, and when, as is often the case, different components show different degrees of structure, a very complicated pattern may result.

As a characteristic example of this phenomenon we may consider the structure of the mercury green line, λ5,461 A.U. which is due to the transition $6^3P_2 \rightarrow 7^3S_1$.‡ This structure is shown in Fig. 4 with intensities of the various components indicated by the lengths of the lines. According to the analysis of Schüler, the components *f*, *g*, *k*, and *l* must be ascribed to the isotopes of even mass-number, ${}_{80}Hg^{198}$, ${}_{80}Hg^{200}$, ${}_{80}Hg^{202}$, and ${}_{80}Hg^{204}$, which presumably possess no spin ($i = 0$) since they give rise to single lines, whilst ${}_{80}Hg^{199}$ is responsible for the three components *c*, *k*, and *p* and ${}_{80}Hg^{201}$ for the eight components *a*, *b*, *d*, *e*, *f*, *m*, *n*, and *o*. The nuclear spins of these species with odd mass-number have been shown to be given by $i = 1/2$ and $i = 3/2$ respectively. Fig. 4 shows the components due to the even-numbered isotopes together with the 'centres of gravity' of those due to the odd-numbered species. The total intensities in the corresponding hyperfine structure patterns are represented in this figure. We see that the relative intensities due to the different components arrived at in this way, 12:11:27:10:32:8 (per cent. of the total intensity),

† H. Schüler and J. Keyston, *Zs. f. Phys.* **70** (1931), 1.

‡ H. Schüler, J. Keyston, and E. Jones, ibid. **72** (1931), 423; **74** (1932), 631.

are in good agreement with the relative abundances of the isotopes as determined by Aston, namely,

$$_{80}Hg^{198} : {}_{80}Hg^{199} : {}_{80}Hg^{200} : {}_{80}Hg^{201} : {}_{80}Hg^{202} : {}_{80}Hg^{204}$$
$$= 9{\cdot}89 : 16{\cdot}45 : 23{\cdot}77 : 13{\cdot}67 : 29{\cdot}27 : 6{\cdot}85.$$

Fig. 4 also shows that the isotope displacement is a monotonic function of the mass-number concerned, though the components due to the odd-numbered isotopes do not fall quite symmetrically

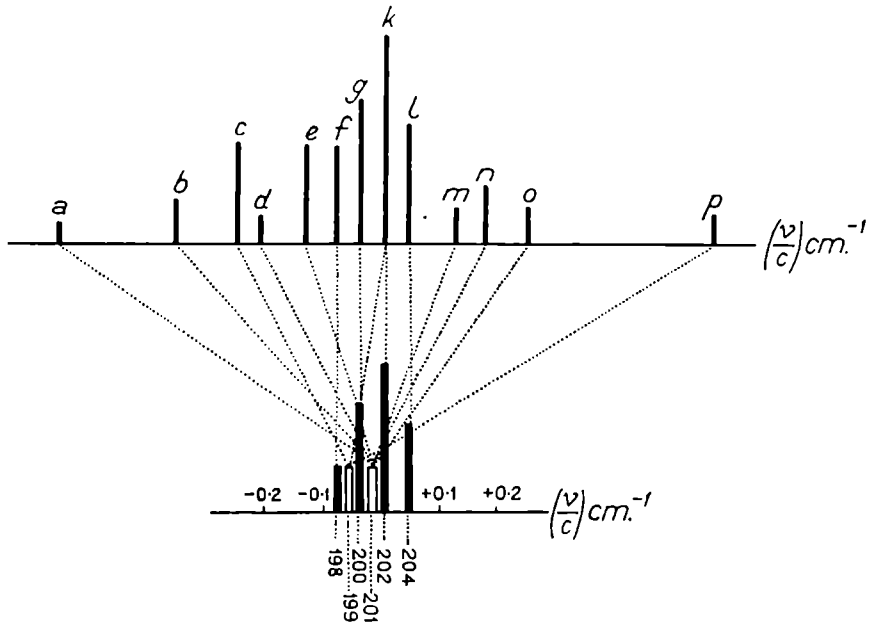


FIG. 4.

between those due to the isotopes of even mass-number. The average wave-number separation for successive even-numbered isotopic components ($\Delta M = 2$), in this particular case, is 28×10^{-3} cm.$^{-1}$

Turning now to the theoretical aspect of the problem it soon becomes evident that we must look for an explanation of isotope displacement in terms of deviations from the simple Coulomb law of force in the immediate neighbourhood of the nucleus. An explanation in terms of nuclear motion alone (compare the original calculations concerning the spectra of hydrogen and singly ionized helium) was shown to be quite inadequate to cover either the magnitude of the effect or the rather particular conditions in which it is observed. A complete discussion of the phenomenon, however, is rather complicated and entails a certain amount of arbitrariness concerning the nature of the modification produced by reason of the finite size of the

nucleus on the forces acting on the orbital electrons. The first calculations in this direction are due to Bartlett,† but by reason of the very rough approximations employed—neglect of relativity corrections and the screening effects of other atomic electrons—they could not be expected to lead to an accurate result. Somewhat more detailed calculations have since been made by Racah‡ on the basis of Dirac's equation for the motion of the electron and taking count of screening; independently, calculations of a similar nature have been carried out by Breit and Rosenthal.§ The method is to assume a reasonable form for the modification of the Coulomb field inside

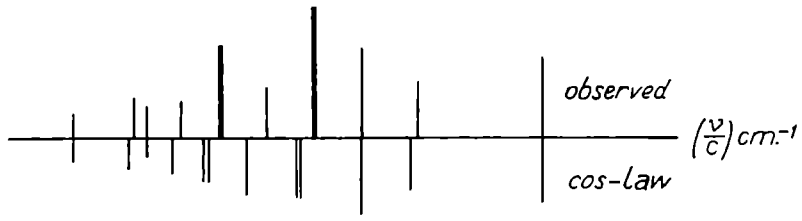


FIG. 5.

the nucleus, estimate the consequent effect on the wave-functions of the atomic electrons, and compute, from the perturbation of the wave-functions, the effect on the electronic energy-levels. Although the accuracy of the results of such calculations is not very great, it is adequate for comparison of the effects of nuclear size among isotopes of the same element and, in fact, Breit finds that the interpretation of the isotope shift in this way is in good agreement with the hypothesis of constant nuclear density.

The second result of further study of the hyperfine structure of spectral lines is the discovery, by Schüler and Schmidt,‖ that the shape of the nucleus is not spherical in certain isotopes. This shows up in the observations on the term system of the atom's hyperfine levels as deviations from the interval rule of Landé (p. 14). Fig. 5 represents the structure of the line λ6,865 ($a^8S_{7/2}$—$z^{10}P_{9/2}$) of europium as measured by these authors. The distribution to be expected from the interval rule (6) is plotted in the lower part of the diagram and we can see clearly that there are systematic deviations between the two sets of components. Schüler found from measurements on this

† J. H. Bartlett, *Nature*, **128** (1931), 408.
‡ G. Racah, ibid., **129** (1932), 723.
§ G. Breit and J. Rosenthal, *Phys. Rev.* **41** (1932), 459.
‖ H. Schüler and T. Schmidt, *Zs. f. Phys.* **94** (1935), 457; **95** (1935), 265; H. Schüler, J. Roig, and H. Korsching, ibid. **111** (1938), 165.

and other spectral lines that the deviations in evidence are rather closely proportional to the square of the cosine of the angle between i and j, so that the complete interaction between the atomic electrons and the nucleus includes a contribution to the energy of the form

$$\alpha \cos(i,j)+\beta \cos^2(i,j),$$

where α is a constant depending upon the magnetic effect in the electron orbit considered and β is a new constant inserted to account for the departures from the interval rule for the first term. These constants have different values for different atoms and for different electron waves in the same atom.

The accepted interpretation of the $\cos^2(i,j)$ term, i.e. in terms of a departure of the nucleus from spherical shape, may be illustrated as follows. Suppose a certain nucleus has a spin (in excess of $\frac{1}{2}\hbar$) and that the surface of the nucleus forms a prolate spheroid with the major axis along the direction of the spin-vector. Since the protons in the nucleus are probably fairly uniformly distributed throughout the nuclear fluid, and since the latter is assumed to be uniform in density, the electric field arising from our prolate nucleus will be slightly stronger in the direction of the spin-axis than in the plane of its equator. Consequently, the binding energy of an orbital electron will vary as the angle between its 'orbital plane' and the axis of nuclear spin is varied, first by reason of the magnetic interaction which has been discussed in the preceding section and which leads to the cosine term, and secondly by reason of the dependence of the nuclear electric field on angle. From the symmetry of the latter it is readily seen that its contribution will vary with $\cos^2$, in first approximation. Quantitatively, the angular dependence of the electric field is expressed by means of the well-known expansion of the field due to a finite, non-spherical charge in terms of the fields of various electric multipoles. The field arising from a non-spherical nucleus is then adequately approximated by adding a quadrupole potential to the monopole potential:

$$V_{\text{coul}} = \frac{Ze}{r} + \frac{3\cos^2\theta - 1}{r^3} qe,$$

where θ is the polar angle relative to the axis of spin, and q is called the *quadrupole moment* of the nucleus. In the example chosen above, of a prolate nucleus, q is positive, for an oblate nucleus it is negative.

As thus defined, q has the dimensions of length squared and is usually expressed in units of 10^{-24} cm.^{2} Its value for a given nucleus is then determinable from the observed hyperfine separations through the expression for V_{coul} and $|\psi|^2$ for the electron states involved. A number of such determinations have been made and the results appear in Table I.

One of the most striking and most important facts concerning quadrupole moments is the result that even the deuteron has one. It was discovered by Rabi and co-workers† in the course of their work on the molecular beam magnetic resonance method of determining magnetic moments, as described in the previous section. The magnetic moments of the proton and the deuteron, as well as certain molecular constants, can be obtained from work on the H_2 and HD molecules. From these, one can predict the resonance peaks for the D_2 molecule in the rotational state, $J = 1$ (para-deuterium). The resonance peaks actually measured for D_2, however, appeared at markedly different values of the homogeneous (second) field than those predicted. The deviations were found to be proportional to the square of the cosine of the angle between the nuclear moment and the molecular axis. Such deviations can be accounted for if the deuteron has an electrical quadrupole moment which interacts with the non-uniform electric field at each nucleus in the molecule. By computing the strength of the electric field, and using the experimental results, Nordsieck found the value of q for the deuteron to be $0{\cdot}00273 \times 10^{-24}$ cm.^{2} This highly significant discovery has a direct bearing on the nature of nuclear forces and will be discussed at greater length in the following chapter.

The geometrical interpretation of q can be derived at once by going back to potential theory and its expression in the form of V_{coul}. It is then seen that q is just the mean value of $\frac{1}{4}Z(3z^2-r^2)$, averaged over the charge density in the nucleus.

5. Nuclear binding energy

A valuable collection of information concerning nuclear structure is presented by the high-precision determinations of many of the nuclear masses. As has been stated already in the beginning of this chapter, one should not expect that the mass of an atom, formed by

† J. M. B. Kellogg, I. I. Rabi, N. F. Ramsey, Jr., and J. R. Zacharias, *Phys. Rev.* **55** (1939), 318; **57** (1940), 677.

the 'condensation' of N hydrogen atoms, must be exactly equal to N times the mass of the hydrogen atom. In fact, the formation of a stable configuration of this type must be accompanied necessarily by the liberation of a certain amount of energy (*binding energy*) which makes the mass of the composite system somewhat smaller than the combined masses of its components. This difference in mass is usually called the *mass-defect*, Δm, and is connected with the binding energy, BE (which is essentially negative) by the relativistic relation:

$$-\Delta m = \frac{BE}{c^2}, \tag{7}$$

where c is the velocity of light in vacuum.

The notion of a mass-defect is, of course, quite general and can be applied also in the case of composite chemical molecules as built from individual atoms. But, whereas in this case the defect of mass (equal to the total dissociation-energy of the molecule divided by c^2) is negligibly small as compared with the total mass of the molecule, atomic mass-defects represent a measurable fraction of the atomic mass, thus affording a valuable method for the direct experimental determination of the binding energies of the various atomic species. Strictly speaking, mass-defects determined by the measurement of the relative masses of different atoms are composed of two parts, corresponding to the internal binding energy of the nucleus and the binding energies of electrons in the atomic envelope; since, however, the binding energy of the atomic electrons is negligibly small compared with the nuclear binding, the experimental results may be interpreted in terms of nuclear energy alone. In speaking of the mass-defect of a composite system it is necessary to have clearly understood with respect to which structural units the mass-defect is to be calculated. Thus, for example, if we defined the mass-defect of helium as the mass-difference between the helium atom and four hydrogen atoms, we should get the binding energy of the helium nucleus as formed from four protons and two electrons. Since, as we have seen above, it is more rational to consider this nucleus as built of two protons and two neutrons, we should speak rather of the part of the energy responsible for the binding together of these particular structural units. This can be done arithmetically either by subtracting from the quantity defined in the preceding sentence the (negative) mass-defects of two neutrons (relative to their proton-electron

constitution), or more directly by calculating this mass-defect as the difference between the mass of the helium atom on one side and two hydrogen and two neutron masses on the other. In general, the nucleus of an isotope with atomic number Z and mass-number A consists of Z protons and $A-Z$ neutrons, so that the conventional mass-defect of the *atom* of this isotope is found by subtracting the mass of the atom from the sum of $(A-Z)$ neutron masses, m_n, and Z hydrogen masses, m_H:

$$\Delta m = Zm_H+(A-Z)m_n-m_{Z,A}, \tag{8}$$

where $m_{Z,A}$ is the atomic mass in question.

The accepted convention in nuclear physics is to express relative atomic masses on an arbitrary scale in which the neutral atom of the O^{16}-isotope has the atomic weight exactly 16·0000.† The unit of this system is called the *mass-unit* (m.u.) and is equivalent to $1{\cdot}483\times10^{-3}$ erg or 932 M.e.v.

Relative masses of naturally occurring isotopes with atomic weights less than 50 can be obtained with great precision by comparing atomic and molecular ions of almost equal masses in a linear-scale mass-spectrograph. An important example of the operation of this 'doublet' method in mass-spectroscopy is given by the work of Bainbridge and Jordan‡ in determining the masses of H^1, D^2, and C^{12} relative to O^{16}. This was done by comparing the deflexions in their spectrograph of the almost equally heavy ions:

$$CH_4^+ \text{ with } O^+$$
$$H_2^+ \text{ with } D^+$$
$$D_3^+ \text{ with } C^{++}.$$

The results of this comparison, and of many others of the same type, are included in Table II. With such direct measurements as a basis it is then possible to compute the masses of the light, radioactive species by simply adding the mass-equivalents of the energy released in the transmutation. These results are given also in Table II (the content of which is taken mostly from the article by Flügge and Mattauch, 'Isotopenberichte 1942', *Phys. Zeits.* (1943) 181). Among the masses deduced from radioactive energies are those for the

† This differs slightly from the conventional chemical system of atomic weights where the atomic weight 16·00000 is ascribed to the natural mixture of oxygen isotopes.

‡ K. T. Bainbridge and E. B. Jordan, *Phys. Rev.* **51** (1937), 384.

TABLE II

Atomic Masses and Binding Energies

Isotope	*Mass*	*Binding energy (M.e.v.)*	*Isotope*	*Mass*	*Binding Energy (M.e.v.)*
n	1·008945	—	Mg^{27}	26·99167	223·48
H^1	1·008131	—	Al^{26}	25·99467	211·73
D^2	2·014724	2·18	Al^{27}	26·98980	224·46
T^3	3·017004	8·39	Al^{28}	27·99017	232·27
He^3	3·016988	7·65	Al^{29}	28·98739	243·36
He^4	4·003860	28·20	Si^{27}	26·99509	218·78
He^5	5·01543	25·76	Si^{28}	27·98663	234·97
He^6	6·0209	28·99	Si^{29}	28·98469	245·12
Li^6	6·016917	31·94	Si^{30}	29·98310	254·92
Li^7	7·018163	39·11	Si^{31}	30·98449	261·96
Li^8	8·024967	41·10	P^{29}	28·98969	239·70
Be^7	7·019089	37·49	P^{30}	29·98740	250·16
Be^8	8·007807	56·31	P^{31}	30·98256	262·99
Be^9	9·014958	57·98	P^{32}	31·98275	271·14
Be^{10}	10·016622	64·76	S^{31}	30·98783	257·33
B^9	9·016104	56·16	S^{32}	31·98089	272·12
B^{10}	10·016169	64·43	S^{33}	31·98014	281·14
B^{11}	11·012901	75·80	S^{34}	33·97699	292·40
B^{12}	12·0168	80·49	S^{35}	34·97897	298·88
C^{10}	10·02086	59·30	Cl^{33}	32·98568	275·23
C^{11}	11·015017	73·07	Cl^{34}	33·98083	286·7
C^{12}	12·003880	91·76	Cl^{35}	34·97884	298·25
C^{13}	13·007561	96·66	Cl^{36}	35·97865	306·75
C^{14}	14·007741	104·82	Cl^{37}	36·97769	315·97
N^{13}	13·009904	93·72	Cl^{38}	37·97997	322·18
N^{14}	14·007530	104·26	A^{35}	34·98465	292·08
N^{15}	15·004870	115·06	A^{36}	35·97790	306·69
N^{16}	16·0118	117·10	A^{37}	..	..
O^{15}	15·0078	111·58	A^{38}	37·97461	326·41
O^{16}	16·000000	127·20	A^{39}	..	..
O^{17}	17·00450	131·30	A^{40}	39·97549	342·24
O^{18}	18·00485	139·30	A^{41}	40·97740	348·79
O^{19}	19·0127	140·49	K^{39}	38·9750	333·6
F^{17}	17·00758	127·68	K^{40}	39·9756	341·4
F^{18}	18·00670	136·82	K^{41}	40·9730	352·1
F^{19}	19·00454	147·16	K^{42}	41·9745	359·1
F^{20}	20·0072	153·63	Ca^{40}	39·9749	341·3
Ne^{19}	19·00798	143·20	Ca^{41}	40·9747	349·8
Ne^{20}	19·998895	159·98	Ca^{42}	41·9717	360·9
Ne^{21}	21·00000	167·28	Ca^{43}	42·9737	367·4
Ne^{22}	21·99858	176·93	Ca^{44}	..	..
Ne^{23}	23·00080	183·19	Ca^{45}	44·9685	388·9
Na^{22}	22·00032	174·55	Sc^{43}	42·9752	365·2
Na^{23}	22·9964	186·53	Sc^{45}	44·9675	389·0
Na^{24}	23·99779	193·56	Sc^{46}	45·9670	397·8
Mg^{23}	23·00052	181·94	Sc^{49}	48·96676	423·1
Mg^{24}	23·99211	198·09	Ti^{45}	44·96976	368·2
Mg^{25}	24·99373	204·91	Ti^{46}	45·9655	398·5
Mg^{26}	25·99037	216·36	Ti^{47}	46·96474	407·5

Isotope	Mass	Binding Energy (M.e.v.)	Isotope	Mass	Binding Energy (M.e.v.)
Ti^{48}	47·9634	417·1	83 RaE	210·0547	1635·05
Ti^{49}	48·9648	424·1	AcC	211·0580	1640·28
Ti^{50}	49·9623	434·7	ThC	212·0631	1643·85
V^{51}	50·9579	446·4	RaC	214·0820	1642·88
V^{52}	51·9561	456·4	84 Po^{210}	210·0526	1636·22
Cr^{52}	51·95424	457·6	AcC′	211·0565	1640·90
Mn^{55}	54·9607	475·6	ThC′	212·0599	1646·04
Mn^{56}	55·9602	484·4	RaC′	214·0678	1655·35
Fe^{56}	55·9571	486·5	84 AcA	215·0722	1659·57
CO^{59}	58·9510	516·4	ThA	216·0758	1664·53
Co^{60}	59·9514	524·4	RaA	218·0844	1673·27
Ni^{58}	57·9597	499·2	86 An^{219}	219·0836	1680·83
Ni^{60}	59·9498	525·1	Tn^{220}	220·0865	1686·34
Ni^{61}	60·9540	529·5	Rn^{222}	222·0942	1695·89
Ni^{62}	61·9496	542·0	87 AcK	223·0959	1701·91
Ni^{64}	63·9474	560·6	88 AcX	223·0937	1702·20
Cu^{61}	60·9562	526·7	ThX	224·0967	1708·75
Cu^{63}	62·9488	550·3	Ra^{226}	226·1030	1719·51
Cu^{64}	63·9493	558·2	McTh I	228·1098	1729·83
Zn^{63}	62·9524	546·2	89 Ac^{227}	227·1052	1725·01
Zn^{64}	63·9486	558·0	MTh 2	228·1089	1729·87
			90 RdAc	227·1042	1725·24
81 AcC″	207·0469	1618·83	RdTh	228·1065	1731·43
ThC″	208·0526	1621·86	Io^{230}	230·1120	1742·94
RaC″	210·0622	1629·60	UY	231·1156	1747·91
82 RaG	206·0429	1613·42	Th^{232}	232·1183	1753·74
AcD	207·0445	1620·30	UX I	234·1253	1763·81
ThD	208·0465	1626·79	91 Pa^{231}	231·1146	1748·07
RaD	210·0555	1634·98	UX 2	234·1243	1764·01
AcB	211·0603	1638·87	92 U II	234·1210	1766·35
ThB	212·0646	1643·23	AcU(U^{235})	235·1244	1771·53
RaB	214·0739	1651·20	U I	238·1337	1787·80

heaviest elements, the naturally radioactive species. Although the relative masses of the heavy elements are quite accurate for comparison of one heavy isotope with another, there is still some uncertainty in the comparison of these masses with that of O^{16}. In other words, the absolute values of the masses and binding energies given for the heavy atoms are not as accurate as those given for the light atoms.

Considerable progress has been made in recent years in determining the masses of isotopes heavier than $A = 50$, principally by Aston, Mattauch, and Dempster. Except for the noble gases, however, the precision of measurement does not compare with that obtained on the lighter elements. A typical determination in this region is the comparison of La^{139} with Ti.† The threefold ionized La atom falls

† A. J. Dempster, *Phys. Rev.* **53** (1938), 869; also ibid., p. 74.

between the singly ionized Ti^{46} and Ti^{47}. By using the separation of the latter as a standard scale the deviation of the La mass from its integral value of 139 is determined from the known deviations of the Ti isotopes. It is readily seen that this method measures most directly the fraction f in the equation:

$$m_{A,Z} = A(1+f).$$

Hence the experimental results are customarily tabulated in terms

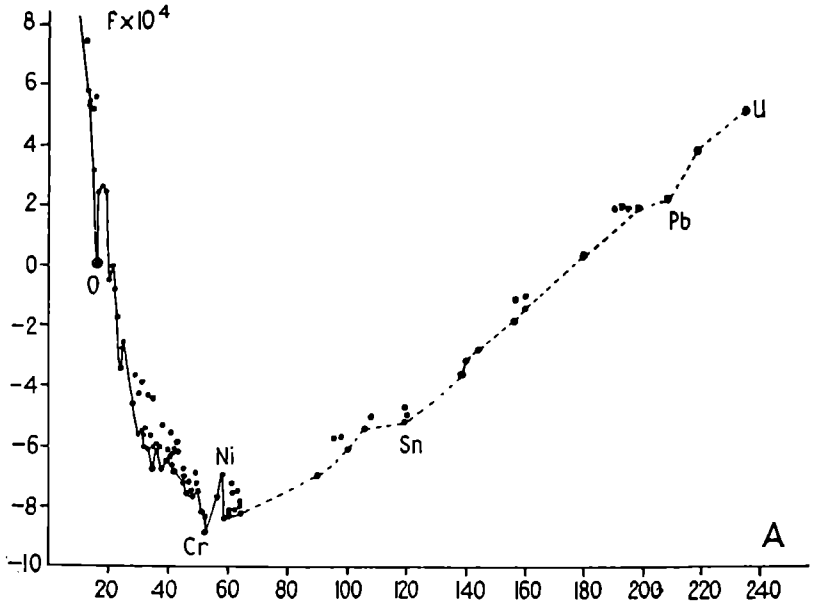


FIG. 6.

of f, known as the *packing fraction*, in the form of $10^4 f$. The binding energy of a nucleus may then be computed by taking the mass-excess of neutron and proton into account, viz.

$$BE = A\,(f - 0{\cdot}00853) - 0{\cdot}0004\,(A - 2Z) \qquad (9)$$

in mass-units. Hence, except for the relatively small and uniformly increasing term in $A-2Z$, the packing fraction plotted as a function of A gives the variation of the average binding energy per nucleon. A graph of $10^4 f$ as a function of A is shown in Fig. 6 for elements heavier than O^{16}. The curve is drawn through the lowest value of f for those isotopes for which several values have been determined. Besides the prominent irregularities in the curve, which are partly due to experimental errors and partly significant, there are several features of the dependence of f on A that are of fundamental importance to the theory of nuclear structure. From the connexion between

f and the binding energy per particle, eq. (9), we see at once that the average binding energy per nucleon is practically the same in all nuclei heavier than O^{16}. Starting at O^{16}, with an average binding of $-8{\cdot}53$ M.m.u., the binding energy decreases to a minimum of about $-9{\cdot}6$ M.m.u. at Cr^{52} (or thereabouts) and then rises with increasing A to about $-8{\cdot}1$ M.m.u. at U^{238}. The binding of nucleons in a large nucleus is then quite similar to the binding of molecules in a condensed phase, such as a crystal or a liquid, i.e. the total energy of binding is proportional to the number of units, in first approximation. This fact, plus the constant value of nuclear density discussed in section 2, forms the foundation of the liquid droplet model of the nucleus. Despite these, and other, points in analogy between nuclei and liquids it must always be borne in mind that nuclei are composed of relatively few particles, so that most of the component particles are on the 'surface' of the droplet. Moreover, the nucleons condense to such a high density that they form a degenerate gas, whereas degeneracy is of no importance to liquids excepting liquid helium; and finally, some of the nucleons carry a permanent electric charge.

The characteristics just mentioned are reflected in the packing-fraction curve, Fig. 6. The decrease of f as A increases from 16 to 52 is evidently due to the fact that a nucleon is more often surrounded by others in the heavier nuclei, i.e. the positive surface energy per nucleon diminishes with increasing A. In ordinary liquids, f would decrease uniformly and approach asymptotically the value for an infinite volume. In the universal nuclear fluid, however, the binding energy per particle is weakened by the electrostatic energy of the protons. Since the Coulomb energy of a nucleus is proportional to $Z(Z-1)/\sqrt[3]{A}$ and the cohesive energy (neglecting the surface effect) varies only as $-A$, the electrostatic effect becomes of major importance in the known heavy nuclei. This accounts for the rise of f with A as A increases above 60. There is, consequently, a most stable finite size of droplet for the nuclear fluid (perhaps Cr^{52}). Furthermore, the elements heavier than U^{238} are so unstable relative to subdivision into smaller nuclei, i.e. *fission*, that they do not occur in nature. Considering energy alone, practically all the heavy nuclei are unstable against fission (and light nuclei unstable against fusion), but the reaction rates for these processes are so low as to be effectively zero. The exceptional cases of neutron induced fission in uranium and of

thermally induced fusion of hydrogen at the centre of stars are discussed in detail in Chapters IX and X.

The gross aspect of the binding energy of a nucleus may be expressed formally as the sum of three terms:

1. The negative *volume energy* which is due to the cohesive forces between nucleons and is partially balanced by the zero-point kinetic energy; both effects are directly proportional to the volume, or weight, of the nucleus.
2. The positive *surface energy* which represents a deficit in the volume energy due to the fact that the particles located on the surface do not have a complete set of neighbours; this energy must be proportional to the free surface of the nucleus, or (because of the constant density) to the $\frac{2}{3}$ power of its weight.
3. The positive electrostatic energy discussed previously.

The rough formula then becomes:

$$BE = -aA + bA^{\frac{2}{3}} + c\frac{Z(Z-1)}{A^{\frac{1}{3}}}, \tag{10}$$

where a, b, c are positive factors, the same for all A. With properly chosen values of the constants the initial drop and subsequent rise of BE as a function of A and Z can be matched reasonably well. It is evident, however, that eq. (10) is not completely satisfactory as it predicts that for a nucleus of given A the lowest energy is obtained with $Z = 0$. This contradiction to observation is removed by taking the Pauli exclusion principle into account in the considerations of the volume energy and is taken up more fully in Chapter IV.

It is evident also that a formula of the simple type of eq. (10) cannot account for the marked fluctuations and kinks in the packing-fraction, or mass-defect, curve. Two kinds of such fluctuations can be distinguished. The first kind shows up as a regular, periodic variation of binding energy having a period of four mass units in the light elements and a period of two in the heavy elements. This is related to the especial stability of α-particle 'shells' mentioned in § 1 and can be predicted when the exclusion principle is taken into account. The second kind of kink is represented by an occasional sharp bend in the general trend of the mass-defect curve. Except for the one that occurs at O^{16} the masses are not well enough known near these kinks to evaluate their true meaning.

Particularly good information about relative atomic masses is

available for the naturally radioactive elements where the energy-differences between the members of the same radioactive family are calculable directly by the energy-liberation in the corresponding α- and β-transformations. Thus, plotting the energies of α-particles emitted by various radioactive substances against the corresponding values of atomic number and against the neutron-protons ratio, N/Z† (Plate Ia), one obtains an energy surface (Plate Ia) which

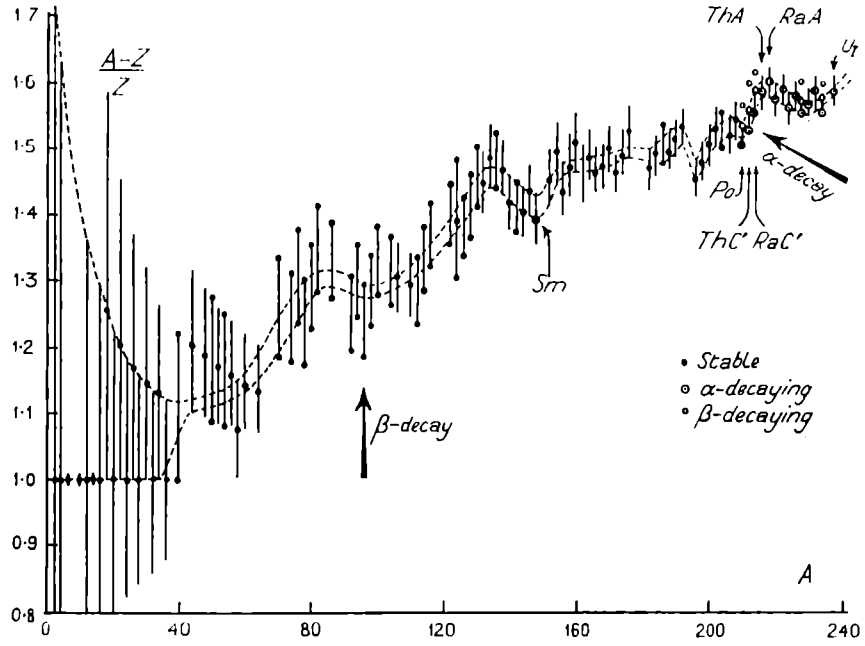


FIG. 7.

shows a sharp peak near $Z = 84$, and also a suggestion that a similar peak exists in the region of the transuranium elements, somewhere near $Z = 100$. These facts prove beyond any doubt that in the case of nuclear structure we encounter the formation of saturated *nucleonic shells*, similar to the familiar electronic shells which are responsible for the periodic properties of atoms. Owing to the absence of exact data concerning the binding energy of intermediate nuclei, it has been necessary to try to establish possible proton and neutron numbers for which closed shells are formed on the basis of other data, mostly connected with the relative numbers of neutrons and protons in the nuclei of stable isotopes.

If the values of neutron-proton ratios $N/Z\{ = (A-Z)/Z\}$ for all known stable isotopes are plotted against the corresponding mass-

† W. Heisenberg, *Rapport du VII^me Congrès Solvay*, 1934.

numbers A (Figs. 7 and 8, which are constructed for even and odd atomic weights, respectively) one notices a well-expressed undulation indicating that at least six or seven shells are being completed between the light and heavy nuclei.† The phenomenon can be represented in a still more striking way by plotting the relative (per cent.) abundances of the isotopes of the even elements as the height of a mountain range. A plot of this kind is shown in Plate I *b*, where the range is mapped in

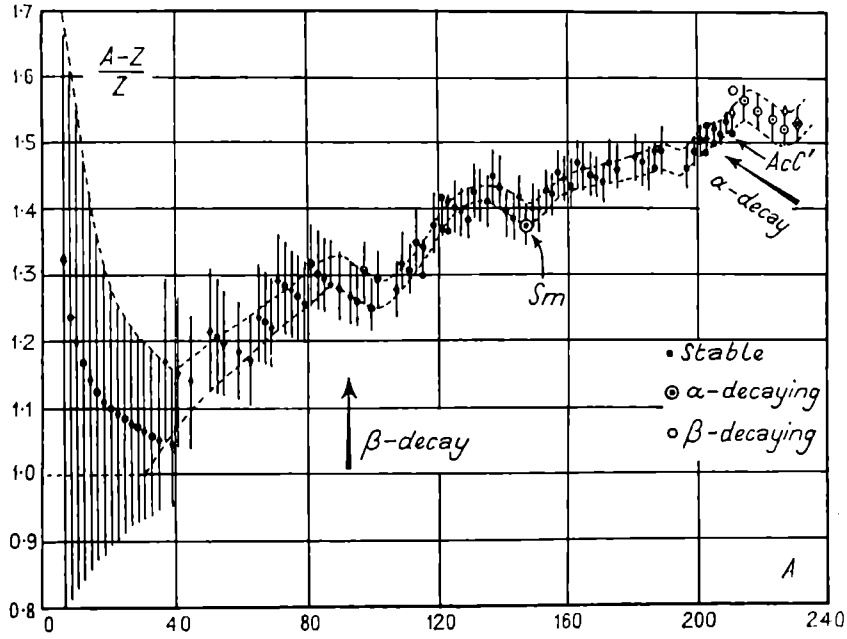


FIG. 8.

the $[N;\ N-Z]$ plane running from $N = 30$ to $N = 128$, and from $N-Z = 6$ to $N-Z = 44$. The peculiar nature of the neutron numbers 50 and 82 is quite evident from the plot. It is probably also significant that the fragments of the fission of heavy elements cluster about these neutron numbers rather than about the midpoint for the division. In the case of protons, the numbers 50 and 82 also play a prominent role which is emphasized by the extraordinary isotopic variety of Sn ($Z = 50$) and the location of the Heisenberg in the energy-plot of Plate I *a*. Many authors have proposed that nucleonic shells are closed at these numbers, as well as other less marked numbers. On the other hand, there is at present no adequate theory of the nuclear shell-structure. We shall return to these questions briefly in Chapter IV.

† G. Gamow, *Zs. f. Phys.* **89** (1934), 592.

II

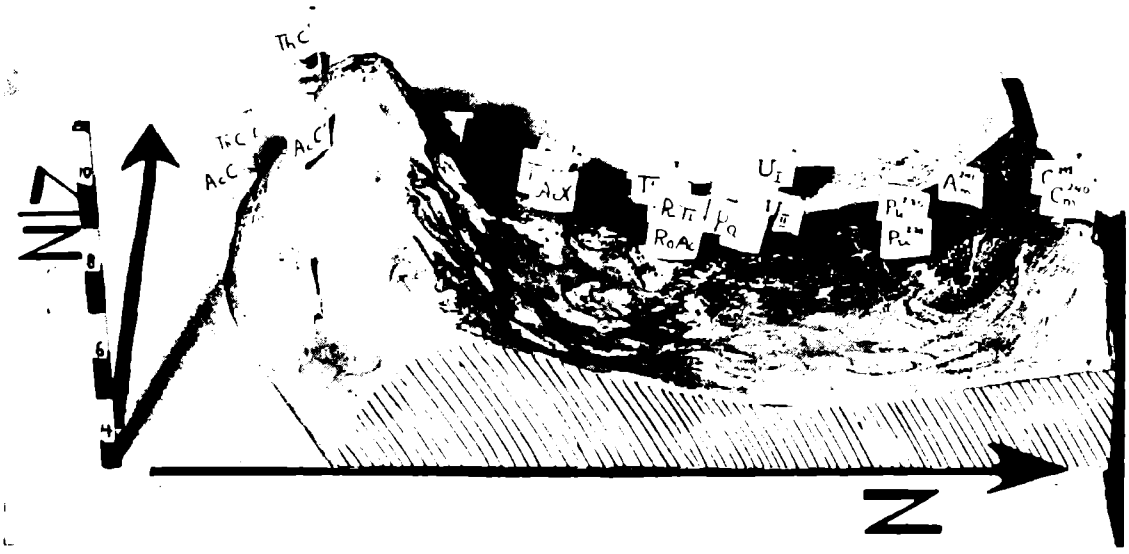
THE FORCES BETWEEN NUCLEONS

1. The forces between protons

As we have seen in the preceding chapter, studies of the general properties of atomic nuclei indicate that they exist as stable mechanical systems due to attractive forces acting between individual nucleons. A more detailed knowledge of the laws of these forces, which is important for the understanding of their intrinsic nature as well as for the development of the complete theory of composite nuclei, can be obtained best from the experimental study of the collisions between individual nucleons. Thus the scattering of a beam of protons by hydrogen gas, which is measured as the scattered intensity as function of angle of scattering, will deviate from predictions made on the basis of electrostatic forces alone. An analysis of the deviation then gives information concerning the nuclear forces exerted between two protons. The forces between neutron and proton can be evaluated directly from the observed scattering of neutrons by hydrogen and by analysis of the stable state of the deuteron. Unfortunately, direct experiments on neutron-neutron collisions lie far outside the limits of possibility, because of the negligibly small collision-probability in the intersection of two neutron beams, as now obtainable. Thus, the only way of evaluating the neutron-neutron force lies in the study of nuclear binding energies; for example, the fact that the binding energies of H^3 and He^3 differ only by 0·74 M.e.v. (see Table II), which is a reasonable estimate of the additional Coulomb energy in the He^3 nucleus, indicates that the n-n forces are essentially equal to the p-p forces.

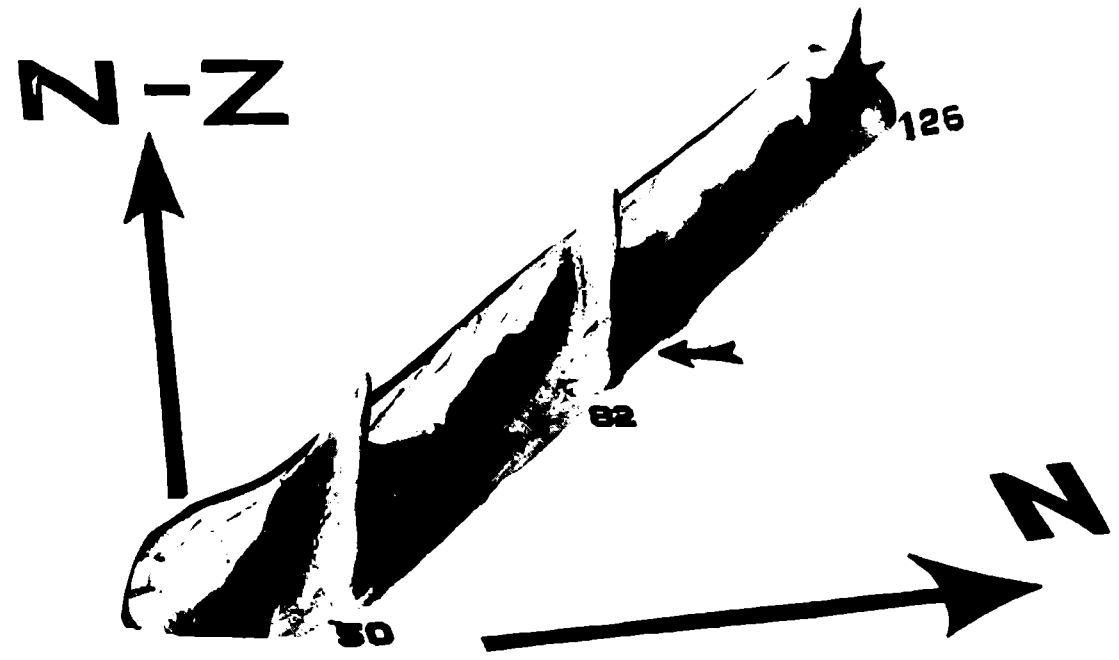
The existence of strong, short-range forces between two protons was first demonstrated by the scattering experiments of Hafstad, Heydenburg, and Tuve,† who showed that there were significant deviations from the scattering as expected for a pure Coulomb field. In Fig. 9 we give the ratios of the observed scattering to that expected from the Coulomb interaction alone (classical Rutherford's formula

† L. Hafstad, N. Heydenburg, M. Tuve, *Phys. Rev.* **49** (1936), 402; **50** (1936), 806; **51** (1937), 1023; **53** (1938), 239; **55** (1939), 603. The energy of collision was extended by the work of R. G. Herb, D. W. Kerst, D. B. Parkinson, and G. J. Plain, *Phys. Rev.* **55** (1939), 603, 998.



(*a*)

The 'Heisenberg'. A plot of the energies of α-decay in the $[Z, N/Z]$ plane, showing a maximum near $Z = 84$, and indicating the possibility of another maximum somewhere beyond $Z = 96$.



(*b*)

A plot of the percentage abundance of isotypes in $[N, N-Z]$ plane showing the distinct maxima at $N = 50$ and $N = 82$, and the rise towards the maximum at $N = 126$.

modified by Mott† to take into account the quantum mechanical symmetry of the wave-function for the two protons making a collision) for different scattering angles, and for different energies of incident protons.‡

It is evident from Fig. 9 that the observed differential cross-sections at low energies (~ 700 k.e.v.) and small angles of deflexion (~ 20°) are smaller than expected from electric forces alone. This indicates that when the nuclear forces between protons first come into

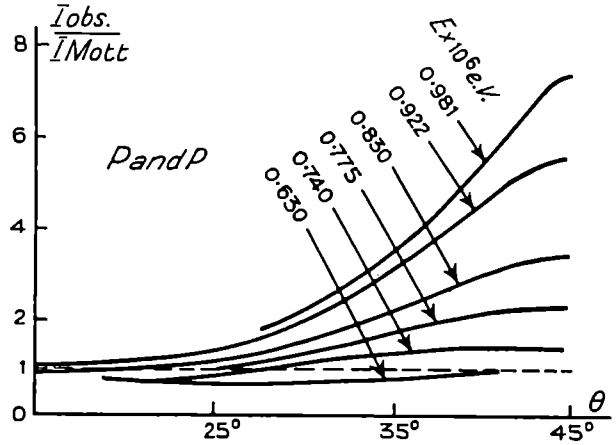


FIG. 9.

play they are attractive and partly annul the repulsive electric forces. At larger angles of deflexion and at higher energies of collision the effect of the nuclear forces predominates and gives rise to differential cross-sections much larger than those due to the Coulomb force alone. Furthermore, if we consider this excess scattering from a frame of reference in which the centre of gravity of the colliding protons is at rest, we find the scattering is nearly isotropic, as compared with the distinctly forward (axial) scattering in a Coulomb field. Thus we conclude from the results shown in Fig. 9, not only that the nuclear

† The formula, eq. (1) of Chapter I, is the correct expression for the differential cross-section for scattering in the Coulomb field alone, if the colliding particles are not indistinguishable. In considering proton-proton scattering, however, the quantum mechanical effect of symmetry must be taken into account, as mentioned above.

‡ In studying nuclear collisions, the distribution of intensity as a function of the angle of scattering is a most valuable source of information concerning the scattering process. This intensity distribution is customarily expressed in terms of the *differential cross-section*, which is defined as the number of particles scattered per unit solid angle, in the direction of scattering being considered, when the incident beam contains one particle per cm.^{2} The total cross-section is then the integral of the differential cross-section over all solid angles. In the case of the Coulomb field this integral diverges, because the field represents forces of infinite range, and one must use the differential cross-section in application to nuclear problems.

forces are attractive between protons, but also that they are 'short-range' forces, since apparently only head-on collisions, i.e. S-waves, are affected by the nuclear forces. The inference is that proton collisions in which the orbital angular momentum is more than zero are too distant for the nuclear forces to be operative, hence the spherical symmetry of the nuclear part of the scattered intensity. This interpretation is borne out by the fact that for very high energy of bombardment, especially for the neutron-proton collisions, the spherical symmetry gives way to a decided backward scattering of the beam, indicating that the higher angular momenta are in evidence in the operation of the nuclear forces.

Since the effect of the nuclear forces between protons evidently depends upon the absolute value of their relative angular momentum, it is no longer possible to interpret the scattering results on classical theory alone. As an introduction we present a brief sketch of the quantum treatment of scattering due to electric forces between particles of charge $+e$, mass M, and relative velocity v when the particles are infinitely distant. If we define the quantities

$$k = \frac{Mv}{2\hbar}, \qquad \eta = \frac{e^2}{\hbar v}, \tag{1}$$

the Schrödinger equation in the *relative coordinates* of the two particles (considering electric forces only) takes the form:

$$\nabla^2\Psi + \left[k^2 - \frac{2k\eta}{r}\right]\Psi = 0. \tag{2}$$

Let the coordinate z be the axis of the bombarding beam. We are interested in the solution of eq. (2) that behaves for large z as e^{ikz}. Now it is readily seen, by direct substitution, that eq. (2) has a solution of the form

$$\Psi = Ce^{ikz}F(r-z), \tag{3}$$

where C is a constant, r is the radial separation of the particles of which z is the component along the axis of bombardment, and $F(q)$ is a function that obeys the differential equation:

$$qF(q)'' + F(q)' - ikqF(q)' - k\eta F(q) = 0, \tag{4}$$

$$q \equiv r - z,$$

where primes indicate differentiation in the usual way. This eq. (4)

may be solved immediately by going over to the 'momentum' space for the variable q, i.e. let

$$\frac{d}{dq} = ip, \qquad q = i\frac{d}{dp}, \qquad F(q) = \int_{-\infty}^{\infty} f(p)e^{ipq}\,dp, \tag{5}$$

and replace q and derivatives with respect to q in eq. (4) accordingly, giving:

$$\frac{d}{dp}(kp-p^2)f(p) = -(p+ik\eta)f(p),$$

which can be integrated at once to give

$$f(p) = (k-p)^{i\eta}p^{-1-i\eta},$$

$$F(q) = \int_{-\infty}^{\infty} (k-p)^{i\eta}p^{-1-i\eta}e^{ipq}\,dp. \tag{6}$$

The functional form of the desired wave-function is then obtained by finding (6) as an explicit function of $q = r-z = r(1-\cos\theta)$. A rigorous treatment of these solutions is beyond the scope of this book,† but the asymptotic behaviour of $F(q)$ is readily determined by the residues at $p = 0$ and $p = k$ which may be evaluated by expanding the integrand in *integral* powers of p, or $(p-k)$, respectively. This may be done by using the appropriate form of the Laurent series for e^{ipq}

$$e^{ipq} = \sum_{m=-\infty}^{\infty} \frac{(ipq)^{m+\zeta}}{\Gamma(m+\zeta+1)},$$

where ζ is, in general, a complex number so chosen as to obtain integral powers about the branch-point being considered. Then by applying Cauchy's theorem in the usual way, $F(q)$ is found as the sum of two (descending) power series in kq, one for each branch-point. For very large r (more exactly, $k(r-z) \gg 1$) we need only the leading terms in these expansions, and one easily obtains the result by the method just outlined:

$$F(r-z) \to i\pi e^{-\frac{1}{2}\pi\eta}\left\{\frac{e^{i\eta \ln k(r-z)}}{\Gamma(1+i\eta)} - \frac{\eta e^{ik(r-z)-i\ln k(r-z)}}{\Gamma(1-i\eta)k(r-z)}\right\}.$$

For convenience in normalization we choose

$$C = \frac{-i\Gamma(1+i\eta)}{\pi}e^{\frac{1}{2}\pi\eta}$$

† See N. F. Mott and H. S. W. Massey, *Atomic Collisions*, Oxford, 1933, for more thorough discussion and further references.

and obtain, for the asymptotic form of the desired solution, equation (3):

$$\Psi \to e^{ikz+i\eta \ln k(r-z)} - \eta \frac{\Gamma(1+i\eta)e^{ikr-i\eta \ln k(r-z)}}{\Gamma(1-i\eta)k(r-z)}. \tag{7}$$

The first term is to be regarded as the incident beam of charged particles, normalized to one particle per unit volume; the second term is the scattered wave at large distances from the scattering centre, i.e. from the centre of gravity of two protons.

The wave-mechanical solution, as developed thus far, applies directly to the Coulomb interaction of non-identical particles (if $\frac{1}{2}M$, in eq. (1) for k, is interpreted more generally as the reduced mass of the colliding particles) and hence, must be equivalent to Rutherford's classical derivation, eq. (1) of Chapter I. From the asymptotic expression, eq. (7), we find the intensity of the scattered part of the beam, at large r and in the direction $\theta \equiv \cos^{-1}(z/r)$,

$$I_{\text{scat}} = \frac{\eta^2}{k^2 r^2 (1-\cos\theta)^2}.$$

The differential cross-section is numerically equal to the intensity per unit solid angle (i.e. $r^2 I_{\text{scat}}$) when the incident beam is normalized per unit volume; hence

$$\sigma(\theta) = \frac{\eta^2}{k^2(1-\cos\theta)^2} = \left(\frac{e^2}{Mv^2}\right)^2 \operatorname{cosec}^4 \tfrac{1}{2}\theta,$$

in agreement with the classical result. The equivalence ot the classical and quantum theoretical results for the scattering is made possible by the fact that the Coulomb field alone does not introduce a characteristic length, since the forces are of infinite range.

Before considering how the Coulomb cross-section is affected by symmetrizing the proton waves, we now introduce a short-range potential in addition to the electric potential and see how it influences the unsymmetrized, asymptotic form of solution, eq. (7). The customary procedure in such a study is to follow the method of Faxen and Holtsmark† and express the wave-function as a superposition of eigenfunctions of orbital angular momentum. The effects of the short-range forces in the states of different angular momentum are then considered independently. In fact, for the experimental work being discussed in this section, it is sufficient to evaluate the effect of the nuclear forces only for the S-wave part of the wave-function.

† H. Faxen and J. Holtsmark, *Zeits. f. Physik*, **45** (1927), 307.

The S-wave component of Ψ in eq. (3) can be obtained by averaging this expression (with F substituted from eq. (6)) over all θ. The resulting integral can then be evaluated by the method used in deriving eq. (7) and leads to the asymptotic expression for the S-component, Ψ_0:

$$\Psi_0 \to \frac{e^{i\beta}}{kr}\sin(kr-\eta\ln 2kr+\beta), \tag{8}$$

where β is defined by the relation

$$e^{2i\beta} = \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}.$$

The expression, eq. (8), for Ψ_0 at large distances is to be considered as the sum of an incoming spherical wave and an outgoing one of equal intensities. The incoming part, $(kr)^{-1}\exp\{-i(kr-\eta\ln 2kr)\}$, remains unaffected, of course, by the short-range forces. As the particles approach to within the range of the nuclear forces, however, both incoming and outgoing parts of the wave will be refracted differently than by the Coulomb field alone. The net result on the asymptotic form of the outgoing part will appear as a certain constant addition to the phase of that wave at large r which we represent by the factor e^{2iK_0}. The angle K_0 is called the phase-shift for S-waves and its magnitude is a function of the energy of collision. The modified form of Ψ_0 is, therefore,

$$\Psi_0^{(m)} \to \frac{e^{i(\beta+K_0)}}{kr}\sin(kr-\eta\ln 2kr+\beta+K_0). \tag{9}$$

The effect of the nuclear forces on the asymptotic expression for the complete wave-function may then be taken into account by adding the difference between eq. (9) and eq. (8) to the form in eq. (7). The result may be written:

$$\Psi^{(m)} \to e^{i\{kr\cos\theta+\eta\ln kr(1-\cos\theta)\}} - \left\{\frac{\eta\,\mathrm{cosec}^{2(1+i\eta)}\frac{1}{2}\theta - 2e^{iK_0}\sin K_0}{2kr}\right\}e^{i(kr-\eta\ln 2kr+2\beta)}. \tag{10}$$

If the colliding particles are distinguishable, the differential cross-section is r^2 times the absolute square of the expression in curly brackets. In classical theory, the cross-section for indistinguishable particles is the sum of cross-sections, $\sigma(\theta)+\sigma(\pi+\theta)$, due to the inability to tell particles that are scattered from those that are knocked on. But in quantum theory there is an additional

consideration to be made. If one proton is at r_a and the other at r_b the properly symmetrized wave-functions for the two protons are

$$\Psi_1(r_a)\Psi_2(r_b)+\Psi_1(r_b)\Psi_2(r_a)$$

in the singlet spin state, and

$$\Psi_1(r_a)\Psi_2(r_b)-\Psi_1(r_b)\Psi_2(r_a)$$

in the triplet. Since $r_a(\theta) = r_b(\pi+\theta)$ for particles of equal mass, and since the scattering at $\pi-\theta$ is the same as at $\pi+\theta$, the scattered wave at $\pi-\theta$ interferes constructively with that at θ in singlet collisions, i.e. $\frac{1}{4}$ of the time, and interferes destructively in triplet collisions, i.e. $\frac{3}{4}$ of the total number of collisions. If we denote by $f(\theta)$

$$f(\theta) = \frac{1}{k}\{\eta \operatorname{cosec}^{2(1+i\eta)}\tfrac{1}{2}\theta - e^{iK_0}\sin K_0\},$$

the differential cross-section then becomes:

$$\sigma(\theta) = \tfrac{1}{4}|f(\theta)+f(\pi-\theta)|^2+\tfrac{3}{4}|f(\theta)-f(\pi-\theta)|^2. \qquad (11)$$

This interference effect was first predicted by Mott† and proved experimentally by Chadwick‡ for the case of the scattering of α-particles in helium (in this case all waves are symmetrical in the two alphas since their spin is zero). The differential cross-section for scattering of protons by the angle θ in the centre of gravity system of coordinates is then:

$$\begin{aligned}\sigma(\theta) = \left(\frac{e^2}{Mv^2}\right)^2 &[\operatorname{cosec}^4\tfrac{1}{2}\theta+\sec^4\tfrac{1}{2}\theta-\sec^2\tfrac{1}{2}\theta\operatorname{cosec}^2\tfrac{1}{2}\theta\cos(\eta\ln\tan^2\tfrac{1}{2}\theta)]- \\ &-\frac{e^2}{Mv^2}\frac{\sin K_0}{k}[\operatorname{cosec}^2\tfrac{1}{2}\theta\cos(\eta\ln\sin^2\tfrac{1}{2}\theta+K_0)+ \\ &+\sec^2\tfrac{1}{2}\theta\cos(\eta\ln\cos^2\tfrac{1}{2}\theta+K_0)]+\frac{\sin^2 K_0}{k^2}. \qquad (12)\end{aligned}$$

The first term in the above expression, eq. (12), is the pure Coulomb scattering, the third term the purely nuclear scattering, and the second term the interference between them. To get the cross-section for the scattering between θ and $\theta+d\theta$, in the centre of gravity system we have to multiply the expression for $\sigma(\theta)$ by the solid angle $2\pi\sin\theta\,d\theta$. The corresponding expressions for the laboratory co-ordinates (in which one of the protons is initially at rest) are obtained by replacing every θ by 2θ. The largest effect of the short-range

† N. F. Mott, *Proc. Roy. Soc.* **126** (1930), 259.
‡ J. Chadwick, ibid. **128** (1930), 114.

forces comes at 90° in the centre of gravity system (45° in the laboratory), and at this angle:

$$\sigma(\tfrac{1}{2}\pi) = 4\left(\frac{e^2}{Mv^2}\right)^2\left\{1-2\frac{\sin K_0}{\eta}\cos(K_0-\eta\ln 2)+\frac{\sin^2 K_0}{\eta^2}\right\}. \qquad (13)$$

Usually when discussing the scattering of indistinguishable particles the differential cross-section is expressed as a ratio of the observed scattering to that predicted for Coulomb forces alone (denoted by $I_{\text{obs}}/I_{\text{Mott}}$ in Fig. 9) and is commonly referred to as 'the ratio to Mott Scattering', $R_M(\theta)$. Thus, $R_M(\frac{1}{2}\pi)$ expressed in terms of an S-wave phase-shift for protons wave is given by the quantity in the curly brackets in eq. (13). At one million electron volts, in the laboratory frame of reference, the value of η is about 1/6 and it becomes smaller as the energy increases. The effect of the phase-shift due to nuclear forces is therefore greatly amplified at these energies. Further, due to the interference, the sign of K_0 is determined.

The application of the formula, eq. (12), to the experimental curves for p-p scattering shown in Fig. 9 has been carried out principally by Breit and his associates.† The procedure is to determine the value of K_0 that must be used in eq. (12) to account for the experimental results at each energy of collision and each scattering angle that has been observed. It was found that the experimental results at various angles of scattering, but for a given energy of bombardment all correspond to one and the same value of K_0; this result substantiates the assumption that, for the energies used in the above experiments, the nuclear, short-range forces influence only the S-wave‡ (since only this wave was modified in derivation of the formula).

From one point of view, the nuclear forces in S-waves are adequately specified by giving K_0 as a function of the energy of collision. This viewpoint has been elaborated, in recent years, in the theory of the *scattering matrix*. For the present it is useful and instructive to interpret the observed values of K_0 on the basis of a particular model for the short-range forces. Following the work of Breit (loc. cit.) we adopt the simplest form of model, viz. the rectangular well potential

† F. Yost, J. Wheeler, and G. Breit, *Phys. Rev.* **49** (1936), 174; G. Breit, E. O. Condon, and R. D. Present, ibid. **50** (1936), 825; G. Breit, H. M. Thaxton, and L. Eisenbud, ibid. **55** (1939), 603, 1018.

‡ It may be noticed that for higher proton-energies modification of the P-waves, as well as of the interference between S- and D-waves will play an essential part. The analysis for the general case is presented in Chapter VIII.

illustrated in Fig. 1, p. 9. The complete specification of the potential acting between two protons is then given by

$$\begin{aligned} U &= -U_{pp} \quad \text{for} \quad r \leqslant a, \\ U &= \frac{e^2}{r} \quad \text{for} \quad r > a, \end{aligned} \tag{14}$$

where a is the 'range' of the nuclear forces. The positive constant, U_{pp}, measures the depth of the proton-proton interaction at small distances. Inside, $r = a$, therefore, the Schrödinger equation becomes

$$\nabla^2\Psi + \frac{M}{\hbar^2}[E + U_{pp}]\Psi = 0. \tag{15}$$

The S-wave solutions of eq. (15) that are bounded at the origin are well known to be

$$\Psi_{\text{in}} = A\frac{\sin r\sqrt{\{M(E+U_{pp})\}/\hbar}}{r}. \tag{16}$$

The amplitude, A, is determined from the continuity of this solution with the solution for $r > a$. For $r > a$ the Schrödinger equation is given by eq. (2) as before, if we identify k^2 with $ME/\hbar^2$ of eq. (16). The complete solution is then available, in principle, from eqs. (3) and (6) above. Instead of the asymptotic forms at large argument, however, we now need the forms near $r = 0$. If we evaluate the spherical part of the bounded solution, eq. (3), without approximation we get a certain function which we shall denote by F_0/kr. This function goes over into $(\sin kr)/kr$ as $\eta \to 0$, and is the proper S-wave solution if the Coulomb field extends unchanged to the origin. Since, however, we are interrupting the Coulomb field at $r = a$ the solution outside must fit smoothly on to the solution inside, eq. (16), and this means, in general, that the solution outside will be a linear combination of F_0/kr and the companion, improper solution, G_0/kr, corresponding to $(\cos kr)/kr$ in the absence of a Coulomb field. Hence we may write the solution for $r > a$ in the form

$$\Psi_{\text{out}} = \frac{F_0\cos K_0 + G_0\sin K_0}{kr}, \tag{17}$$

where K_0 has the same meaning as before. The logarithmic derivatives of the functions (16) and (17) must be equal at $r = a$ and this determines $\tan K_0$ as a function of E, U_{pp}, and a:

$$\left[\frac{\Psi'_{\text{in}}}{\Psi_{\text{in}}} = \frac{F'_0 + G'_0\tan K_0}{F_0 + G_0\tan K_0}\right]_{r=a}, \tag{18}$$

where primes mean differentiation with respect to r. Taking into account the general relation between F_0, G_0, and their first derivatives:

$$G_0 F_0' - F_0 G_0' = k,$$

we can write eq. (18) in the form:

$$\sqrt{\{M(E+U_{pp})\}}/\hbar \cot a\sqrt{\{M(E+U_{pp})\}}/\hbar = \frac{F_0'}{F_0} - \frac{k}{F_0 G_0 + F_0^2 \cot K_0} \quad (r = a). \quad (19)$$

Except for very high energy of bombardment, E, the correct functions, F_0, G_0, and F_0' differ very appreciably from their field-free

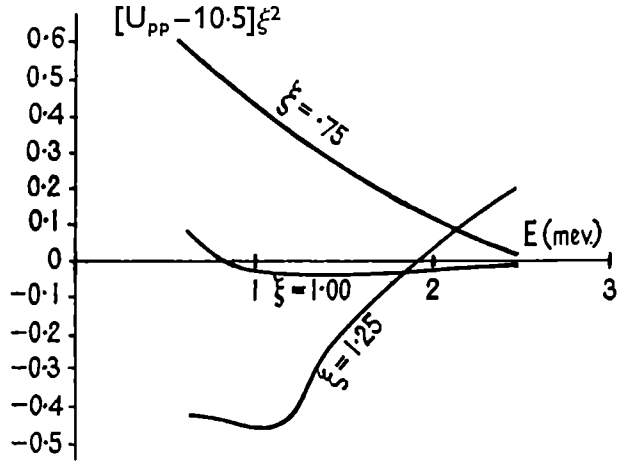


FIG. 10.

counterparts, $\sin kr$, etc., and they have been computed in the papers cited by Breit et al. for various values of k and r. From these tabulated functions, and the experimental values of K_0, the right-hand side of eq. (19) can be computed for the energies at which each K_0 was observed and for an arbitrary choice of the range of forces, a. The corresponding values of U_{pp} can then be determined by solving the left-hand side of this equation.

The values of U_{pp} computed from the experimental results at different energies are plotted as $U_{pp}\xi^2$ in Fig. 10 for three different ranges of force: $a = \xi(e^2/mc^2)$, viz. $\xi = 0{\cdot}75$, $1{\cdot}00$, and $1{\cdot}25$. It is apparent from this plot that U_{pp} is practically independent of energy if $\xi = 1$, i.e. $a = e^2/mc^2$, whereas for $a \gtrless e^2/mc^2$, U_{pp} respectively increases or decreases with energy. Therefore, $a = e^2/mc^2$ is the only range of forces consistent with a velocity independent, square-well potential. Assuming $a = e^2/mc^2 = 0{\cdot}283 \times 10^{-12}$ cm. we obtain:

$$U_{pp} = 10{\cdot}5 \text{ M.e.v.} \qquad a = e^2/mc^2 \quad (20)$$

as the most acceptable representation of the short-range forces between two protons. In the following section we shall compare this value with similar deductions from the neutron-proton interactions. In making the comparison, however, it must be remembered that the results expressed in eq. (20) are derived by cutting off the Coulomb field altogether at $r = a$. If we add to U_{pp} the (volume) average of the Coulomb repulsion inside the nuclear potential well: $e^2\left(\int_0^a r\,dr \Big/ \int_0^a r^2\,dr\right) = 0{\cdot}8$ M.e.v. we get $\overline{U}_{pp} = 11{\cdot}3$ M.e.v. for the effective depth of the nuclear potential well.

2. Forces between neutron and proton

The most direct manifestation of attractive forces between neutron and proton is the formation of the stable deuteron. This nucleus is to be regarded, from the point of view of quantum mechanics, as a stationary state formed by the mutual attraction between neutron and proton in somewhat the same way that the normal hydrogen atom is a stationary state formed by the Coulomb forces between the proton and the electron. The nature of nuclear forces appears to be quite different from Coulomb forces, however, as evidenced in the preceding analysis of proton-proton scattering data.

We shall approximate the nuclear potential by the same type of rectangular well that we used for the interpretation of the proton-proton forces. In fact, we shall assume the range of the forces, i.e. the radius of the well, to be the same as for the p-p forces and use the known binding energy to determine the depth of well in the deuteron.

Let r be the coordinate of the proton relative to the position of the neutron, M the mass of a nucleon (assuming proton and neutron to have the same mass), and E the energy of the system. Then the wave-equation for the relative motion becomes:

$$\begin{aligned} &\frac{\hbar^2}{M}\nabla^2\Psi(r,\theta,\phi)+[E-V(r)]\Psi(r,\theta,\phi) = 0, \\ &\qquad V(r) = -U_1 \qquad r \leqslant a, \\ &\qquad V(r) = 0 \qquad\quad r > a. \end{aligned} \tag{21}$$

Since the potential that has been assumed has spherical symmetry the state of lowest energy will be an S-state. In fact, due to the short range of the potential, states of higher orbital angular momentum would be hardly affected by the potential if their wave-lengths are

long compared with a. It is for the same reason, of course, that the physical results of the calculation are not particularly sensitive to the shape of potential well assumed. Accordingly, we let

$$\Psi(r,\theta,\phi) = \frac{u(r)}{r}$$

and the wave equation becomes:

$$\frac{d^2u}{dr^2}+\frac{M}{\hbar^2}(E-V)u = 0. \tag{21a}$$

Outside $r = a$, $V(r) = 0$ and the bounded solution for u is

$$u(r) = A\exp\left\{-\frac{\sqrt{(-ME)}}{\hbar}r\right\} \qquad r > a. \tag{22}$$

Inside $r = a$, $V(r) = -V_1$ and the bounded solution is

$$u(r) = B\sin[r\sqrt{\{M(E+V_1)\}}/\hbar] \quad r \leqslant a. \tag{23}$$

In eqs. (22) and (23), A and B are constant factors. The value of V_1 is then determined as a function of E, which is known to be $E = -2{\cdot}18$ M.e.v., and of $a = e^2/mc^2$, which is assumed on the basis of the proton-proton results, through the fact that solutions (22) and (23) must join smoothly at $r = a$. Equating logarithmic derivatives of $u(r)_a$:

$$[M(E+V_1)/\hbar^2]^{\frac{1}{2}}\cot a[M(E+V_1)/\hbar^2]^{\frac{1}{2}} = -(-ME/\hbar^2)^{\frac{1}{2}} \tag{24}$$

determines the value $V_1 = 41mc^2 = 21$ M.e.v.

The value of V_1 thus obtained is about twice as great as the corresponding value, U_{pp}, for the forces between protons, even though the range has been assumed to be equal. The disparity is well known to be connected with the fact that the protons interact in S-collisions only when they form the singlet spin-state (because of the Fermi–Dirac statistics), whereas the spins of the neutron and proton forming the deuteron must be parallel (triplet state) since the angular momentum of the system is unity. Hence the difference between the values of V_1 and U_{pp} is a manifestation of the *spin-dependence of nuclear forces*. Furthermore, the wave-function of the ground state is spherically symmetric so that no electrical quadrupole moment can be obtained. In spite of this defect, however, the simple rectangular well assumption will now be extended to the calculation of the scattering of neutrons by protons in order to determine how the forces between neutron and proton in the *singlet* state fit into the picture.

Scattering by short-range forces alone is easier to compute than when combined with the Coulomb field and it may be deduced from the results of § 1 by going to the limit of $e \to 0$. Nevertheless, we shall sketch the customary treatment of the problem for the sake of completeness. The wave equation is eq. (21) and we are looking for appropriate solutions with positive values of E. Consider a plane wave $e^{ikr\cos\theta}$ describing neutrons falling upon protons. We need be concerned only with the part of this wave that is spherically symmetric about the centre of gravity, since this part alone will be influenced by the nuclear forces if, as we assume, the 'wave-length', k^{-1}, is long compared with the range of forces, a. The spherical component of the plane wave is readily found by averaging over all θ

$$\Psi_0^0 = \frac{\sin kr}{kr}.$$

Owing to the nuclear forces this wave will become strongly refracted inside $r = a$, and the effect on the wave at large r will be a certain, constant phase-shift, δ. Since the incoming half of Ψ_0^0 cannot show the effect of the refraction, the general form of the spherical wave as influenced by the nuclear forces is then:

$$\Psi_f^0 = e^{i\delta}\frac{\sin(kr+\delta)}{kr}. \tag{25}$$

The difference between Ψ_f^0, eq. (25), and Ψ_0^0 is the scattered wave of the form $S(e^{ikr}/kr)$ with

$$S = e^{i\delta}\sin\delta.$$

Since the incident wave is normalized per unit volume, the total cross-section for scattering will be the number of scattered particles per unit radius,

$$\sigma_1 = 4\pi r^2|\Psi_f^0 - \Psi_0^0|^2 = 4\pi\left|\frac{S}{k}\right|^2 = \frac{4\pi}{k^2}\sin^2\delta. \tag{26}$$

The determination of the phase-shift, δ, as a function of energy is made as follows. Outside $r = a$ the wave-function is given by eq. (25), inside $r = a$ by eq. (23) as before, except that E is now positive. Equating the logarithmic derivatives at $r = a$

$$\frac{1}{\hbar}\sqrt{\{M(E+V_1)\}}\cot\frac{a}{\hbar}\sqrt{\{M(E+V_1)\}} = k\cot(ka+\delta), \tag{27}$$

analogous to eq. (24) in which E stands for the binding energy of the deuteron, $-\epsilon$. At low energies of bombardment, E, V_1 is much larger

than both E and ϵ so that the left-hand sides of eqs. (24) and (27) are equal, in first approximation. At low energy, also, ka may be neglected as compared with δ. In this approximation we get, equating the right-hand sides of eqs. (24) and (27), the rough estimates:

$$\cot\delta \cong \frac{\sqrt{(M\epsilon)}}{\hbar k}, \qquad \sigma_1 \cong \frac{4\pi\hbar^2}{M}\frac{1}{E+|\epsilon|}. \tag{28}$$

For more careful work, one solves eq. (27) for δ using the correct bombarding energy (in centre of gravity system, so this value is one-half the laboratory energy of the neutrons) and substitutes in eq. (26). Using the latter method with $V_1 = 21$ M.e.v. and $a = e^2/mc^2$, we find a total cross-section for very slow neutrons, i.e. $E = 0$,

$$\sigma_1(0) = 4{\cdot}43\times10^{-24}\ \text{cm.}^2 \tag{29}$$

This result, eq. (29), is to be compared with the observed cross-section for the scattering of thermal neutrons by hydrogen,† viz. $\sigma(0) = 20\times10^{-24}$ cm.^{2} By way of interpreting the discrepancy between the observed value and that calculated from the theory of the stable deuteron, Wigner pointed out that the calculation is based upon what is known about the triplet state of the deuteron only, and that it is possible to choose a value of $|\epsilon'|$ for the singlet state in such a way as to account for the experimental results. Let the cross-section for the singlet state, and vanishing energy of collision, be $\sigma_0(0)$. Since there is no correlation between the spins of the colliding particles, three out of four collisions will form the triplet state, on the average, and one out of four the singlet. The formula for the total cross-section is then

$$\sigma(0) = \tfrac{3}{4}\sigma_1(0)+\tfrac{1}{4}\sigma_0(0). \tag{30}$$

Substituting 20×10^{-24} for $\sigma(0)$ and $4{\cdot}4\times10^{-24}$ for $\sigma_1(0)$,

$$\sigma_0(0) = 67\times10^{-24}, \tag{31}$$

which is 15 times larger than $\sigma_1(0)$. From the rough formula eq. (28), putting $E = 0$, one would say that the 'binding energy' in the singlet state must be one-fifteenth the binding in the stable deuteron, or around 140 k.e.v. However, it is impossible to tell from the simple scattering formulae whether this binding energy is negative, as in the triplet case, or positive, in which event the singlet energy-level is 'virtual' rather than 'real'. In either case the value of ϵ' is so small that the depth of the singlet well can be estimated at once from

† V. W. Cohen, H. H. Goldsmith, and J. Schwinger, *Phys. Rev.* **55** (1939), 106.

eq. (24) by setting $E = 0$, replacing V_1 by V_0 and putting $a = e^2/mc^2$. This gives

$$V_0 \cong (\tfrac{1}{2}\pi)^2 \frac{\hbar^2}{Ma^2} \cong 25mc^2.$$

The nature of the S-wave scattering at low energy is most readily illustrated by solving eq. (21 a) for the simple case of vanishing E. The solution at radii larger than a is then a straight line, i.e. a wave of infinite length, and at radii smaller than a is the sine wave,

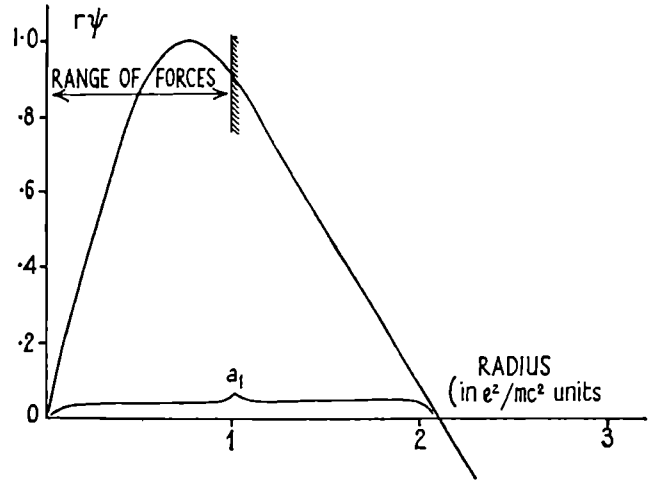


FIG. 11. Wave-function (times r) for n-p triplet state and zero energy.

$\sin r\{(\sqrt{-MV})/\hbar\}$. In Fig. 11 we show the graph of such a solution for the triplet state collisions. The straight-line part of the solution intersects the r-axis at the distance a_1 from the origin. The distance a_1 is obviously the *limiting value of wave-length divided by* 2π *times phase-shift* as the wave-length is increased indefinitely. Since in this case, however, a given phase of the wave is shifted to larger r the value of δ in eq. (25) is negative. Hence

$$a_1 = -\lim_{\lambda\to\infty} \delta_1 \frac{\lambda}{2\pi} = -\lim_{k\to 0} \frac{\delta_1}{k}. \tag{32}$$

From eq. (26) we see that also in the limit of vanishing E

$$\sigma_1(0) = 4\pi a_1^2. \tag{33}$$

In Fig. 12 is shown the analogous solutions for the singlet interaction. Both possibilities for the sign of ϵ' are represented. If there should exist a bound state for the singlet state, the wave inside the radius a would be curving downward at the range of forces and the intercept a_0 will be positive. The actual situation with the deuteron

is the opposite, viz. the nuclear forces are not quite strong enough to advance the phase of the inside wave past 90°. No bound state can be formed and the intercept a_0 is on the negative axis. This means that the phase shift, δ_0, is positive. The fact that the singlet cross-section is 15 times larger than the triplet then means:

$$a_0 = -3{\cdot}9a_1. \tag{34}$$

The extraordinarily large value of a_0 is obviously connected with the 'accident' that the straight-line solution has to be joined to the sine

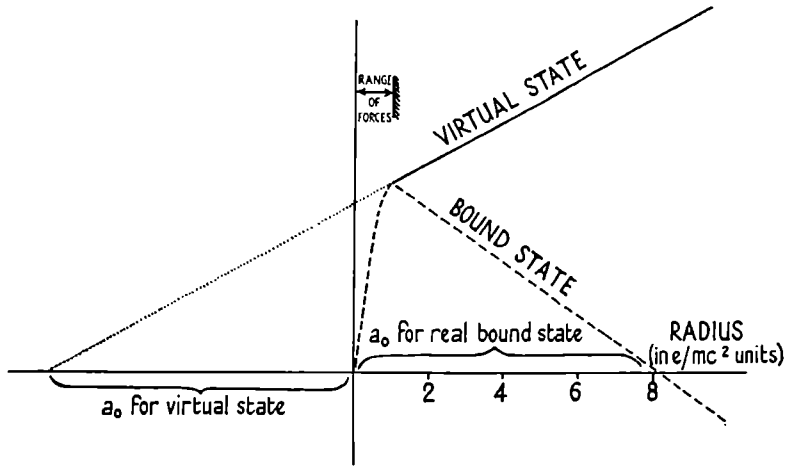


FIG. 12. Wave-function (times r) for n-p singlet state and zero energy.

wave so close to a maximum in the latter at 90°. This is the simplest illustration of the importance of wave-mechanical *resonance* phenomena to nuclear physics, but by no means the most striking example. The nearness to resonance in the singlet interaction between slow nucleons plays some part, also, in the proton-proton scattering.

It was stated that the phase-shift in the singlet collision is known to be opposite in sign to that in the triplet, as expressed in eq. (34). In other terminology, the singlet state of the deuteron is 'virtual' rather than 'real'. This has been very beautifully demonstrated by scattering slow neutrons on hydrogen. If hydrogen molecules are bombarded by neutrons that have energies characteristic of liquid air temperatures (0·012 e.v.), the wave-length of the neutrons will be 10 times as long as the separation of the protons in the molecule. The scattered wave will thus be the coherent superposition of amplitudes from the two protons (assuming elastic scattering). In the particular case of scattering from para-hydrogen the proton spins form the singlet state and, if the neutron spin is parallel to the spin

of one of the protons, thus forming the triplet, it forms half-triplet and half-singlet with the other proton. In terms of a_0 and a_1, the cross-section for inelastic scattering from a molecule of para-hydrogen will be proportional to

$$\sigma_{pp} \sim (3a_1+a_0)^2. \tag{35}$$

Hence the size of this cross-section, σ_{pp}, is vastly different, depending upon which sign a_0 has, relative to a_1.† For a_0 and a_1 of opposite sign, as given in eq. (34), the value of σ_{pp} is only one-sixtieth as large as if the a's had the same sign. The decisiveness of this comparison thus results from the fact that the singlet cross-section is so much larger than the triplet that a_0 almost cancels $3a_1$.

The experiment consists in scattering liquid air neutrons (0·012 e.v.) from para-hydrogen which is mostly in its ground state of zero rotation ($J = 0$). The energy required to raise the molecule to the ortho-state, $J = 1$, is 0·023 e.v., hence the scattered neutrons must leave the hydrogen in the para-state. The scattering observed under these conditions is then compared with the scattering from normal hydrogen ($\frac{3}{4}$ ortho and $\frac{1}{4}$ para). The result found is that the scattering cross-section per molecule in the usual mixture is many times greater than that in para-hydrogen alone. This proves conclusively that the singlet state of the deuteron is virtual.

Knowing that the singlet state is not stable the depth of the singlet well can now be computed unambiguously. For a range of forces, e^2/mc^2, the depth V_0 turns out to be

$$V_0 = 23mc^2 = 11{\cdot}6 \text{ M.e.v.}$$

in very good agreement with the depth of the singlet well for two protons, after the correction for the Coulomb energy is applied. This result suggests that nuclear forces are independent of the electric charge of the nucleon.

An analysis of the scattering of neutrons in ortho- and para-hydrogen has been made, by Schwinger, under the assumption that the spin of the neutron is $3/2\hbar$ instead of $\frac{1}{2}\hbar$. The level of the deuteron that lies near zero would then be a quintet. In this case, however, the fortuitous cancellation of terms does not occur and all cross-sections

† This theory is due to J. Schwinger and E. Teller, *Phys. Rev.* **52** (1937), 286, and the experiments have been carried out by F. G. Brickwedde, J. R. Dunning, H. J. Hoge, and J. H. Manly, ibid. **54** (1938), 266; W. F. Libby and E. A. Long, ibid. **55** (1939), 339, and by L. W. Alvarez and K. S. Pitzer, ibid. **58** (1940), 1003.

of molecular hydrogen are of the same order of magnitude. Thus, the experiments prove also that the spin of the neutron is $\frac{1}{2}\hbar$.

3. Non-central nuclear forces

The theory of the deuteron that has been developed in the preceding section is generally satisfactory except that it does not predict an electrical quadrupole moment for the ground state (see Chap. I). This defect arises from the assumption that the potential energy of the nucleons depends only on the separation, r, so that the lowest quantum state is an S-state. On the other hand, we have found that even such central forces depend upon the relative spin orientation of neutron and proton in the sense that the potential wells for the two spin-states must be given different depths for the same range. In Yukawa's description of the nuclear potential, as a generalization of the electromagnetic potential (cf. Chap. III), this simple type of spin-dependence would be expressed by choosing different strengths of source, g, for singlet and triplet states.

By generalizing the electromagnetic field a much more interesting and satisfactory possibility for introducing the spin-dependence presents itself, viz. to assume that each nucleon carries a *mesic dipole moment* parallel to its spin $\boldsymbol{\sigma}$,

$$\mathbf{m} = \frac{\mathfrak{f}}{\varkappa}\boldsymbol{\sigma}, \tag{36}$$

in addition to its monopole strength g. In eq. (36) $\mathfrak{f}$ has the same dimensions as g but is not necessarily equal to g numerically. The forces between nucleons will then depend not only upon their separation but also upon the orientation of the dipoles relative to each other and relative to the radius vector joining the positions of the nucleons. In electromagnetic theory, the potential energy due to two dipoles, $\mathbf{m}_1$ and $\mathbf{m}_2$ separated by the radius vector r takes the form

$$-V_{12} = \frac{3(m_1, r)(m_2, r) - (m_1, m_2)r^2}{r^5}. \tag{37}$$

The formula (37) contains the well-known fact that bringing two parallel dipoles together along a line perpendicular to their orientation, $(m_1, r) = (m_2, r) = 0$, creates the repulsive potential $m_1 m_2/r^3$; and bringing them together along a line coinciding with their axes develops an attractive potential of $-2m_1 m_2/r^3$. Under such forces alone, therefore, dipoles with parallel moments find their state of

lowest energy by lining up on an axis containing the moments. Under combined central forces and dipole forces there will be a tendency for the nucleons to be lined up along the axis of their resultant spin, i.e. the ground state of the deuteron should be somewhat 'cigar-shaped'. This is precisely what is required to account for the electrical quadrupole moment of the deuteron.

Although dipole-dipole nuclear forces follow naturally in associating nuclear forces with mesons of spin $\hbar$ and assuming that the nucleons carry mesic dipole moments, an exact application of the results of meson theory to the deuteron is not possible because the dipole potential diverges to minus infinite values with r^{-3}, cf. eq. (37). Given a (cigar-shaped) wave-function that falls to zero at $r > a$, the average potential energy in that state will be proportional to $-a^{-3}$, whereas the average kinetic energy will be of the order $\hbar^2/Ma^2$. Thus the total energy becomes lower the smaller a is chosen and there is no solution for a lowest energy, at least for the Schrödinger equation. Since the dipole potential must, therefore, be modified at small r and since it falls off exponentially at large r, in meson theory, we shall approximate the radial dependence of this part of the potential by the rectangular well. Let the radial dependence be $J(r)$ and the dependence on the angles included between the vector r and the spins of the nucleons be designated by S_{12}. The spin-dependent potential is then

$$\left.\begin{aligned} U_{12} &= -J(r)S_{12}\,(\text{spin, angles}) \\ S_{12} &\equiv \frac{3(\sigma_1, r)(\sigma_2, r) - (\sigma_1, \sigma_2)r^2}{r^2} \end{aligned}\right\}, \tag{38}$$

and the wave equation for the relative motion of neutron and proton becomes:

$$\left[\frac{\hbar^2}{M}\nabla^2 + E + J_0(r) + J(r)S_{12}\right]\Psi = 0, \tag{39}$$

where $J_0(r)$ is the part of the potential that does not depend upon angles, i.e. proportional to $\mathfrak{g}^2$. If the neutron and proton spins form the singlet state, S_{12} can be set equal to zero (since there can be no spin versus r correlation in a singlet state) and the equation solved as before. In the triplet state, let us start by assuming a wave-function with spherical symmetry in its space dependence:

$$y(r)\alpha(n)\alpha(p),$$

where $\alpha(n)$ designates the state of upward spin for the neutron, and

so on. β will be used to designate the state of downward spin component. Operating on this function with S_{12} leads to:

$$\begin{aligned} S_{12}y(r)\alpha(n)\alpha(p) = y(r)\{&(3\cos^2\theta-1)\alpha(n)\alpha(p)+ \\ &+3\cos\theta\sin\theta e^{i\phi}[\alpha(n)\beta(p)+\beta(n)\alpha(p)]+ \\ &+3\sin^2\theta e^{2i\phi}\beta(n)\beta(p)\} \\ \equiv\ & y(r)P_2(\theta,\phi,\alpha,\beta), \end{aligned} \tag{40}$$

where θ and ϕ are the usual spherical angles with the spin-axis as pole. The result, eq. (40), can be shown easily to be a ${}^3D_1$-wave. The wave-equation for $y(r)$ is coupled, therefore, with the wave equation for the radial dependence of a D-wave which we may assume to have the form: $w(r)P_2(\theta,\phi,\alpha,\beta)$. Putting this form in the wave-equation (39), the dependence on spin and angles in P_2 is preserved in all terms except that containing S_{12} which mixes it with a spherical wave, in addition to reproducing the same function with the coefficient $-2J(r)$. The wave-function for the triplet state of the deuteron must be considered as the sum of two components, an S-wave for one and a D-wave for the other. We get the differential equations that the radial functions $y(r)$ and $w(r)$, introduced above for the S-wave and D-wave, must obey by substituting in eq. (39) and factoring out the dependence on angles and spin:

$$\left.\begin{aligned} \frac{d^2y}{dr^2}+\frac{M}{\hbar^2}[E+J_0(r)]y+\sqrt{8}\frac{M}{\hbar^2}J(r)w &= 0, \\ \frac{d^2w}{dr^2}-\frac{6w}{r^2}+\frac{M}{\hbar^2}[E+J_0(r)-2J(r)]w+\sqrt{8}\frac{M}{\hbar^2}J(r)y &= 0. \end{aligned}\right\} \tag{41}$$

Part of the D-wave, cf. eq. (40), has the same spin polarization as the S-wave, viz. $\alpha(n)\alpha(p)$, and these parts will interfere to give the cigar-shape to the deuteron (provided, of course, that the sign of interaction is such that this is the state of lowest energy). The density distribution in the D-wave alone is readily computed by squaring the independent terms in the angular distribution, and adding, and is seen to be proportional to $5-3\cos^2\theta$ which is decidedly oblate. The amount of D-wave in the ground state of the deuteron turns out to be so small, however, that the interference between S- and D-waves far outweigh the effect of the D-wave alone. Rarita and Schwinger† have solved the equations (41) assuming $J(r)$ and

† W. Rarita and J. Schwinger, *Phys. Rev.* **59** (1941), 436.

$J_0(r)$ to be square wells of range $0{\cdot}28\times10^{-12}$ cm. and with depths in the ratio $\gamma = [J(r)/J_0(r)]_{r<a}$. They find that the binding energy and the observed value of the quadrupole moment are accounted for by choosing $\gamma = 0{\cdot}775$ and $J_0(r) = 13{\cdot}89$ M.e.v. $(r < a)$.† The intensity of the D-wave in their (numerical) solution is only 4 per cent. The amplitude is then about 1/5 as large as that of the S-wave and has the same sign, so that the interfering terms produce a charge density proportional to $3\cos^2\theta - 1$ as required.

† Slightly different values of J_0 were used for singlet and triplet interaction. If the range of force had been taken just a little smaller (5 per cent.) the values would have come out equal and the entire spin-dependence would be contained in the dipole-dipole interaction.

III

THEORIES OF NUCLEAR FORCES

1. Various hypotheses concerning nuclear forces

In the chapter just preceding this one we have dealt with the forces between nucleons by describing such forces essentially in terms of 'action at a distance' and by choosing the functional dependence of force on separation of the nucleons solely on the basis of convenience, i.e. using 'square well' potentials. Certain characteristics of the nuclear forces are put into evidence in the, rather empirical, methods of that chapter, namely, the short-range and great strength of these forces† and also their dependence upon spin orientation, especially the dipole-dipole nature of the interaction between neutron and proton. In this chapter we shall review the attempts to 'explain' the nuclear forces in the sense that one explains the forces between two electric charges by introducing the concept of an electric field. There, instead of the charges acting directly on each other, when they are not in contact, one charge acts as a source of electric field strength by distorting the field at a certain point in space; this distortion is then transmitted by the field, which is to be considered a separate physical system, to the position of the other charge. The interaction between the field and the second charge then produces the observed acceleration. Hence we are concerned with possible *field theories* of the nuclear forces.

A field theory of nuclear forces must be able to account for the characteristics of the interaction between two nucleons, mentioned above, and it must also lead to the formation of complex nuclei in which the binding energy is simply proportional to the number of nucleons and in which the density is roughly independent of the number of nucleons. This feature of the heavy nuclei has been presented in Chapter I and discussed in its relation to condensed phases of ordinary matter. It is readily seen that one cannot assume attractive potentials between all pairs of nucleons, of the type used in Chapter II, and at the same time account for the structure of the heavy nuclei. Forces between nucleons that obtain if the same potential applies between all possible pairs of nucleons in a complex nucleus

† It will be recalled that the depth of the nuclear potential between two protons was $11{\cdot}3/0{\cdot}8 \simeq 14$ times the average potential due to their electrostatic repulsion.

are called *ordinary forces*.† If all nuclear forces were ordinary forces, the average potential energy per nucleon would be proportional to the number of other nucleons that are to be found within the little sphere described about that nucleon with a radius equal to the range of nuclear forces. Consequently, the greater the density, the greater the (negative) potential energy, $\bar{V} \sim \rho$. As the density increases, however, the volume available to each nucleon becomes less, and the average kinetic energy must increase also. Effectively, if the density increases from ρ_0 to ρ_1 the average wave-length of a nucleon decreases by a factor $(\rho_0/\rho_1)^{\frac{1}{3}}$ and the average kinetic energy (non-relativistic) increases by the factor $(\rho_1/\rho_0)^{\frac{2}{3}}$. This is to be compared with the decrease in energy due to nuclear forces by the factor ρ_1/ρ_0. Thus, with ordinary forces, the energy of the nucleus can always be made lower by increasing the density (the Coulomb energy increases only as $(\rho_1/\rho_0)^{\frac{1}{3}}$) up to the point at which all nucleons lie within the range of nuclear forces of each other. The radii of all nuclei would be substantially equal to the range of forces and the binding energy would be proportional to the square of the total number of nucleons. This is in complete contradiction to the observed properties of nuclei, the radii of which increase with $A^{\frac{1}{3}}$ and the binding energy of which increases only with the first power of the total number of nucleons, A.‡

It is evident from the arguments just given that the nuclear forces must have the additional property that a given nucleon will be attracted to only a *limited number* of other nucleons at one time, i.e. the nuclear forces must show a *saturation character*. The saturation of nuclear forces could be accounted for in a number of conceivable ways, for example, by supposing that the depth of the attractive potential decreases as the velocity of the nucleons increases (at least as the first power), or by supposing that the attractive potential gives way to a very strong repulsive potential as the distance between nucleons is decreased. Neither of these possibilities is indicated by the study of the two-body systems presented in Chapter II, but data on S-wave scattering alone (i.e. relatively low energy) is hardly sufficient to exclude them. The tendency in developing a theory of the nuclear forces, however, has been to avoid these mechanisms as too arbitrary and too complicated to enter into the description of so fundamental and so simple (?) a phenomenon. This philosophy was

† Ordinary forces in nuclei are sometimes called Wigner forces.

‡ This argument is due originally to Heisenberg, *Solvay Congress*, 1933.

somewhat more tenable when it originated than it is now, since the discovery that the forces between nucleons depend upon angles indicates that the nucleons are not inspired particularly by simplicity. The real foundation of the doctrine of simplicity was, of course, the paucity of experimental results. Extension of the study of forces between protons and between neutron and proton to higher energy and high accuracy, now in progress, will produce the real basis for a description of the saturation character, as well as the other properties, of the nuclear forces. Such studies might lead to the necessity for giving the nucleon a 'structure', for example, that leads to a repulsion when two particles are too close together but to attraction when they are at certain separation in analogy to the forces between molecules that form an ordinary liquid.

The theory of the saturation character of nuclear forces that has been pursued with greatest vigour since the discovery of the neutron is based essentially on Heisenberg's (loc. cit.) original suggestion that when a neutron interacts with a proton the single electric charge jumps from one nucleon to the other so that, in the first such jump, the original proton changes into a neutron and the neutron into a proton. In this way, a neutron may interact with only one proton at a time and the forces have a saturation character that is somewhat related to ordinary chemical valence. This simple form of the idea of *exchange forces* does not give an adequate account of the equality of proton-proton and neutron-proton forces, since the former requires a double jump which is not only less probable but leads to a repulsion, but we shall use it to introduce the concepts underlying this approach to the saturation character.

A simple, atomic example of the exchange forces is found in the ionized molecule of hydrogen H_2^+. This is a stationary state of two protons, numbered 1 and 2, and one electron, and we shall consider in particular the state in which the molecule is not rotating. In an attempt to get an *approximate* description of this stationary state we may think of the electron being in the normal $1s$ orbit $\psi_1(\mathbf{x})$ about proton number one and proton number two being bare, and call this state $H(1)p(2)$, with the obvious meanings for $H(1)$ and $p(2)$. A second approximate state of exactly the same energy would be $p(1)H(2)$. Neither of these is a stationary state, so that if at a certain time the configuration is given by $H(1)p(2)$ the electron will subsequently 'jump' to the second proton, forming the state $p(1)H(2)$,

then jump back, etc. The stationary states in this case, as is well known, are either of the functions:

$$H(1)p(2)\pm p(1)H(2). \tag{1}$$

In this approximation, the forces holding the ion together can be computed by first finding the total energy of the electron plus the electrostatic repulsion of the protons as averaged in each of the states (1). This energy is then the effective potential for the ionic forces and usually is denoted $(C\pm A)/(1\pm S)$, the $\pm$ sign being determined by choosing one of the two states. Let V' = kinetic energy of the electron+Coulomb attraction to each proton+Coulomb repulsion of protons; then for given positions of the protons, and writing $H(1) = p(1)\psi_1(\mathbf{x})$,

$$S = \int \psi_1(\mathbf{x})\psi_2(\mathbf{x})\, d\mathbf{x},$$

$$C = \int \psi_1(\mathbf{x})^2 V'\, d\mathbf{x} = \int \psi_2(\mathbf{x})^2 V'\, d\mathbf{x},$$

$$A = \int \psi_1(\mathbf{x})\psi_2(\mathbf{x}) V'\, d\mathbf{x}.$$

We are particularly interested in the *exchange part* of the potential, A, which depends upon the symmetry of the wave-function. The part C is an *ordinary* potential. If we choose the plus sign in eq. (1), A represents the extra potential due to the fact that $\psi_1(\mathbf{x})$ and $\psi_2(\mathbf{x})$ interfere constructively in the region between the two protons. This enhances the electron density in the region of most attraction (and also gives a relatively long wave-length to the electron state so that the kinetic energy is lowered) and, in fact, accounts for the binding of the H_2^+ ion. If we had taken the minus sign, the electron waves interfere destructively, reduce the negative charge density in the most favourable region, and elevate the average kinetic energy of the electron with the result that, in this state, a hydrogen atom and a proton repel each other and there is no bound state of this type.

Heisenberg's proposal for nuclear forces abstracts from the elementary example given above, formally replacing hydrogen atoms H by neutrons n in eq. (1), considering *only* the exchange part of the interaction, and (what is the same thing) simplifying the interaction between the nucleons and the *field*. This proposal is of the general type of those that *would have* been made in atomic theory if a knowledge of molecular structure, band-spectra, etc., had been acquired far in advance of the structure of the atom. Then, being ignorant

of the intrinsic electromagnetic origin of the forces in the H_2^+ ion, one would account for the dependence of the potential between proton and hydrogen atom by supposing that when the potential acts, an electric charge jumps from atom to proton, thereby effectively exchanging the positions of atom and proton. From further information about the intrinsic angular momenta of H-atom and proton plus the observation, say, that H-atoms can be transformed into protons by irradiation which 'creates' electrons simultaneously, one would conclude that the atom-proton forces are characterized by exchange of a single electron. The electron in this case is the *field particle*, and since we are supposing that both the H-atom and the proton are elementary particles, i.e. we are disregarding the true structure of the atom, we should describe the interaction by saying that the atom 'creates' a free electron momentarily, and on borrowed energy, which is subsequently annihilated by the proton. In this process the atom changes to proton and the proton to atom and we have a simple manifestation of an exchange force which makes no reference to structural details *except* that different assumptions about the amount of energy that has to be borrowed will lead to different effective ranges of the force.

In applying the simple, general description of unknown forces to neutron and proton, we start with the facts that both these particles have spin $\frac{1}{2}\hbar$ and that neutron can change into a proton by simultaneous emission of an electron and a neutrino (and proton can change into a neutron by emitting a positron and neutrino, i.e. the β-decay). Thus Heisenberg suggested that nuclear forces are exchange forces in which electrons (positrons) and neutrinos act simultaneously as the field particles. In this event, however, the strength of the nuclear forces can be determined from the observed probability of β-decay. In Chapter V we shall see that the interaction between nucleons and the electron-neutrino field which will account for β-decay is characterized by the 'Fermi constant', $g \simeq 2 \times 10^{-49}$ erg cm.[3] The potential energy between two nucleons will then be proportional to the square of g, will depend upon the separation of the nucleons, r, and, since the electron states involved are mostly in the extreme relativistic region, the potential will be independent of the rest-mass of the electron. Exact calculations of this potential were first made by Ivanenko and Tamm,† but we shall consider here only an analysis

† D. Ivanenko and I. Tamm, *Nature*, **133** (1934), 981.

on the basis of physical dimensions. The problem is simply to construct a function of the dimension of energy out of g^2 and certain powers of r, $\hbar$, and c, $\hbar$ and c being the universal constants other than g that enter into the β-decay theory. Except for a dimensionless proportionality factor α we find†

$$U(r) = -\alpha \frac{g^2}{\hbar c} \frac{1}{r^5} \tag{2}$$

for the potential energy that is proportional to g^2. The exact calculations give $\alpha = 1/16\pi^3$. Substituting the known values of g, $\hbar$, and c and choosing a value of $r = 10^{-13}$ cm. we obtain

$$-U \simeq 2 \times 10^{-19} \text{ erg} \cong 10^{-7} \text{ e.v.}$$

as compared with a value of the order of 10^7 e.v. that would be expected at this separation, according to Chapter II. Nuclear forces that should arise from electron-neutrino exchange are then weaker by a factor $\sim 10^{-14}$ at $r = 10^{-13}$ cm., than required by a *square well*. On the other hand, the potential, eq. (2), is by no means a square well but rather diverges as r^{-5} as $r \to 0$.

In Chapter II we found that if one has an attractive potential that diverges as r^{-3} near the origin there is no lowest stationary state of the interacting particles and it is necessary to 'cut-off' the potential at a finite radius. This result holds *a fortiori* for a r^{-5} attractive potential and means that, despite our best intentions, the 'structure' of the nucleon must affect our considerations in order to produce a sensible result. Suppose, for example, that the potential (2) becomes modified at $r = a$ in such a way (and by unspecified physical forces) that $U(r)$ becomes constant at r smaller than a. Then the average potential energy of a neutron-proton system in an S-wave of linear dimensions a will be roughly $U(a)$ and the average kinetic energy will be of the order of $\hbar c/a$ (since a turns out to be much less than the Compton wave-length of a nucleon, so that the relativistic formula must be used). The radius of the deuteron will be that value of a for which $\hbar c/a \cong |U(a)|$, and this leads to

$$a_0 = \tfrac{1}{2}\pi^{-\frac{1}{4}} \sqrt{\left(\frac{g}{\hbar c}\right)} \simeq 1{\cdot}6 \times 10^{-17} \text{ cm.} \tag{3}$$

The range of force indicated by (3) is smaller by the factor $0{\cdot}6 \times 10^{-4}$ than the range found necessary to interpret scattering data, etc., in

† We have $[g^2\hbar^x c^y r^z] = [\text{erg}]$; or $[\text{erg}.L^3]^2.[\text{erg}.T]^x.[L.T^{-1}]^y.[L^z] = [\text{erg}]$ which gives $x = -1$, $y = -1$, $z = -5$.

Chapter II. Finally, the neutrino-electron hypothesis for nuclear forces gives no forces between two protons (or two neutrons) in the approximation proportional to g^2, since this approximation is characterized by a single exchange of charge, and the equality of all nuclear forces does not follow at all from the simple content of the theory. In fact, the terms in the potential proportional to g^4 give *repulsion*.

Owing to the failure of the attempt to describe nuclear forces in terms of electron-neutrino fields, numerous alternative proposals for the nature of the field particles have been made. These proposals fall naturally into two categories according to whether *the field particles have half-integral spin* and obey Fermi–Dirac statistics or *the field particles have integral spin* and obey Bose–Einstein statistics.

In the first category, to which the electron-neutrino theory belongs, the field particles must be created and absorbed by the nucleons in pairs in order to conserve spin and statistics. An alternative theory in this category, proposed by Wentzel,† and independently by Gamow and Teller,‡ assumes that nucleons exchange electron pairs (e^+e^-), and possibly neutrino pairs, as well as electron-neutrino pairs. If the nuclear forces are characterized by the exchange of electron pairs the value of g_c is no longer related to the Fermi constant and may be chosen arbitrarily in eqs. (2) and (3). We may choose this value so as to obtain an acceptable value of r, say a_1, at which the r^{-5} potential should be cut off. Such a value is $a_1 = e^2/mc^2$, and, since nucleon waves of this length represent kinetic energies in the non-relativistic region, we set $-U(a_1)$ equal to $\hbar^2/Ma_1^2$; this gives

$$g_c^2 \simeq \frac{16\pi^3\hbar^3 a_1^3 c}{M} \simeq 3\times 10^{-82},$$

i.e. a value of the interaction constant that is 10^8 times the Fermi constant. Thus, although the quantitative difficulty found for the electron-neutrino theory may be removed in this way and one should expect neutron-proton, neutron-neutron, and proton-proton forces to be equal in the electron-pair theory, the cut-off trouble remains and the original idea of introducing exchange forces, to obtain saturation through the exchange of electric charge has been lost entirely. We shall return to the pair-theories (including meson-pair theories) briefly in a later section.

† G. Wentzel, *Helv. Phys. Acta*, **10** (1936), 107.
‡ G. Gamow and E. Teller, *Phys. Rev.* **51** (1937), 289.

The only well-established 'elementary particle' having integral spin is the *photon* which has spin $\hbar$ and zero rest-mass. The photon is the field particle for the retarded interaction between moving charges.† The motion of a charge e_1 may be thought of as creating a photon which travels over to, and becomes absorbed by, another charge e_2.‡ Formally, a similar description of the electrostatic interaction of two charges can be made by inventing 'longitudinal photons', but this description just leads back to the classical formula for the electrostatic potential ϕ_e. For a point-source of strength e at the origin of coordinates, the classical equation is

$$\nabla^2\phi_e = 0, \quad r > 0, \tag{4}$$

with the well-known solution

$$\phi_e = \frac{e}{r}. \tag{5}$$

As pointed out in Chapter II, electrostatic forces are of infinite range. In order to produce forces of finite range between nucleons, Yukawa§ suggested that these forces be derived from a potential (static) that obeys the differential equation

$$\nabla^2\phi - \varkappa^2\phi = 0, \quad r > 0 \tag{6}$$

instead of eq. (4). The solution of eq. (6) for a point-source of strength g is readily seen to be‖

$$\phi = \frac{\mathfrak{g}}{r}e^{-\varkappa r}. \tag{7}$$

From the relativistic extension of Yukawa's theory (cf. § 3) we find that the parameter $\varkappa$ is related to the *rest-mass of the field particle* m by

$$mc = \hbar\varkappa. \tag{8}$$

† W. Heitler, *Quantum Theory of Radiation*, chapter iii (Oxford, Clarendon Press, 1936).

‡ The frequency of the temporary, or virtual, photon may take any value since the energy for its creation is 'borrowed'. The energy of the intermediate state, E_i, is then higher than that of the initial and final states, $E_a(= E_f)$. Let H_{ai} be the matrix element of the interaction energy between the charge e_1 and the radiation field that is represented by one photon of energy $h\nu_i$. Similarly, let H_{if} be the matrix element for the charge e_2 interacting with the same quantum. According to perturbation theory, the retarded potential between the two charges is then obtained by summing over all possible intermediate states:

$$V = \sum_i \frac{H_{ai}H_{if}}{E_a - E_i}.$$

§ H. Yukawa, *Phys. Math. Soc. Japan*, **17** (1935), 48.

‖ The reader may recognize this form of potential as that which occurs in other physical problems, e.g. the Debye–Hückel theory of electrolytic solutions.

In other words, the range of nuclear forces, $\varkappa^{-1}$, is just the Compton wave-length of the field particle. Substituting e^2/mc^2 for the range of forces we obtain for the mass of the particle $m \cong 137$ electron masses. Hence, Yukawa proposed the existence of an entirely new hypothetical particle, of mass intermediate between electronic and nucleonic rest-masses.† The subsequent discovery of such a particle in cosmic radiation by Anderson and Neddermeyer‡ stimulated a major interest in Yukawa's theory. However, the more detailed experimental investigations concerning the properties of cosmic-ray mesons seem to throw again the monkey-wrench into the elegant theory of the meson-exchange forces. In fact, if the mesons interact with atomic nuclei strongly enough to give the rise to the observed nuclear forces, one would expect that a beam of mesons passing through the matter will be subject to a strong nuclear scattering; it turns out, however, that the observed scattering of cosmic-ray is much smaller than would be expected on the basis of such mesons theory.§

Another still more striking piece of evidence is provided by the experiments of Conversi, Pancini, and Piccioni, and others,‖ on the absorption of cosmic-ray mesons by various nuclei.

When a beam of cosmic-ray mesons is slowed down in some material we should expect that the positive mesons, being electrically repelled by the atomic nuclei, will decay as free particles into the positive electrons and a neutrinos. And, indeed, the observation has established the presence of high-energy positive electrons resulting from such a process.

In the case of negative mesons the situation is, however, expected to be entirely different. Being attracted by the nuclear electric field, these particles must be strongly absorbed by the nuclei leading to violent nuclear excitation. According to the calculations of Fermi, Teller, and Weisskopff,†† and also of Wheeler,‡‡ the mean life of a slow negative meson against nuclear absorption must be of the order

† In the years just following its introduction this particle was variously dubbed heavy electron, heavy quantum, barytron, yukon, Japanese electron, mesotron, and meson. As indicated in Chapter I, we use the last mentioned name.

‡ C. Anderson and S. Neddermeyer, *Phys. Rev.* **51** (1937), 884.

§ R. F. Christy and S. Kusaka, ibid. **59** (1941), 414.

‖ M. Conversi, E. Pancini, and O. Piccioni, ibid. **71** (1947), 209; T. Sigurgeirsson and A. Yamakawa, ibid. **71** (1947), 319; R. Valey, ibid. (in publication 1948).

†† E. Fermi, E. Teller, and V. Weiskopff, ibid. **71** (1947), 314.

‡‡ J. A. Wheeler, ibid. **71** (1947), 320.

of magnitude of 10^{-18} sec. (if the sufficiently strong interaction, necessary for the explanation of nuclear forces, is assumed). Since the mean decay-time of a free meson amounts to only a few microseconds, we must expect that all slow negative mesons would be captured by the nuclei, and no emission of the negative electrons should be observed. The above-mentioned experiments indicate, however, that whereas this theoretical expectation is actually fulfilled in the case of heavy elements like iron, a comparatively small number of captures takes place in such light element as carbon. This seems to indicate that the probability of nuclear capture is of the same order of magnitude as the probability of free decay, being somewhat larger in the case of iron and somewhat smaller in the case of carbon (such variation of the relative capture probability is to be expected on account of the difference in nuclear electric charge). Thus, unless one would be able to give to this observation some other interpretation,† we should be forced to admit that the interaction between the negative (and most probably also the positive) mesons and the nuclear particles is too small by a factor of 10^{12} to explain the nuclear exchange forces.

An alternative solution (comp. Appendix I) is to admit the existence of two different kinds of charged meson (one kind for cosmic rays and another for nuclear forces) with rather different properties, or as a possible remaining avenue of escape ascribe nuclear forces to the exchange of hypothetical *neutral mesons*, which must be supposed to interact with the nucleons 10^{12} times stronger than the charged ones. Such an assumption, which is in principle not very different from the above-mentioned hypothesis of electron-pair exchange (Gamow and Teller, loc. cit.), would, however, lead again to the difficulties in explaining to saturation phenomenon of nuclear forces.

Inasmuch as none of the proposed field theories are satisfactory and an exposition of all of them would fill a book twice the volume of this one, the main objective of this chapter is to introduce the reader to the language and methods used in approaching such theories. With this in mind we turn, in the section immediately following, to the language that has been used most in those attempts. As field

† It was, for example, suggested by V. Weiskopff (*Phys. Rev.* **72** (1947), 155), that the failure of carbon nuclei to capture negative mesons is due not to the small probability of that process but rather to the unfavourable energy balance.

theories, these attempts belong to the second category mentioned above, i.e. the field particles have integral spin. At the same time, however, we shall develop the more general character of exchange forces to a degree that will be useful in Chapter IV. This general theory emphasizes the effect of the *symmetry* of the wave-functions of the nucleons on the *sign* of the potential and is rather independent of the particular field theory except that the meson theory used below provides many useful examples of what might be going on physically.

2. The formalism of meson theory

In this section we shall go on the simplifying assumption that there is just one kind of meson, i.e. a unique mass, spin, and half-life, independently of whether the meson occurs in cosmic radiation or in nuclei, but that the electric charge can be positive or negative (one electronic charge) or, perhaps, even zero. It is reasonable to hope that this assumption is also fact, but that is certainly not established at the present time when there is not even a single determination of the spin of a meson. Regarding the spin, we shall assume first that the spin is zero and see to what trouble the assumption leads. This is also the assumption originally made by Yukawa, as evidenced by the fact that the wave-equation (6) contains only a single component for the ϕ-function of the meson. With regard to electric charge, we shall assume first that a meson carries either a positive or negative charge of one unit. Introduction of neutral mesons will be made at the end of this section.

The assumption of charged mesons has three theoretical advantages. One is the possibility of obtaining saturation of nuclear forces, through exchange of charge, as discussed at length in the preceding section. In connexion with that discussion, it will be noted that the potential (7) does *not* have to be 'cut-off'; unfortunately, this is not generally true of meson theories. The second point is that, owing to the observed instability of the meson against decay into an electron, one automatically includes the possibility of β-decay of a nucleon. This comes about because the ϕ-function of eq. (7) represents, in a certain sense, the presence of a free meson ($g^2/\hbar c$ of the time) which, if the energy balance is right, yields to its natural propensity to disintegrate. This phenomenon is considered quantitatively in Chapter V with the regrettable conclusion that the agreement with observation is not good. The third point is that if the ϕ-function for the meson is

generalized to unit spin, and finite magnetic moment for the meson, the 'anomalous' magnetic moments of neutron and proton might be accounted for as contributions from the meson field.†

For the present, we concern ourselves with the exchange nature of the forces between neutron and proton. Let the two nucleons be in orbits a and b and let 1 denote the proton and 2 the neutron. The potential, V, between them will then be some function of radius $J(r)$ times the exchange character. If the nucleons are 'symmetrically coupled':

$$V[a(1)b(2)+a(2)b(1)] = -J(r)[a(2)b(1)+a(1)b(2)], \tag{9}$$

i.e. the effective potential is $-J(r)$. On antisymmetrically coupled particles, however,

$$\begin{aligned} V[a(1)b(2)-a(2)b(1)] &= -J(r)[a(2)b(1)-a(1)b(2)] \\ &= J(r)[a(1)b(2)-a(2)b(1)] \end{aligned} \tag{10}$$

and the effective potential is $+J(r)$. Hence, a potential that is attractive in symmetric states becomes repulsive in antisymmetric states.

The exchange nature of the neutron-proton interaction is substantiated by recent experiments on the scattering of high-energy neutrons by hydrogen. If the nuclear forces were not of the exchange type one would expect, as the energy of bombarding neutrons is increased, that the scattering should become predominantly forward, owing to the persistence of momentum. At 17 M.e.v., however, there is a slight *back scattering* of the neutrons. At this energy one must take into account not only the S-wave collisions but also the P-wave. If the forces are attractive in both states the scattering will be forward. The observed backward scattering thus indicates that the forces in the P-wave are repulsive, since the forces in the S-wave are attractive (Ch. II). The exchange effect may be pictured as follows: the incident, high-energy neutron beam will be expected to go mostly forward, but if in the interaction with the hydrogen these neutrons are changed to protons, by meson exchange, it will appear as if the incident particles are scattered backward. In general, the potential energy of neutron and proton colliding in a state of orbital angular momentum, $L\hbar$, will be (as far as the considerations to this point go)

$$V_m = -(-1)^L J(r). \tag{11}$$

† This point was made first by Wick, *Accad. Lincei Atti*, **21** (1935), 170, the electron-neutrino theory of nuclear forces.

Forces derived from a potential of this type are called *Majorana forces*, after their inventor.

The situation is actually a little more complicated. Consider the interaction of the neutron and proton in the ^3S state of the stable deuteron. In both spin and space coordinates this state is symmetric to the exchange of the two nucleons. On the other hand, if the nuclear forces are accompanied by mesons jumping from one nucleon to the other, for certain short intervals of time there will have to be two protons present in this state (negative meson jumping) or two neutrons (positive meson), in violation of the exclusion principle. It was to avoid this contradiction that Heisenberg proposed that nucleons be given an additional coordinate which, like the spin-coordinate, can take one of only two possible values. Because of the formal similarity to the ordinary spin, the new coordinate is called the *isotopic spin*. The required antisymmetry of the nuclear wave-function, even during meson exchange, can then be achieved by making *all* wave-functions antisymmetric when considered as functions of space, spin, *and* isotopic spin. The triplet deuteron is then thought of as made up of nucleon 1 and nucleon 2 in the S-wave: $S(12)$, and in the symmetric spin state, $\alpha(1)\alpha(2)$, but in the anti-symmetric isotopic spin state, $(1/\sqrt{2})[n(1)p(2)-p(2)n(1)]$. This means that we write a function of space and spin for particle 1 as the neutron and particle 2 as the proton and subtract the same function with particle 1 as the proton and 2 as the neutron.

$$^3\Psi_0 = S(12)\alpha(1)\alpha(2)\frac{1}{\sqrt{2}}[n(1)p(2)-p(2)n(1)]. \qquad (12)$$

In a similar way, the wave-function for the singlet state of the deuteron is deduced to be

$$^1\Psi_0 = \tfrac{1}{2}S'(12)[\alpha(1)\beta(2)-\alpha(2)\beta(1)][n(1)p(2)+p(2)n(1)] \qquad (13)$$

In the course of the exchange of a meson between them, the nucleons 1 and 2 remain at relatively fixed positions and, since the meson we are considering does not have a spin, the spins remain the same. The effect of the exchange potential on the wave-functions (12) and (13) is then simply to interchange n and p. Hence, if we choose a potential that is attractive in the singlet S-state it will be repulsive in the triplet S, and Yukawa's original proposal is unsuited to account for the observations.

It is evident from the structure of the wave-functions (12) and (13)

that one can obtain negative potentials simultaneously in these states if both charge and spin are exchanged when the meson jumps. Since all wave-functions are antisymmetrical, the potential in this case may be described uniquely in terms of the Majorana potential, eq. (11). There is a convenient formulism for expressing the exchange properties of the various potentials which is based on the fact that the scalar product of the Pauli spin-vectors for particles 1 and 2, viz. $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$ has the eigenvalue -3 for the antisymmetric, singlet state and $+1$ for the symmetric, triplet state. Therefore, the expression, $\frac{1}{2}[1+(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)]$ is $+1$ for symmetric spin-states and -1 for antisymmetric spin-states. We define an analogous, formal expression for the isotopic spin (of course, the three components of the isotopic spin-'vector' do not refer to axes in ordinary space), i.e. $\frac{1}{2}[1+(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)]$, where $\boldsymbol{\tau}$ is the isotopic spin-vector. The type of potential just described as equivalent to the Majorana potential may then be written:

$$V_m = -P_{12}\, J(r) = \tfrac{1}{4}[1+(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)][1+(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)]J(r). \qquad (14)$$

The symbol P_{12} is customarily used to denote this particular type of exchange, $-\frac{1}{4}$ (etc.). Potentials that involve an exchange of charge alone, i.e. proportional to $\frac{1}{2}\{1+(\tau_1,\tau_2)\}$, are known as Heisenberg potentials, and those in which the spin alone is exchanged as Bartlett potentials; both are obviously of opposite sign for symmetric and antisymmetric spin-states (holding the orbital angular momentum the same) and may be used to account for the spin-dependence of nuclear forces. It is often useful to apply these central forces as an approximation even though one is sure that the spin-dependence arises from the non-central forces discussed in the preceding chapter.

In order to make the meson carry angular momentum, as well as electric charge, it is necessary either that the meson have an intrinsic spin, or that it be emitted into a P-state. In Yukawa's theory the latter possibility is formally forbidden by the conservation of parity, but this difficulty is circumvented by calling the meson wave-function an antisymmetrical tensor of rank four (relativity theory) instead of a scalar. The theory is called the 'pseudoscalar meson theory'. The other possibility, of giving the meson a spin of $\hbar$, leads to the 'vector' theory. This theory is just the generalization of electromagnetic theory in which the light quantum is replaced by a particle of finite mass, i.e. the meson. The vector meson field may be

described in terms of a four-vector potential $\mathbf{A}_\nu$ analogous to the electromagnetic four-potential. The time component of $\mathbf{A}_\nu$ is governed by the static source-strength ρ and obeys eq. (6). This equation may be written formally for every point, including $r = 0$, by defining a singular source-density $\mathfrak{g}\,\delta(r)$, where $\delta(r)$ is Dirac's (three-dimensional) δ-function:

$$\nabla^2\mathbf{A}_4 - \varkappa^2\mathbf{A}_4 = -4\pi\mathfrak{g}\,\delta(r). \tag{15}$$

In a similar way, we define a singular current density in terms of the mesic dipole moment, viz.

$$\mathbf{j} = -\frac{\mathfrak{f}}{\varkappa}\operatorname{curl}[\boldsymbol{\sigma}\,\delta(r)]. \tag{16}$$

The equation for the space components of $\mathbf{A}_\nu$ then becomes:

$$\nabla^2\mathbf{A} - \varkappa^2\mathbf{A} = 4\pi\frac{\mathfrak{f}}{\varkappa}\operatorname{curl}[\boldsymbol{\sigma}\,\delta(r)]. \tag{17}$$

Green's functions for eqs. (15) and (17) are of the form of $e^{-\varkappa r}/r$; the solutions are therefore,

$$\begin{aligned}\mathbf{A}_4 &= \mathfrak{g}\int\frac{\delta(r')}{|r-r'|}e^{-\varkappa|r-r'|}\,d\mathbf{r}' = \frac{\mathfrak{g}}{r}e^{-\varkappa r},\\ \mathbf{A} &= -\frac{\mathfrak{f}}{\varkappa}\int\frac{\operatorname{curl}[\boldsymbol{\sigma}\,\delta(r')]}{|r-r'|}e^{-\varkappa|r-r'|}\,d\mathbf{r}' = -\frac{\mathfrak{f}}{\varkappa}\operatorname{curl}\left[\boldsymbol{\sigma}\frac{e^{-\varkappa r}}{r}\right].\end{aligned} \tag{18}$$

The potential energy of interaction between a nucleon with mesic charges, $\mathfrak{g}_1, \mathfrak{f}_1$ and a nucleon with mesic charges $\mathfrak{g}_2, \mathfrak{f}_2$ is then, in analogy with electromagnetic theory:

$$\begin{aligned}V_v &= \frac{\mathfrak{g}_1\mathfrak{g}_2}{r}e^{-\varkappa r} + \frac{\mathfrak{f}_1\mathfrak{f}_2}{\varkappa^2}(\boldsymbol{\sigma}_1, \operatorname{curl}\mathbf{A}_2)\\ &= \left\{\mathfrak{g}_1\mathfrak{g}_2 + \mathfrak{f}_1\mathfrak{f}_2\left[-\frac{3(\sigma_1,r)(\sigma_2,r)-r^2(\sigma_1,\sigma_2)}{r^2}\left(\frac{1}{3}+\frac{1}{\varkappa r}+\frac{1}{\varkappa^2r^2}\right)+\right.\right.\\ &\qquad\left.\left.+\tfrac{2}{3}(\sigma_1,\sigma_2)\right]\right\}\frac{e^{-\varkappa r}}{r} \quad \text{for } r \neq 0.\end{aligned} \tag{19}$$

The corresponding calculation for the pseudoscalar theory leads to

$$V_{ps} = \mathfrak{q}_1\mathfrak{q}_2\left\{\frac{3(\sigma_1,r)(\sigma_2,r)-r^2(\sigma_1,\sigma_2)}{r^2}\left(\frac{1}{3}+\frac{1}{\varkappa r}+\frac{1}{\varkappa^2r^2}\right)+\tfrac{1}{3}(\sigma_1,\sigma_2)\right\}\frac{e^{-\varkappa r}}{r} \quad \text{for } r \neq 0, \tag{20}$$

where $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are the mesic charges on the nucleons.

The forms of potential, eqs. (19) and (20), give the predicted space

and spin-dependence of the interaction of two nucleons if the field particle is a meson of spin one, or spin zero, respectively. It will be noted that the non-central forces discussed in Chapter II appear in both cases. As pointed out in that section, however, the fact that the tensor forces diverge as r^{-3} near the origin means that there are no stationary solutions for two nucleons in such potentials. This may be averted by arbitrarily 'cutting off' the potentials at some small value of r. Actual calculation shows, however, that the necessary cut-off radius is of the order of (and generally a little larger than) $1/æ$. Hence the field aspect of the theory is seriously compromised and must be considered unsatisfactory. In order to avoid the r^{-3} terms, Møller and Rosenfeld proposed that one set $\mathfrak{q}$ equal to $\mathfrak{f}$ and use the sum of V_v and V_{ps}. Thus, the nucleon interacts simultaneously with two kinds of meson-fields. The difficulty with this proposal is that the entire tensor force is subtracted out in this way, and there is no direct way of accounting for the quadrupole moment of the deuteron. Schwinger further proposed† that the r^{-3} alone be subtracted away by choosing different masses for the vector and pseudoscalar meson and assuming

$$\frac{\mathfrak{q}}{m_{ps}} = \frac{\mathfrak{f}}{m_v}. \tag{21}$$

Required values of the mass ratio m_v/m_{ps} and of the coupling constants can be determined from the deuteron states. Using these values,‡ and taking $m_{ps} = 177$ electron masses, the quadrupole moment can then be computed. The result is only one-third the observed value, even when no static force is assumed, i.e. $\mathfrak{g} = 0$, and becomes smaller when $\mathfrak{g}$ is finite. The limiting case in which $m_v \to m_{ps}$ as $\mathfrak{f} \to \infty$ has been investigated also,§ but with not much improvement. Hence the present status of this development of the meson theory, in which the nucleons are supposed to carry mesic charges and dipoles, is completely unsatisfactory with regard to quantitative results. The field-theoretic aspects of Yukawa's proposal will be reviewed in § 3.

It has been assumed, in the preceding discussion of the nuclear forces in the deuteron, that the mesons are charged and that the resulting potential is of the Majorana type. The mechanism by

† J. Schwinger, *Phys. Rev.* **61** (1942), 387.
‡ J. M. Jauch and Ning Hu, ibid. **65** (1944), 289.
§ L. L. Foldy, ibid. **72** (1947), 125.

which the forces are attended by the exchange of a charged meson fails, however, for the interaction between two protons or between two neutrons. A purely *charged meson theory* gives no attraction between like nucleons in the first approximation, i.e. according to eq. (4), and leads to repulsion in the second approximation. The observed equality of nuclear forces in the singlet interaction of all possible pairs of nucleons suggests, therefore, that there exist ordinary forces in addition to the exchange forces. The ordinary forces may be thought of as accompanied by the exchange of electrically neutral mesons. Thus a proton may emit not only a positive meson but also a neutral one; and the relative probability of emission of the two kinds is open to choice. A representative mixture of this type that is commonly considered is that due to Kemmer† in which the ordinary forces and the Majorana forces are present in equal amounts. This assumption is known as the *symmetrical theory*. Finally, Bethe‡ has worked out the theory for the assumption that all mesons connected with the nuclear forces are neutral. In this *neutral theory* the nuclear forces are strictly ordinary forces, and the saturation feature is absent. Even in the symmetrical theory the exchange nature of the forces is not sufficient to give saturation, for, as we shall see in Chapter IV, the ordinary forces may not be stronger than one-fourth the strength of the Majorana forces in order to achieve saturation. Nevertheless the three theories, neutral, charged, and symmetrical, provide a convenient reference system for discussing the possible mixtures of ordinary and exchange forces. The most general forms of potential for the three cases (not relating to any particular meson field) are summed up in Table III.

TABLE III

Forms of potential

Neutral theory	$-[J_0(r)+(\sigma_1,\sigma_2)J_s(r)+S_{12}J_t(r)]$
Charged theory	$\frac{1}{2}[1+(\tau_1,\tau_2)][J_0(r)+(\sigma_1,\sigma_2)J_s(r)+S_{12}J_t(r)]$
Symmetrical theory	$\frac{1}{3}(\tau_1,\tau_2)[J_0(r)+(\sigma_1,\sigma_2)J_s(r)+S_{12}J_t(r)]$

The operator S_{12} is defined in eq. (38), Chapter II.

3. Field theories of elementary particles

Although the meson theory of nuclear forces is in a completely unsatisfactory state at the present time, its development has established

† N. Kemmer, *Proc. Camb. Phil. Soc.* **34** (1938), 354.
‡ H. A. Bethe, *Phys. Rev.* **55** (1939), 1261.

certain general aspects of field theory of elementary particles that are of considerable interest in themselves. In particular, electrons and mesons connected with nuclear and cosmic-ray phenomena usually have energies far in excess of their 'rest energies', so that a relativistic treatment of their wave-equations is essential. This part of the theory has been greatly developed in recent years.

An elementary particle that is subject to no forces whatever is assumed to obey the relativistic energy relation:

$$E^2 = c^2p^2+m^2c^4, \tag{22}$$

where m is the rest-mass of the particle. From this equation we derive the Klein–Gordon equation

$$\nabla^2\psi-\frac{\partial^2\psi}{c^2\,\partial t^2}-æ^2\psi = 0, \tag{23}$$

by making the familiar substitutions:

$$E = i\hbar\frac{\partial}{\partial t}, \qquad p = -i\hbar\,\mathrm{grad}, \qquad æ = \frac{mc}{\hbar}.$$

If the free, elementary particle has a spin s there will be $2s+1$ independent components to its ψ-functions each of which obeys the eq. (23). The question then arises whether these $2s+1$ components are completely independent, as they would be if each had only to satisfy eq. (23), or whether there are, for example, some relations between their first derivatives. A classical example of this question may be posed by choosing $m = 0$ and considering the six equations of type (23) for the components E_x, E_y, E_z, H_x, H_y, H_z of an electromagnetic field without sources. In addition to the second-order equations, Maxwell has shown that these vectors obey the relations:

$$\begin{aligned} \mathrm{div}\,E &= 0, \qquad \mathrm{curl}\,E+\dot{H} = 0, \\ \mathrm{div}\,H &= 0, \qquad \mathrm{curl}\,H-\dot{E} = 0, \end{aligned} \tag{24}$$

where the dot over E and H indicates differentiation with respect to ct. These equations may be regarded as the wave-equations for the photon, and the photon density per unit volume is

$$\frac{E^2+H^2}{8\pi h\nu},$$

where ν is the frequency of the radiation. In this case, as in all cases where the 'particle' has zero rest-mass, there are only two independent states of polarization.

Dirac has argued that all relativistic wave-equations must be put in the form of first-order differential equations so as to define the first-time derivatives of the ψ-functions. He met this requirement for a particle of finite rest-mass by the well-known relativistic equation for the electron wave which has the four components $\psi_1, \psi_2, \psi_3, \psi_4$.

$$\begin{aligned} c[(p_x - ip_y)\psi_4 + p_z\psi_3] &= (E - mc^2)\psi_1, \\ c[(p_x + ip_y)\psi_3 - p_z\psi_4] &= (E - mc^2)\psi_2, \\ c[(p_x - ip_y)\psi_2 + p_z\psi_1] &= (E + mc^2)\psi_3, \\ c[(p_x + ip_y)\psi_1 - p_z\psi_2] &= (E + mc^2)\psi_4. \end{aligned} \tag{25}$$

It has been shown by Dirac† that the form of the equations (25) is invariant under Lorentz transformations. Under rotations in ordinary space the new ψ_1' and ψ_2', representing the two directions of the spin ($\frac{1}{2}\hbar$) are linear functions of the old ψ_1 and ψ_2 only, similar to the situation with the polarization of light. Under such rotations, also, the components ψ_3' and ψ_4' are functions of ψ_3 and ψ_4 alone. Under acceleration (i.e. transformation to a moving frame of reference) ψ_1' and ψ_2' are linear functions of ψ_3 and ψ_4 as well as of ψ_1 and ψ_2. In this respect the two components, ψ_1 and ψ_2, are analogous to the three components of the electric field vector, E_x, E_y, E_z and ψ_3, ψ_4 analogous to H_x, H_y, H_z. The analogy extends in a way to reflections of the coordinate axes. From eq. (25) it is evident that the components ψ_1 and ψ_2 acquire a different sign than do ψ_3, ψ_4 under the inversion $x' = -x$, $y' = -y$, $z' = -z$. In fact, if we suppose ψ_1 to be a spherically symmetric state (S-state), ψ_3 and ψ_4 are first derivatives with respect to z and $x - iy$, hence they are P-states. The space dependence of these states is even and odd, respectively, so that there is a difference in parity similar to that for the axial and polar vectors, H and E. The behaviour under reflections of the spin-part of the wave-function is not so simply interpreted.

As is well known, eq. (25) has plane-wave solutions for all values of $E \geqslant mc^2$ and also for all values of $E \leqslant -mc^2$. In order to prevent electrons with positive energy making radiative transitions to states of negative kinetic energy, Dirac proposed that normally all such states of negative energy are completely occupied by electrons. In the perfect vacuum all negative energy-levels are filled and all

† P. A. M. Dirac, *Principle of Quantum Mechanics* (Oxford, Clarendon Press, 1935), chap. xii.

positive energy-levels in the universe are empty. Then if an electron is added it must be put in a state of positive energy and it will be forbidden to make a transition to a state of negative energy because of the Pauli exclusion principle. If an electron is subtracted from the vacuum it is removed from a state of negative energy. The 'hole' in the vacuum then appears as a particle of positive energy and also of positive electric charge. It was by such a line of reasoning that Dirac predicted the existence of the positron.

The Dirac theory is of interest to nuclear theory for several reasons. In the first place, the relativistic theory of electrons and positrons is of direct importance to the interpretation of β-activity. This is discussed further in Chapter V. In the second place, the nucleons have spin $\frac{1}{2}\hbar$ and must ultimately be described relativistically by an equation of the Dirac type. Finally, the spin of the meson is still unknown, and if it turns out to be $\frac{1}{2}\hbar$ the same form of equation will be applicable to them. Some study of nuclear forces for which the field particles are mesons of spin $\frac{1}{2}\hbar$ has been made and will be referred to at the end of the chapter.

Throughout the preceding section, i.e. in line with Yukawa's theories, we have considered the meson to have an integral spin, either $s = 0$ or $s = 1$. In the general case of integral spin, the components of the wave-functions are components of vectors, tensors, etc., in four-space, which are familiar from the usual tensor calculus. In the case of the half-integral spins one must resort to some form of the spinor calculus.† Again, the most familiar example of integral spin is afforded by the Maxwell equations, eq. (24), which, however, are specialized to vanishing rest-mass. The simplest example with finite rest-mass is that for the 'scalar' meson for which we express eq. (23) as two first-order equations:

$$\chi^\alpha = D^\alpha \phi \qquad \varkappa^2 \phi = D_\beta \chi^\beta, \tag{26}$$

where repeated indices are to be summed from 1 to 4 and

$$D^1 = D_1 = \frac{\partial}{\partial x}, \text{... etc.}, \qquad D^4 = -D_4 = \frac{\partial}{c\,\partial t}.$$

The 'pseudoscalar' meson alluded to in the preceding section obeys essentially the same equations, for a free particle, although formally one should replace the scalar function ϕ by the tensor of fourth rank $\chi^{\alpha\beta\gamma\delta}$ that is antisymmetric in every pair of indices (i.e. a scalar

† O. Laporte and G. Uhlenbeck, *Phys. Rev.* **37** (1931), 1380.

times the dual tensor), and replace the vector χ^α by the antisymmetric tensor of rank three, $\phi^{\alpha\beta\gamma}$.

For mesons of spin one, or vector mesons, the appropriate first-order equations have been found first by Proca† and may be written in terms of the vector ϕ^α and the antisymmetric $\chi^{\alpha\beta}$. As remarked before, these equations are the generalizations of the Maxwell equations for finite rest-mass.

$$\begin{aligned} \chi^{\alpha\beta} &= D^\alpha \phi^\beta - D^\beta \phi^\alpha \\ \varkappa^2 \phi^\alpha &= D_\beta \chi^{\alpha\beta}. \end{aligned} \tag{27}$$

By going over to the dual tensors one gets the 'pseudo-vector' meson equations.

Generalizing this procedure, one can assume an antisymmetric tensor of rank, $s+1$, and relate it to a tensor of rank s by taking the divergence, and in this way construct wave-equations for any spin. The form of such equations is extraordinarily simple when written in the spinor notation. Although some attention has been given to the equations for higher spin (the equations for spin 2 and vanishing rest-mass apply to weak-field gravitational quanta), the attempts to account for nuclear forces have centred around spins 0, $\frac{1}{2}$, and 1. These attempts are characterized by various assumptions about the nature of the interaction between the waves of the mesons and those of the nucleons.

Before taking up the specifically nuclear interactions we should note the effect of ordinary electric forces on the relativistic wave-functions. If the four-potential of the electromagnetic field is A_x, A_y, A_z, V, and the electric charge on the elementary particle is e, the interaction between particle and field is inserted into the wave-equations by the replacements:

$$\begin{aligned} \frac{\partial}{\partial x} &\to \frac{\partial}{\partial x} - \frac{ie}{\hbar c} A_x, \\ \frac{\partial}{\partial y} &\to \frac{\partial}{\partial y} - \frac{ie}{\hbar c} A_y, \\ \frac{\partial}{\partial z} &\to \frac{\partial}{\partial z} - \frac{ie}{\hbar c} A_z, \\ \frac{1}{c}\frac{\partial}{\partial t} &\to \frac{1}{c}\frac{\partial}{\partial t} + \frac{ie}{\hbar c} V. \end{aligned} \tag{28}$$

† A. J. Proca, *J. de Radium et Physique*, **7** (1936), 347.

Putting $\mathbf{A} = 0$ and $V = -Ze/r$, where r is measured from the position of a heavy nucleus of charge Ze, one obtains from eq. (25) the well-known equations for the electron in the atom. Solutions of this equation for positive energy will be considered in more detail in the discussion of the β-decay (Chap. V). Similarly, one can find the stationary states of a meson of spin zero by inserting (28) into eq. (26) and solving for ϕ.

If the Coulomb potential is inserted into the equations (27) for the meson of spin one, the acceptable solutions of the equations do not form a complete set of eigenfunctions.† This comes about, apparently, because the meson spin interacts so strongly with the magnetic field owing to its orbital motion that a r^{-3} law of attractive potential results. As shown in § 2, there is then no finite state of lowest energy. Thus, in principle, the charged meson-proton interaction presents an insoluble problem in quantum mechanics. If mesons of spin *one* exist, therefore, the $1/r^3$ attractive potential must become modified at some radius or other. Landau and Tamm have pointed out† that if the attraction is 'cut-off', by some means or other, at a distance of $\sim (e^2/mc^2)$ stable meson-nucleon systems will be formed under the action of such forces. The forces between nucleons could then be understood as electromagnetic interplay, just as in the case of the formation of molecules from neutral atoms. On the other hand, the physical nature of such a cut-off, and the details of the interaction in case such states exist, are by no means clear at the present time.

4. General theory of interaction

In the preceding section we have considered the field theories for free particles, with a few remarks on the interaction of charged particles with a central Coulomb field. In this section we take up the more general question of interactions which must be interpreted quantum-mechanically by the emission and absorption of particles. The prototype of all such theories is the quantum theory of electromagnetic radiation which has been formulated successfully in terms of the *quantization of the wave-equation* and of *relativistic invariance*. It is beyond the scope of this book to deal extensively with these two doctrines, but we shall give brief sketches of them in order to present

† I. Tamm, *Phys. Rev.* **58** (1940), 952; H. C. Corben and J. Schwinger, ibid., p. 953; L. Landau and I. Tamm, ibid., p. 1006 (L).

some of the attempts that have been made to account for nuclear interactions. It must be borne in mind, however, that there is at the present time no satisfactory theory of nuclear forces. And in the light of this realization §§ 2, 3, and the present section are intended as an introduction to current parlance, rather than as an exposition of an established theory.

We have been describing nuclear forces, and β-activity, in terms of the emission and absorption of mesons, etc. The creation and annihilation of particles is a quantum-mechanical concept having no parallel in classical theories. As stated above, the introduction of the purely quantum-mechanical feature was effected in the theory of the emission of light quanta. In this theory, the combined wave-functions of all the light quanta in the radiation field are written as a single 'functional' which, because of the Bose–Einstein statistics, is symmetric to the interchange of any two quanta. The functional is therefore uniquely specified by giving the number of quanta in each 'individual particle orbit', ψ_1, ψ_2,..., ψ_a..., etc. (there being no interaction between the 'particles'). The functional may then be designated:

$$\chi = \chi(n_1, n_2, \ldots, n_a, \ldots), \tag{29}$$

corresponding to n_1 quanta in the state ψ_1, n_2 in ψ_2 and so on.

The number of quanta per 'individual particle state' (considered as a dynamical variable) has, therefore, the same eigenvalues as does the action variable in the theory of the harmonic oscillator, except for a factor $\hbar$. Furthermore, the creation or annihilation of a quantum in a given state corresponds exactly to the transition from one quantum level of the oscillator to an immediately neighbouring level. Accordingly, Dirac in the 'second quantization' and Jordan and Pauli in the 'quantization of the wave equation' invented the operators ξ_a and ξ_a^+ that cause a decrease by one and an increase by one, respectively, in the number of quanta in the normalized, individual particle state ψ_a. The operators are then similar to $p - im\omega q$ and $p + im\omega q$ of harmonic oscillator theory. The operators ξ_a and ξ_a^+ then produce

$$\begin{aligned} \xi_a \chi(n_1, n_2, \ldots, n_a, \ldots) &= \sqrt{n_a}\, \chi(n_1, n_2, \ldots, n_a - 1, \ldots), \\ \xi_a^+ \chi(n_1, n_2, \ldots, n_a, \ldots) &= \sqrt{(n_a + 1)}\, \chi(n_1, n_2, \ldots, n_a + 1, \ldots), \end{aligned} \tag{30}$$

where the factors $\sqrt{n_a}$ and $\sqrt{(n_a+1)}$ take account of the fact that the χ are normalized and thus represent a sum of $n!$ terms for every state

containing n-particles. The symmetric character of the functionals is also illustrated by the commutation rules for the ξ's:

$$\begin{aligned} \xi_a \xi_b - \xi_b \xi_a &= 0, \\ \xi_a^+ \xi_b^+ - \xi_b^+ \xi_a^+ &= 0, \\ \xi_a \xi_b^+ - \xi_b^+ \xi_a &= \delta_{ab}, \end{aligned} \tag{31}$$

which follow from eq. (30) and state that the order in which particles are added to, or subtracted from, the states ψ_a and ψ_b $(a \neq b)$ is immaterial. This symmetry is characteristic of all particles that obey Bose–Einstein statistics, so that this formulation may be generalized at once to include the description of mesons of spin zero and spin one.

The ξ_a, introduced above, obey the same equation of motion and the same transformation laws as the ψ_a to which each belongs.† The energy-operator for the functional χ is then related to the Hamiltonian function for individual particles by

$$E\chi = \sum_{a,b} \xi_a^+ H_{ab} \xi_b \chi. \tag{32}$$

The H_{ab} are the matrix elements of the Hamiltonian between the states ψ_a and ψ_b of the individual particle description. In the general application of the theory the Hamiltonian function for the individual particles is usually expressed as a function of the coordinates and their derivatives. In such a representation the operators ξ refer to a particular point in space, i.e. the ψ's are δ-functions. These functions are written simply as $\psi(\mathbf{x})$ and the form of the Hamiltonian for the functional of all particles is then

$$\int \psi(\mathbf{x})^+ H \psi(\mathbf{x})\, d\mathbf{x}\, \chi = E\chi. \tag{33}$$

Jordan and Wigner discovered the method of quantization of the wave-equation for particles obeying Fermi–Dirac statistics.‡ The functional for these particles is antisymmetric to the exchange of any two particles so that 0 and 1 are the only allowed occupation numbers for individual particle states. The characteristic values for the number of particles in a given state are then the same as those of $\frac{1}{2}(1-\sigma_z)$ of quantized spin theory. The operators giving rise to

† Proof of these properties will be omitted, but the reader is referred to Dirac's *Principles of Quantum Mechanics*, 2nd ed. (Oxford, 1935), chaps. xi and xiii, or to the original papers: P. A. M. Dirac, *Proc. Roy. Soc.* **114** (1927), 243, 710; P. Jordan and W. Pauli, *Zs. f. Phys.* **47** (1928), 151.

‡ P. Jordan and E. Wigner, ibid, p. 631.

creation (ξ^+) and annihilation (ξ) in a particular state are then equivalent to $\frac{1}{2}(\sigma_x+i\sigma_y)$ and $\frac{1}{2}(\sigma_x-i\sigma_y)$, respectively, and we have

$$\xi_a\xi_a^+ + \xi_a^+\xi_a = 1. \tag{34}$$

Furthermore, the creation of a particle in state ψ_a followed by the creation of one in ψ_b leads to a functional of opposite sign to one formed by creating one in ψ_b first and then one in ψ_a (since the particles are effectively interchanged in this way). Hence

$$\left.\begin{aligned}\xi_a\xi_b+\xi_b\xi_a &= 0\\ \xi_a^+\xi_b^+ + \xi_b^+\xi_a^+ &= 0\end{aligned}\right\} \quad (a \neq b). \tag{35}$$

Also, it was proved that the ξ's in this formulism obey the same wave-equation and transformation laws as the individual particle functions to which they belong. The construction of a Hamiltonian function for a system of many particles then proceeds as for the case of Einstein–Bose particles.

In the theory of the quantization of the wave-equation it is assumed that the individual particle states are independent of the number of particles present. This assumption is certainly admissible in the case of light quanta, for which the theory was developed, but in the case of charged particles it must be considered as an approximation only. The customary procedure in considering the interaction between meson waves $\phi(x)$, say, and nucleon waves, $\psi(x)$, is to assume, in zeroth approximation, that the Hamiltonian functions for each particle has no interaction terms of any kind. Thus the Hamiltonian, H_N, for a nucleon may be deduced at once from the wave-equation (25) for a free electron by changing the rest-mass and solving for $E\psi_\nu$. We shall write the result in the usual shorthand notation in which the four components of ψ_ν are considered as the components of a vector (in the algebraic sense, not the relativistic sense). The coupling between the various components in eq. (25) is then represented by the linear operators α_x, α_y, α_z, the elements of which are made up of 0, ± 1, $\pm i$, in such a way that eq. (25) takes the form

$$\left.\begin{aligned}E\psi_\nu &= c(\alpha_x p_x+\alpha_y p_y+\alpha_z p_z+\beta Mc)_{\nu\mu}\psi_\mu\\ \text{or}\qquad H_N &= c(\boldsymbol{\alpha},\mathbf{p})+\beta Mc^2.\end{aligned}\right\} \tag{36}$$

For the methods of constructing Hamiltonian functions for particles of integral spin, the reader is referred to the articles of Kemmer†

† N. Kemmer, *Proc. Roy. Soc.* A **166** (1938), 127.

and of Pauli.† The simplest example is the familiar one for light quanta for the H in eq. (33) becomes $(E^2+H^2)/8\pi$. In the following, the explicit form of the unperturbed Hamiltonian functions is of no direct interest, and we shall symbolize this part of the energy of a system of two kinds of particles by

$$H_0 = \int \Psi(\mathbf{x})^+ H_N \Psi(\mathbf{x})\, d\mathbf{x} + \int \phi(\mathbf{x}) H_m \phi(\mathbf{x})\, d\mathbf{x} \qquad (37)$$

and turn our attention to questions of relativistic invariance and the possible forms of interaction between the fields.

From the usual interpretation of a wave-function, plus the fact that the operators, $\Psi(x)$, etc., appearing in eq. (37) have the same space dependence and transformation properties as wave-amplitudes, the integrands in the terms on the right-hand side of eq. (37) represent energy densities. In relativity theory, therefore, the integrands must have the transformation properties of the 44-component of a symmetric tensor, viz. the stress-energy-momentum-density tensor, T_{ik}. The left-hand side of eq. (37), being energy, transforms like the 4-component of a polar vector. Thus, the differential $dxdydz$ (which is formally the 123-component of an antisymmetrical tensor of rank three) must be regarded as the 4-component of a polar vector. In general relativity, this would be accomplished by including the factor $\sqrt{(-|g_{\mu\nu}|)}$, where $|g_{\mu\nu}|$ is the determinant of the metric coefficients in the integrand. Moreover, it is evident from transformation theory of special relativity that the right-hand side of eq. (77) should include terms of the type $\int T_{14}\, dydzdt + \int T_{24}\, dxdzdt + \int T_{34}\, dxdydt$ in order to be properly covariant. In the case of light quanta, for example, this would represent $\int (E \times H)\, dSdt$. In any case, it is just the time integral of flux of energy out of the volume being considered. In eq. (37) all space is included in the volume considered so that this term does not appear, but if a finite volume were considered, any contribution to H_0 coming from this term would be properly interpreted as energy carried by particles that have left or entered the volume.

The relativistic theory of the *interaction* between two kinds of particles requires that they interact only if they occupy the same position in space simultaneously.‡ Thus, for example, in the expres-

† W. Pauli, *Rev. Mod. Phys.* **13** (1941), 203.

‡ Even the static Coulomb interaction between charged particles may be formally expressed as 'point' interactions between charges and 'longitudinal' quanta. However, the phase relations between the waves representing the scalar potential and those representing the longitudinal part of the vector potential are predetermined

sion for the energy of a collection of nucleons and mesons, eq. (37), we should represent the interaction by a term that contains the product of $\Psi(x)$ and $\phi(x)$ at each point x. The interaction energy per given volume will then be determined uniquely by the integral over $d\mathbf{x}$.

The form of the interaction term that is to be added to the energy, such as eq. (37), is to be chosen so that the integrand is a relativistic *scalar* (rather than a symmetric tensor). Interpreting *dxdydz* as the 4-component of a polar vector, as before, the integral will then have the desired covariance. Since we shall always be concerned with interaction in which one of the interacting particles has spin $\frac{1}{2}\hbar$, we shall start by writing down all the covariants that can be constructed as bilinear forms from the four components of ψ^+ and of ψ which describe the relativistic motion of such a particle. These are presented in Table IV along with the shorthand notation in terms of the linear operators, α_x, α_y, α_z, β, and their matrix products, and, in order to be more general, the forms are given for the creation of one kind of particle, Ψ^+, say a proton, and the annihilation of another kind, Φ, i.e. the neutron. Interactions in which the particle does not change charge are of the same types, but with one kind of wave, Ψ^+ and Ψ, say, instead of Ψ^+ and Φ.

As pointed out at the beginning of this section, the prototype of nuclear interactions is the electron-photon interaction. The photon field is described by the vector-potential $\mathbf{A}$ and taking the scalar product of this vector with the current four-vector for the electron (polar vector in Table IV) we get the familiar interaction energy

$$H' = e \int \psi(\mathbf{x})^+(\boldsymbol{\alpha}, \mathbf{A})\psi(\mathbf{x})\, d\mathbf{x}. \tag{38}$$

Since $\mathbf{A}$ occurs linearly, the number of photons is either one greater or one smaller after operation by H', whereas the number of electrons remains the same (the electron is absorbed and then re-emitted, so to speak). The same type of interaction applies to nucleons interacting with neutral mesons of spin *one* except that one must add the 4-component to the field vector and also allow for the possibility that the scalar product contains the tensor form, in Table IV, contracted on $\chi^{\lambda\mu}$. Hence

$$H'_v = \sum_{k=1}^{4} \mathfrak{g} \int \{\Psi(\mathbf{x})^+ v_k \Psi(\mathbf{x})\phi^k(\mathbf{x})\}\, d\mathbf{x} + \\ + \mathfrak{f} \sum_{i,k} \int \Psi(\mathbf{x})^+ \tau_{ik} \Psi(\mathbf{x})\chi^{ik}(\mathbf{x})\, d\mathbf{x} \text{ plus terms in } \phi(\mathbf{x})^+, \text{ etc.} \tag{39}$$

in such a way that nothing new is added in the quantum theory (see Heitler, *Quantum Theory of Radiation*, Oxford, Clarendon Press, 1936), so that the static Coulomb interaction is usually expressed in the classical form of action at a distance.

TABLE IV

Covariant forms for particle of spin $\frac{1}{2}\hbar$

Covariant	*Matrix*	*Product*
Scalar	β	$\psi_1^+\phi_1+\psi_2^+\phi_2-\psi_3^+\phi_3-\psi_4^+\phi_4$
Polar vector	α_x	$\psi_1^+\phi_4+\psi_2^+\phi_3+\psi_3^+\phi_2+\psi_4^+\phi_1$
v_k	α_y	$-i\psi_1^+\phi_4+i\psi_2^+\phi_3-i\psi_3^+\phi_2+i\psi_4^+\phi_1$
	α_z	$\psi_1^+\phi_3-\psi_2^+\phi_4-\psi_4^+\phi_2+\psi_3^+\phi_1$
	1	$\psi_1^+\phi_1+\psi_2^+\phi_2+\psi_3^+\phi_3+\psi_4^+\phi_4$
Tensor	$-i\beta\alpha_y\alpha_z$	$\psi_1^+\phi_2+\psi_2^+\phi_1-\psi_3^+\phi_4-\psi_4^+\phi_3$
τ_{ik}	$-i\beta\alpha_z\alpha_x$	$-i\psi_1^+\phi_2+i\psi_2^+\phi_1+i\psi_3^+\phi_4-i\psi_4^+\phi_3$
	$-i\beta\alpha_x\alpha_y$	$\psi_1^+\phi_1-\psi_2^+\phi_2-\psi_3^+\phi_3+\psi_4^+\phi_4$
	$+i\beta\alpha_x$	$i\psi_1^+\phi_4+i\psi_2^+\phi_3-i\psi_3^+\phi_2-i\psi_4^+\phi_1$
	$+i\beta\alpha_y$	$\psi_1^+\phi_4-\psi_2^+\phi_3-\psi_3^+\phi_2+\psi_4^+\phi_1$
	$+i\beta\alpha_z$	$i\psi_1^+\phi_3-i\psi_2^+\phi_4+i\psi_4^+\phi_2-i\psi_3^+\phi_1$
Axial vector	$-i\alpha_y\alpha_z$	$\psi_1^+\phi_2+\psi_2^+\phi_1+\psi_3^+\phi_4+\psi_4^+\phi_3$
s_k	$-i\alpha_z\alpha_x$	$-i\psi_1^+\phi_2+i\psi_2^+\phi_1-i\psi_3^+\phi_4+i\psi_4^+\phi_3$
	$-i\alpha_x\alpha_y$	$\psi_1^+\phi_1-\psi_2^+\phi_2+\psi_3^+\phi_3-\psi_4^+\phi_4$
	$-i\alpha_x\alpha_y\alpha_z$	$\psi_1^+\phi_3+\psi_2^+\phi_4+\psi_3^+\phi_1+\psi_4^+\phi_2$
Pseudoscalar	$\beta\alpha_x\alpha_y\alpha_z$	$i\psi_1^+\phi_3+i\psi_2^+\phi_4-i\psi_3^+\phi_1-i\psi_4^+\phi_2$

If the meson has a positive charge, the terms in $\phi(\mathbf{x})$ and $\phi(\mathbf{x})^+$ have to be treated differently, i.e. ϕ must occur only in products of the type $\Psi^+\Phi$ (Ψ^+ for proton, Φ for neutron), etc., in order to conserve electric charge. The corresponding expressions in neutral scalar and pseudoscalar theories are:

$$H'_s = g_s \int \Psi(\mathbf{x})^+\beta\Psi(\mathbf{x})[\phi(\mathbf{x})+\phi(\mathbf{x})^+]\,d\mathbf{x},$$
$$H'_{ps} = \sum_k g_{ps} \int \Psi(\mathbf{x})^+ s_k \frac{\partial}{\partial x_k}(\phi+\phi^+)\Psi(\mathbf{x})\,d\mathbf{x}. \tag{40}$$

As a specific, and the simplest, example of a charged meson interaction, there will be two kinds of scalar mesons, ϕ_+ for positive charge and ϕ_- for negative, and the interaction will take the form

$$H''_s = g \int \{\Psi(\mathbf{x})^+\beta\Phi(\mathbf{x})[\phi_-(\mathbf{x})^+ + \phi_+(\mathbf{x})] + \\ + \Phi(\mathbf{x})^+\beta\Psi(\mathbf{x})[\phi_+(\mathbf{x})^+ + \phi_-(\mathbf{x})]\}\,d\mathbf{x}. \tag{41}$$

If the meson has spin $\frac{1}{2}\hbar$, rather than integral values, both the nucleon and meson field covariants will be of the type presented in Table IV. This means that such mesons must be emitted and absorbed in pairs. Such meson theories have received a certain amount of attention as such† and the general form of the theory is

† R. E. Marshak, *Phys. Rev.*, **57** (1940), 1101; C. L. Critchfield and Lamb, ibid. **58** (1940), 46; also **59** (1941), 48.

of interest in other connexions. Thus, if instead of assuming the field particles to be mesons we take them to be electrons and neutrinos and such that the interaction with the nucleon gives simultaneous creation (or absorption) of an electron and a neutrino, we get the formulation of the theory of β-decay as originally presented by Fermi. If the interaction constant is chosen so as to account for the observed β-decay, however, the resulting forces between nucleons are too small by a factor 10^{-12}. This result, plus the fact that there are attractive forces between like nucleons, led Gamow and Teller (loc. cit.) to suggest that nuclear forces are accompanied by an exchange of electron-positron pairs. In order to achieve saturation of nuclear forces in this theory (which obviously does not give 'exchange forces') Teller† suggested that, instead of emitting the particles in a δ-function at the position of the nucleon, the field particles are emitted in a state of finite extension $u(\mathbf{x})$. This assumption compromises the relativistic invariance of the form of the theory but it presumably has the merit of approaching the whole subject of divergent self-energies, etc., more directly. The state $u(\mathbf{x})$, having a finite radius, has a finite kinetic energy. It is then possible to choose the interaction energy to be much stronger than this kinetic energy and thus guarantee that the state $u(\mathbf{x})$ is almost perpetually filled with electron-positron (or meson) pairs. Calculation of the energy of the interacting system may then be done by perturbation methods treating the *coupling* as the main energy and the kinetic energy of the emitted particles as the perturbing term. This procedure applies as well in the integral spin meson theories, if one assumes a source-function, and all such theories are referred to as *strong coupling* theories. In the *weak coupling* theories the kinetic energies of the emitted particles are the large terms and the coupling energy is considered as a perturbation. This is the method used in the customary treatment of emission of light quanta, retarded interaction of electric charges, etc.‡ It ignores the divergences predicted for point charges, as they do not enter directly into the calculations. Applying this method to the vector, pseudoscalar, or pair theory of mesons, however, one runs into divergences at once, as illustrated above, and it is

† C. L. Critchfield and E. Teller, *Phys. Rev.* **53** (1938), 812.

‡ If an electron is given a finite radius r_0, the coupling with the electric field is essentially the self-energy e^2/r_0, and the average kinetic energy of the 'emitted photons' is $\hbar c/r_0$. The ratio of coupling to the kinetic energy is then $e^2/\hbar c$ which is much less than unity.

generally necessary to 'cut-off' the forces between two nucleons. Thus (except for the scalar theory and the 'mixtures') even the weak coupling approximation does not avoid the question of the structure of the elementary particles.

It is beyond the scope of this book to continue into the field of methods of solution for the various types of meson theory. With the introduction that has been given, however, the interested reader is referred to the pertinent articles in the archive journals.† By way of comment, we can only repeat that none of the theories is satisfactory.

† 'Recent Research in Meson Theory', G. Wentzel, *Rev. Mod. Phys.* **19** (1947), 1; 'Strong Coupling Mesotron Theory of Nuclear Forces', R. Serber and S. M. Dancoff, *Phys. Rev.* **63** (1943), 143; 'On the Interaction of Mesotrons and Nuclei', J. R. Oppenheimer and J. Schwinger, ibid. **60** (1941), 150; 'Electron–Positron Theory of Nuclear Forces', E. P. Wigner, C. L. Critchfield, and E. Teller, ibid. **56** (1939), 530; 'Meson Theory of Nuclear Forces (Neutral)', H. A. Bethe, ibid. **55** (1939), 1261; N. Kemmer, *Proc. Roy. Soc.* A **166** (1938), 127; for general presentation of integral spin, weak coupling theory.

IV

GENERAL THEORY OF NUCLEAR STRUCTURE

1. Statistical nuclear model

THE universal features of nuclear systems such as the evidence for the saturation character of the forces of attraction, the (roughly) constant nuclear density, and the absolute magnitude of the average binding energy due to nuclear forces can be understood only on the basis of a complete theory of the forces between nucleons. There is no adequate theory of these forces at the present time. Nevertheless, on the basis of these features as empirical facts, a great deal can be understood through further application of quantum mechanics and statistics to nuclear systems. It is not even necessary to adopt a particular 'model' for the nuclear structure to systematize both the general trend of the binding energy as a function of the number of constituent particles, A, and the short period fluctuations of the binding-energy curves for the ground states of nuclei. This general theory of nuclear structure is similar in its content to the general theory of the lowest terms of atomic wave-functions.

There are three main steps, of decreasing importance to the energy of binding, in the estimation of the lowest term for electronic wave-functions in *atoms*: (1) the electrostatic interaction between electron and the nucleus, (2) the electrostatic repulsion between electrons, and (3), the spin-orbit interaction. The first step accounts for the bulk of the binding energy of the electrons and is based on the well-known theory of the motion of an electron in the field of a point charge. The second effect decreases the magnitude of the binding but by different amounts according to whether the space-part of the electron waves are symmetrically or antisymmetrically coupled. Since the total wave-function for electrons must be antisymmetric to the interchange of any two, the states that are antisymmetric in their space-dependence are symmetric in spin-dependence, and vice versa. Now a wave-function that is antisymmetric in the positions of two electrons, e.g.

$$\psi_a(r_1)\psi_b(r_2)-\psi_a(r_2)\psi_b(r_1),$$

vanishes for coinciding electrons and, in general, the average distance between electrons in such a state is greater than in the corresponding symmetric state in which the minus sign is replaced by a plus

sign. The electrostatic repulsion between electrons is therefore less in antisymmetric space wave-functions or, what is the same thing, in symmetric spin states. For this reason the lowest states of atomic wave-functions are as symmetric in electron spins as is compatible with conditions (1); hence the deepest lying state has the largest spin quantum number that can be obtained with those conditions. This is the familiar rule for the order of the multiplets. Finally, (3), the magnetic interaction between the spin-state thus determined and the orbital motion of all electrons is taken into account and the order of terms in the fine structure determined.

In setting up an analogous series of approximations for nuclear systems, it is first of all evident that the first step for electron functions has no counterpart in nuclei. There is no central field so that the bulk of the binding energy must arise in, (1), the interaction between nucleons due to nuclear forces. As a first approximation it is assumed that these forces are the same between all nucleons, i.e. independent of spin-orientation and electric charge. Then as step (2) for the nuclei we shall take account of: 2(*a*), the electrostatic repulsion between protons, 2(*b*), the spin-dependence of the nuclear forces and, 2(*c*), the neutron-proton mass-difference. For step (3) we take, again, the spin-orbit coupling. This method of approach to the study of nuclei has been developed principally by Wigner† and is commonly referred to as the theory of the symmetric Hamiltonian because of the assumed symmetry of forces in step (1).

Since the forces between nucleons give a net attraction, instead of repulsion as in step (2) for electrons, the rule for multiplets will be reversed and we shall expect nuclei to form as many symmetric space-wave couplings as possible so that the average distance between interacting nucleons is as small as possible. Thus, in adding two neutrons to the ground state of O^{16} to form O^{18} we should predict that the space part of their wave-functions is symmetric, hence the spin part is antisymmetric, forming the singlet state. In fact, it would appear reasonable that the two extra neutrons in O^{18} not only have opposite spins but occupy the same orbit. Since, however, there is no satisfactory way of specifying orbits for nucleonic wave-functions it is not, in general, possible to label the orbits with definite quantum numbers even in the approximation that assumes symmetry of forces. Exceptions to this rule are provided by the nuclei of mass 4

† E. Wigner, *Phys. Rev.* **51** (1937), 106, 947.

and lower, where, if the spin-forces are disregarded, the orbits are almost certainly of the $1s$ type. Other possible exceptions may be provided by 'closed shells' in which all orbits of certain quantum numbers are filled. Thus O^{16} is probably a closed system of $1s$ and $2p$ orbits.

In this general theory of nuclear structure it is assumed that the average effective interaction between a pair of nucleons depends critically upon the symmetry of their orbital wave-function. If the forces are of the ordinary type we shall designate the average potential energy between two nucleons in the nucleus A, disregarding symmetry for the moment, by $-L_o$. Since nuclear forces are short-range and since nuclei have constant density, the value of L_o decreases as A increases roughly as A^{-1}. To this average potential must be added a correction depending upon whether the pairs of particles considered are symmetrically or antisymmetrically coupled. This correction is just the exchange part of the ordinary forces and will be designated by L_o^{ex}, which is defined to be a positive energy. Then for symmetrically coupled pairs the average potential is $-L_o-L_o^{ex}$ and for antisymmetrically coupled pairs $-L_o+L_o^{ex}$. A basic assumption of this theory is then that L_o^{ex} is comparable to (though always smaller than) L_o.

There is convincing evidence that ordinary forces do exist between nucleons. On the other hand, as proved by Heisenberg, they cannot be the only forces because they do not show saturation (assuming that the attraction does not give way to a shorter range repulsion, in which event, of course, there will be no difficulty with saturation). In addition to ordinary forces, therefore, we assume the existence of nuclear exchange forces that are attractive for symmetrically coupled nucleons and repulsive for antisymmetrically coupled ones. These forces, known as Majorana forces, have been discussed in Chapter III on the basis of an exchange of charge between neutron and proton. Here, however, it is important to extend the principle to the interaction between like nucleons as well, thus retaining complete symmetry of the Hamiltonian. Between two neutrons, for example, the Majorana forces cannot be expressed in terms of exchange of electric charge but may be regarded in other ways, such as a special type of velocity dependent interaction.† Since the overall sign of the potential depends upon the nature of the coupling, the sum of

† J. A. Wheeler, ibid. **50** (1936), 643.

interactions between all pairs of nucleons contains contributions of both algebraic signs and it will be seen that, for large nuclei the repulsive interactions predominate. The Majorana forces therefore give saturation.

In determining the average effect of Majorana forces between a pair of nucleons we have, again, to take account of the fact that the average distance between nucleons will depend upon the symmetry of their coupling, i.e. the exchange part of the Majorana forces must be represented. Let L_m be the average (positive) magnitude of these forces and let L_m^{ex} be the magnitude arising from the exchange integral. For symmetrically coupled nucleons we get an average potential of $-L_m-L_m^{ex}$ and for antisymmetrically coupled pairs $+L_m-L_m^{ex}$. Combining these results with those for ordinary forces we note that the contributions due to L_m and L_o^{ex} change sign with symmetry while those due to L_o and L_m^{ex} are independent of symmetry. Hence, it is convenient to define two new parameters

$$L = L_m+L_o^{ex},$$
$$L' = L_o+L_m^{ex},$$

for the purpose of discussing the effect of symmetry on nuclear binding. The results for the average potential energy between pairs of nucleons are summarized in Table V.

TABLE V

Average potential between pairs of nucleons

Coupling	*Types of forces*		*Total*
	Ordinary	*Majorana*	
Symmetric	$-L_o-L_o^{ex}$	$-L_m-L_m^{ex}$	$-L-L'$
Antisymmetric	$-L_o+L_o^{ex}$	$L_m-L_m^{ex}$	$L-L'$

With these definitions of average interaction between pairs of nucleons we are now prepared to write down an expression for the total potential energy in a nucleus. The part that is independent of the symmetry of coupling contributes $-L'$ times the number of distinct pairs of particles, viz. $\frac{1}{2}A(A-1)$, in a nucleus of A particles. The part contributed by L, however, will be $-L\Xi$ where Ξ is the number of symmetric couplings *minus* the number of antisymmetric couplings in the nucleus. In general, the number Ξ is to be computed from the theory of the permutation group (cf. ref. to Wigner above).

For most of our purposes, however, the value of Ξ can be derived from the physical content of the Majorana forces. For example, in the helium and hydrogen nuclei all nucleons may be considered to be in the same orbit, so that all couplings are symmetric and Ξ equals 1 for H^2, 3 for He^3, and 6 for He^4. Let us designate this orbit by the letter a and write subscripts as follows:

a_1 neutron with spin up,
a_2 neutron with spin down,
a_3 proton with spin up,
a_4 proton with spin down.

The wave-function for He^4 may then be symbolized by

$$a_1 a_2 a_3 a_4 \quad (He^4).$$

The *exclusion principle forbids putting any more nucleons in this orbit* and we shall call such orbits closed. To write a wave-function for He^5, therefore, we choose a new orbit b and put a neutron in it with, say, its spin up:

$$a_1 a_2 a_3 a_4 b_1 \quad (He^5).$$

Majorana forces acting between this neutron and any of the nucleons in a are accompanied by an exchange of subscripts. However, the operation of the exclusion principle in orbit a forbids the exchange of the subscript 1 with those of the a's except with a_1. Otherwise, after the exchange, there would be two neutrons with spin up in orbit a. Hence the contribution to Ξ vanishes for the interactions between b_1 and the a's except for a_1. Since the spin-states of these two neutrons are the same, the space wave-functions must be antisymmetric, so that this interaction contributes -1 to Ξ. The Ξ for He^5 is 5, consequently. Using the same arguments, the wave-function for Be^8 may be written

$$a_1 a_2 a_3 a_4 b_1 b_2 b_3 b_4 \quad (Be^8).$$

There are six symmetric couplings in each of the closed orbits a and b, and four antisymmetric couplings, one for each subscript. The Ξ is then 8. It is to be noted that symmetric couplings come from within the same orbit, whereas antisymmetric couplings arise between different orbits. It is one of the basic assumptions of this approximation that the magnitude of the interaction is the same, but differs in sign. Modifications of this assumption will be considered when specific models of nuclei are taken up in the next section.

The formula for Ξ that applies to closed orbits can be generalized

at once. If there are k closed orbits, there are $6k$ symmetric couplings and $4\times\frac{1}{2}k(k-1)$ antisymmetric couplings. The Ξ_k for k closed orbits, i.e. for the α-particle nuclei, is therefore

$$\Xi_k = 8k-2k^2.$$

Any heavy nucleus with an even number of protons and an even number of neutrons can be considered as composed of $\frac{1}{2}Z$ closed orbits and $\frac{1}{2}(A-2Z)$ pairs of excess neutrons. For reasons that will be evident presently the number of pairs of excess neutrons is called the ζ-component of the isotopic spin and designated

$$T_\zeta = \tfrac{1}{2}(A-2Z).$$

Every even-even nucleus can be characterized by its mass-number A and an integral value of T_ζ. The greatest potential energy for such a nucleus, due to nuclear forces, will then be obtained if the particle waves are grouped into k closed orbits and T_ζ half-filled orbits containing two neutrons each. Under these conditions the number of symmetric couplings will be $6k+T_\zeta$ and the antisymmetric couplings will be $k(k-1)$ between protons and $(k+T_\zeta)(k+T_\zeta-1)$ between neutrons. Substituting $\frac{1}{4}(A-2T_\zeta)$ for k, the value Ξ for the greatest potential in an even-even nucleus becomes

$$\Xi_{ee} = 2A-\tfrac{1}{8}A^2-\tfrac{1}{2}T_\zeta(T_\zeta+4).$$

We can find the Ξ_{oo} for the lowest state of a given odd-odd nucleus by changing one of the neutrons in a half-filled orbit into a proton and putting the latter into another (half-filled) orbit. The value of T_ζ then decreases by unity, the new proton contributes $2-k$ to Ξ, and the loss of the neutron contributes $k+T_\zeta-1$. The formula for Ξ_{oo} is then

$$\Xi_{oo} = 2A-\tfrac{1}{8}A^2-\tfrac{1}{2}T_\zeta(T_\zeta+4)-\tfrac{3}{2}.$$

By similar arguments, the ground state Ξ for odd-even nuclei can be shown to be

$$\Xi_{oe} = 2A-\tfrac{1}{8}A^2-\tfrac{1}{2}T_\zeta(T_\zeta+4)-\tfrac{3}{4}.$$

Collecting these derivations and adding the ordinary forces we may write the total potential energy due to nuclear forces in the lowest state of a nucleus as

$$\begin{aligned} PE &= -\tfrac{1}{2}A(A-1)L'-\Xi L \\ &\cong -\tfrac{1}{2}A(A-1)L'-[2A-\tfrac{1}{8}A^2-\tfrac{1}{2}T_\zeta(T_\zeta+4)-\tfrac{3}{4}\delta_A]L, \end{aligned} \qquad (1)$$

where

$\delta_A = 0$ for even-even nuclei,
$= 1$ for odd-even nuclei,
$= 2$ for odd-odd nuclei.

The assertion that the greatest nuclear binding will occur in those states that have as many orbits occupied by four nucleons as is possible and the remaining nucleons paired off, two to an orbit, has been made only on intuitive grounds. It can be proved, however, by application of the theory of the symmetric group to this problem, that the couplings so chosen do lead to the lowest potential. Some of the more general results of the theory may be summed up as follows. Given a set of orbits, not necessarily filled and all coupled in some definite way, it is usually possible, i.e. not a violation of the exclusion principle, to change the spin directions of the nucleons and to change neutrons into protons, or vice versa, and keep the same symmetry in space wave-functions. In the approximation that the energy does not depend upon charge or spin, all these states obtained by such changes belong to the same energy. A complete collection of the states that are consistent with a given symmetry is called a *supermultiplet* in analogy to the $(2S+1)$ states of an atom that have the symmetry characterized by the spin quantum-number S. In atomic theory the multiplet is characterized by the largest value of the z-component of the spin alone. In nuclear theory, different members of the same multiplet may be reached by three operations:

1. Rotation of axes so as to obtain the largest consistent value of S_z. This gives a quantum number S.
2. Transformation of protons into neutrons so as to obtain the largest consistent value of T_ζ. This gives a quantum number called T, the isotopic spin. This operation corresponds to positron β-radiation obeying Fermi selection rules.
3. Simultaneous transformation of protons into neutrons and change of spin direction, so as to obtain the largest consistent value of $S_z T_\zeta$. This gives a quantum number Y. The operation is positron emission obeying Gamow–Teller selection rules.

A nuclear supermultiplet is then characterized by the three numbers (STY) instead of just S as a result of the fact that four particles may occupy the same orbit in the nuclear case instead of two in the electronic case. Furthermore, spins may be exchanged for isotopic spins without changing the symmetry. Hence, if we denote by P the largest of the three numbers, S, T, Y, by P' the next largest, and by P'' the remaining one, the multiplet is characterized by the 'partition

quantum numbers' ($P\,P'\,P''$). In terms of these numbers, the general formula for Ξ had been shown by Wigner to be

$$\Xi = 2A - \tfrac{1}{8}A^2 - \tfrac{1}{2}[P(P+4)+P'(P'+2)+P''^2].$$

In addition to the potential energy, PE, due to nuclear forces we have to take into account the kinetic energy of the nucleons and also the electrostatic repulsion of the protons. These quantities will be designated KE and CE, respectively. Both of these energies will be estimated on the assumption that the density of nuclei is a constant. An estimate of KE can then be made by applying the well-known theory of a degenerate Fermi gas to *free* particles at the same density. The actual kinetic energy is probably somewhat higher. It is convenient at this stage to adopt the 'milli-mass-unit' as the unit of energy in which to express PE, KE, and CE.

The kinetic energy will depend upon the symmetry of the nuclear wave-function. If, for a given wave-function, one changes as many protons into neutrons (with and without change of spin) and as many downward spins into upward spins as possible, one gets the composition of four independent Fermi gases with

Λ_1 neutrons with spin up,
Λ_2 neutrons with spin down,
Λ_3 protons with spin up,
Λ_4 protons with spin down.

The kinetic energy is then calculated for each number of particles moving in the nuclear volume and the sum taken. This has been done by Wigner and expressed as an expansion in the partition quantum numbers:

$$P = \tfrac{1}{2}(\Lambda_1+\Lambda_2-\Lambda_3-\Lambda_4),$$
$$P' = \tfrac{1}{2}(\Lambda_1-\Lambda_2+\Lambda_3-\Lambda_4),$$
$$P'' = \tfrac{1}{2}(\Lambda_1-\Lambda_2-\Lambda_3+\Lambda_4).$$

The result, in milli-mass-units, is:

$$KE = (3\pi^2)^{\frac{1}{3}}\left(\frac{15\hbar}{Mcr_0}\right)^2 A\left\{1+\frac{20(P^2+P'^2+P''^2)}{9A^2}+\ldots\right\}. \qquad (2)$$

The Coulomb energy will also depend upon symmetry to some extent, but the dependence will be neglected along with other minor effects such as non-uniform distribution of protons and a possible deviation from the $1/r$ law for protons. The estimate is then just the

electrostatic energy of a uniformly charged sphere of radius $A^{\frac{1}{3}}r_0$, and in M.m.u.

$$CE = 600\frac{e^2}{r_0 Mc^2}\frac{Z(Z-1)}{A^{\frac{1}{3}}}. \tag{3}$$

Let $$C_1 = 600\frac{e^2}{r_0 Mc^2} = 0{\cdot}65\frac{e^2}{2r_0 mc^2}.$$

The numerical coefficient of KE is then $36{\cdot}4C_1^2$. In the stable nuclei, we may assume $P = T_\zeta$ with $P' = P'' = 0$ and find for the total binding energy of these nuclei

$$\begin{aligned} BE &= PE+KE+CE \\ &\cong -A^2\frac{4L'-L}{8}+(36{\cdot}4C_1^2-\tfrac{1}{2}L+2L')A+ \\ &\quad +81C_1^2\frac{T_\zeta^2}{A}+C_1 Z(Z-1)A^{-\frac{1}{3}}+\tfrac{1}{2}L[T_\zeta(T_\zeta+4)+\tfrac{3}{2}\delta_A]. \end{aligned} \tag{4}$$

Application of this formula to nuclei will be discussed in detail in the next section, but there are several general features of the result for BE that are apparent without evaluating the parameters C_1, L, and L'. The main part of the nuclear binding must come from the first term, since at constant density the L's should decrease as A^{-1}, approximately. This means that, under the conditions existing in nuclei:

$$L' > \tfrac{1}{4}L.$$

On the other hand, if the nuclear density be increased so that all particles interact simultaneously, the L's become independent of A and the coefficient of A^2 must be positive in order to preserve the saturation character. Under these collapsed conditions, however, the forces are no longer sensitive to the average distance between nucleons (all couplings are within the range of the nuclear forces), so that the exchange integrals in L and L' vanish. This means that in order to guarantee saturation:

$$L_0 < \tfrac{1}{4}L_m$$

under collapsed conditions; or, in other words, the ordinary forces are not more than one-fourth as strong as the Majorana forces between pairs of particles.

The second general point is that, if we consider all possible nuclei of a given value of A (isobars), the binding energies are quadratic functions of Z or T_ζ alone, except for the term $\frac{3}{4}L\delta_A$. For odd values of A this is a constant term, so that BE as a function of Z, for given A,

is a parabola. The nucleus with this A that has the greatest binding energy will then have the value of Z that lies closest to the minimum

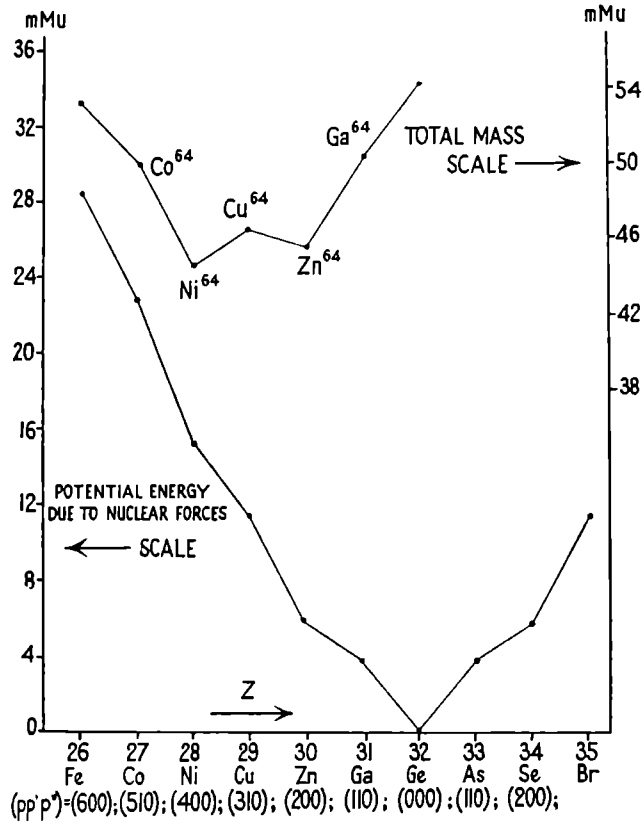


FIG. 13. Energy diagram for $A = 64$. The exact expressions for Ξ, and values of C_1 and L derived in § 2 were used in computing potential and kinetic energies. The theory predicts Zn^{64} and Ni^{64} to be stable as observed. The zero of the mass-scale is arbitrarily chosen.

in the parabola. The most stable nucleus of given A, on the other hand, is determined by the minimum in the mass curve,

$$BE+(M_n-M_H)T_\zeta,$$

which is also a parabola. In principle, all other isobars can transform by β-emission into the one with the most stable value of Z.

For even values of A, $\delta_A = 0$ if Z is even and $\delta_A = 2$ if Z is odd. Hence the binding energy-curve, or the mass-curve, must be represented by two parabolas, one displaced along the energy axis by approximately $\frac{3}{2}L$ compared to the other. In general, certain odd-odd nuclei will be less stable than either of its even-even neighbours

(see Fig. 13). Among the heavy nuclei, therefore, no odd-odd nucleus is stable against β-decay. An even-even nucleus can transform into another even-even nucleus of the same A only by simultaneous emission of two β-particles, and this is such an extremely improbable event that if both are more stable than the intervening odd-odd nucleus, both will be found in nature. There are, in fact, fifty-three such isobaric pairs known and at least three cases of isobaric triples.

2. Comparison with observation

Application of the general theory of the previous section to the observed atomic masses has been made by Barkas† and by Wigner.‡ It is assumed that r_0, i.e. C_1, is the same constant for all nuclei. The constant is evaluated by comparing the binding energies of nuclei having one more proton than neutron with those having one more neutron than proton, but with the same total mass-number. Pairs of these (Nordheim) nuclei are known for every odd A up to 27 (except $A = 5$) and, of course, one of each pair is β-active. From the energy of the β-radiation one can derive the difference in binding energy. Since A and the partition quantum numbers are the same for members of a pair, the expression (4) for the binding energy differs only in the Coulomb contribution and a value for C_1 can be computed for each case. The best value for C_1 obtained in this way is

$$C_1 = 0{\cdot}635,$$

corresponding to $r_0 = 1{\cdot}45 \times 10^{-13}$ cm. This is a little larger than values of r_0 indicated from scattering data, etc., as presented in Chapter I, and might be taken as evidence for the protons having a tendency to concentrate near the surface.

With a definite value for C_1 the kinetic energy is calculable and the value of L can be deduced from comparison of isobars for which the partition quantum numbers ($P\ P'\ P''$) are known. This has been done for two series of nuclei among the light elements, viz. pairs of the type C^{12}—B^{12}, having quantum numbers (000) and (110), respectively (and pairs of the type Na^{23}—Ne^{23}, with numbers $(\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2})$ and $(\frac{3}{2}\ \frac{1}{2}\ \frac{1}{2})$, respectively). Using the exact expressions for PE and KE,

† W. H. Barkas, *Phys. Rev.* **55** (1939), 691.

‡ E. Wigner, *University of Pennsylvania Bicentennial Conf.* (1941), Univ. of Penn. Press, Philadelphia.

the difference in the former is $4L$ (which is readily seen by counting couplings) and the difference in the latter is $65/A$. Then computing the difference in Coulomb energies, the value of L is determined from the experimental mass differences in each pair. In this way, very good agreement with experiment is obtained by setting

$$L = \frac{43}{A} \text{ (M.m.u.)} \quad (A < 40).$$

As A becomes large, say over 100, the values of Ξ become of the order of $-1{,}000$ and the exact dependence P, P', and P'' becomes of minor importance. In this region, therefore, the neglected effects such as spin-dependence and the influence of the Coulomb forces on symmetry may cause mixing of many states belonging to different partition numbers. Nevertheless, we may assume the general form for BE of the last section and compute values for L at higher A from isobaric pairs, one of the pair being radioactive. The values for AL obtained for five nuclei (Wigner, loc. cit.) together with Ξ and T_ζ for each is given in Table VI.

TABLE VI

A	T_ζ	Ξ	AL
12	0	6	43
20	0	−10	43
50	3	−225	55
116	8	−1500	77
197	19½	−4850	89

It will be noted that AL increases with A so that for gold it is twice as large as for the light elements. Most of this behaviour is probably due to a surface effect. In the light nuclei very few nucleons are completely surrounded by other nucleons at a given time, but in the heaviest nuclei about half of the particles are completely surrounded. With short-range forces acting, the effective nuclear potential per particle will then be greater for larger A. The ratio of the surface contribution to the volume contribution will be $A^{\frac{2}{3}}/A = A^{-\frac{1}{3}}$ and the results for L in Table VI are fairly well represented by the empirical formula

$$L = \frac{152}{A}\left[1 - \frac{2{\cdot}4}{A^{\frac{1}{3}}}\right] \quad (A > 50).$$

Finally, we consider the absolute magnitude of the binding in the nuclei shown in Table VI in order to estimate the magnitude of L'.

It will be seen that it is necessary to make L' as large as reasonable to account for the observed binding. Hence, we assume the ordinary forces to be just one quarter as strong as the Majorana forces, the maximum allowable by the saturation requirement. Further, we assume as a reasonable upper limit to the ratio of the exchange part of the forces to the average part, the value 0·6, i.e. $L^{ex}_{m,o} = 0{\cdot}6L_{m,o}$. Then $L' \simeq \frac{3}{4}L$. The comparison with experiment is shown in Table VII.

TABLE VII

A	$-BE$	CE	KE	$-PE_{obs}$	$-PE_{est}$
12	98	8	175	282	200
20	171	21	293	485	285
50	465	76	737	1,278	775
116	1,056	320	1,713	3,089	2,325
197	1,740	675	2,683	5,098	4,350

It is evident that the general theory does not give an adequate account of the absolute binding energies. In order to make the theory fit the data one must assume larger values of L' than those deduced above as reasonable upper limits. An empirical formula for L' that fits the data in Table VII for $A > 50$ is

$$L' = \frac{96}{A}\left[1 - \frac{1{\cdot}28}{A^{\frac{1}{3}}}\right] \quad (A > 50).$$

Thus the surface effect in L' is quantitatively different than in L. If we substitute the empirical expressions for L and L' into eq. (4) for BE we get one form of a 'semi-empirical' formula for the binding energies of heavy nuclei. Such formulae have been presented in a variety of forms and are generally useful for studying questions of relative stability and particularly for the interpretation of fission.

If we add to the considerations above the fact of the spin-dependence of nuclear forces, the manner in which stable nuclei may be formed by successive addition of one nucleon can be understood in a qualitative way. In the series H^1–He^4 all nucleons may be considered, in first approximation, to occupy the same s-orbit. The deuteron is the most stable two-nucleon system, mainly because it alone can exist in the triplet state and thus get benefit of the greater potential energy in that state. He^3 is more stable than H^3, although it has greater Coulomb energy and hence less binding, simply because the mass of the neutron is greater than that of the hydrogen atom. And

He^4 is probably the only stable system of four nucleons since it is the only nucleus in which all nucleons can occupy the same orbit.

For each value of A up to 36 there is just one stable nucleus and the binding energies show a marked period of four. This is evident in Fig. 14, in which the minimum energy required to remove one

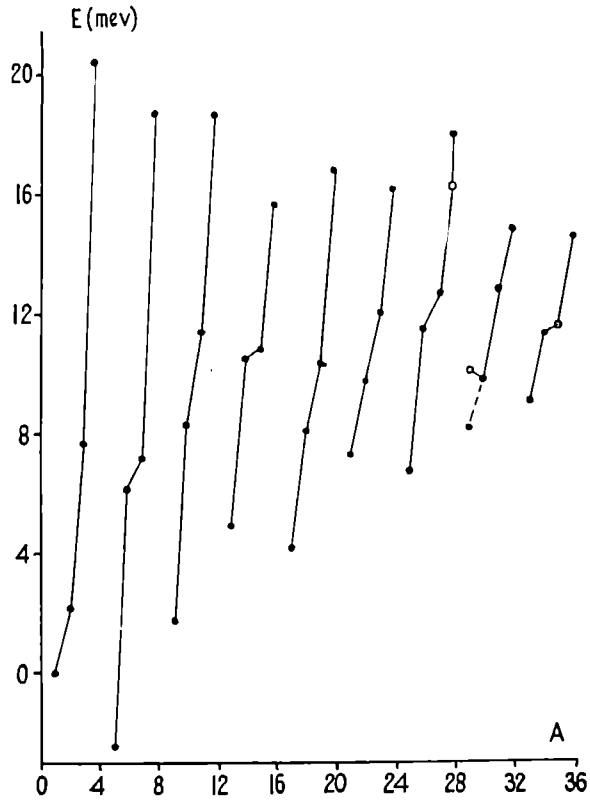


FIG. 14. Binding energy of 'last' neutron.

neutron from the stable nuclei is plotted as a function of A. The neutron is most tightly bound in the α-particle nuclei and least bound in nuclei that are just one heavier than α-particle nuclei. This is what one would expect from the symmetry of the couplings in these nuclei. Although the period of four is perhaps the most prominent feature of the binding of stable nuclei up to Ca^{40}, it is instructive to divide these nuclei into two groups. The first group includes the nuclei from He^5 to O^{16}. Of these, He^5 and Be^8 do not exist because their potential energy is not great enough, but counting them as the most stable systems of $A = 5$ and 8, this series of nuclei is built up by adding one neutron and one proton alternately. Starting with He^4, one gets a more stable system by adding a neutron and forming He^5 than by

adding a proton (Li^5) because the electrostatic energy of the proton more than offsets the neutron-hydrogen mass-difference. To He^5, however, it is more advantageous to add a proton, to form Li^6. The reason for this is characteristic for nuclei lighter than O^{16} and is based on the spin-dependence of the nuclear forces. Since the triplet interaction between two nucleons is so much stronger than the singlet interaction there is more binding in Li^6 than in He^6 in spite of the

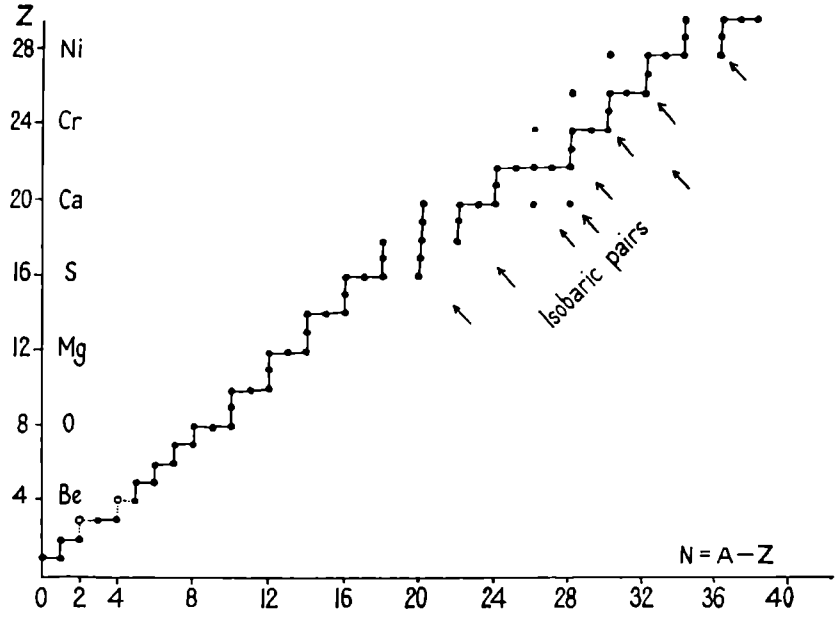


FIG. 15.

larger Coulomb energy of the former. The neutrons in He^6 must be in the singlet state. There are four examples of this phenomenon, H^2, Li^6, B^{10}, and N^{14}, and the interpretation is borne out by the fact that each of these nuclei has spin $\hbar$. It will be noted also that each of these nuclei has an odd number of neutrons and an odd number of protons. They are the only stable nuclei of this type and they appear to constitute exceptions to the general rule discussed at the end of the preceding section. That rule does not apply here as it was derived for heavy nuclei in which the odd neutron and odd proton must occupy different orbits in the state of lowest potential energy.

In the region between O^{16} and Ca^{40} the binding to be gained by forming triplet, odd-odd nuclei, is evidently not great enough to offset the Coulomb repulsion of the proton. The nuclei in this region can be built up, therefore, by adding two neutrons in succession, then two protons, and so forth (Fig. 15). This leads to the obvious result that

every even element in this interval has three stable isotopes and every odd element only one. Conversely, there are three elements for even neutron numbers. The *nn*, *pp*, *nn*, *pp*, etc., sequence continues through A^{36} and then breaks down, presumably because of the increasing importance of the Coulomb energy. According to the rule that holds below A^{36} one should expect A^{37} to be the next stable nucleus, but it

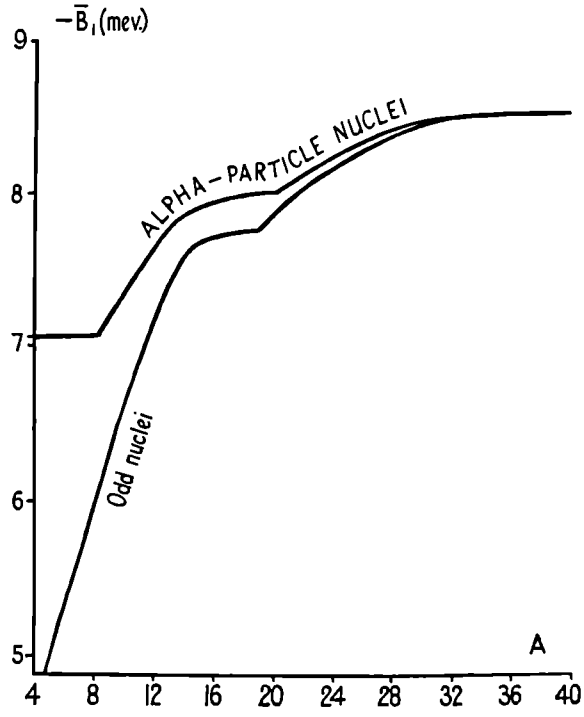


FIG. 16. Binding energy (negative) per particle, as the function of A.

is preferable to break up one of the α-units and form Cl^{37} instead, for which the value of Ξ is -102 as compared with -99 for A^{37}. The change-over at this point may be partly influenced by the possibility that a system containing twenty neutrons is particularly stable, i.e. a closed shell. This possibility is substantiated by the fact that there are five adjacent elements with this neutron number (S^{36}, Cl^{37}, A^{38}, K^{39}, and Ca^{40}). Furthermore, the proton number 20 appears to be particularly stable, also, since there are six known isotopes of Ca as compared with three, at most, for lighter elements. Among the lighter elements, in addition to the obvious case of He^4, there is strong evidence that a shell is completed at O^{16}. This is shown by the break in the slope of the curve for total binding as shown in Fig. 16.

For nuclei heavier than Ca the dominant effect in deciding the most stable nuclei among isobars is the Coulomb repulsion. This is apparent for several reasons. In the first place, the period of four is abandoned, it often being energetically more favourable to add four neutrons in succession than to follow two neutrons by two protons. The Coulomb effect overbalances the tendency of the nuclear forces to reach maximum symmetry. This is illustrated in Fig. 13 and, as pointed out in § 1, the effect will entail a number of cases of stable isobaric pairs of even A. Isobaric pairs are very common above Ca. Finally the Coulomb repulsion becomes so important that it reduces the average binding energy so that only 5 M.e.v. is required to remove a neutron from the very heavy elements (above lead) as compared with 9·5 M.e.v. from elements in the neighbourhood of calcium. It is this reduction in binding energy, of course, that accounts for the α-radioactivity and the fissionability of the very heavy elements.

It would be wrong to leave the impression that above Ca the behaviour of the elements can be accounted for by the orderly influence of the Coulomb energy. There are, for example, neutron and protons numbers for which no stable nuclei exist (there are no nuclei with 61 of either). Also, the phenomenon of five adjacent elements having the same neutron number occurs for the numbers 50 and 82 as well as for 20. Other evidence of the peculiar nature of these neutron numbers has been presented in Chapter I. The present theory does not account for these irregularities.

Of course, in an assembly of a rather limited number of particles there is a finite probability that there will be accidental anomalies. Concrete examples of such are afforded by the existence of four or five pairs of neighbouring isobars. These all have odd A and so far as experiment can decide both members of the pair are stable. An example is Sn^{115}–In^{115}. According to very general arguments, such as presented in § 1, it is not possible for both of these nuclei to be stable, since there is no intervening nucleus to block single β-decay. If the masses of the atoms are accidentally equal, or nearly so, however, both will appear to be stable because of the extremely long lifetime for the β-decay at low energy. There are also four cases of isobaric triples, $A = 96$, 124, 130, and 136, which must be considered as accidental or, perhaps connected with 'shell-structure'.

3. The more detailed nuclear models

The model of the nucleus that has been adopted in the preceding sections is essentially that of a degenerate Fermi gas enclosed in a sphere. It is not expected of such a model to account for the finer features of nuclear structure such as magnetic dipole and electric quadrupole moments, positions of excited levels, and the shell structure. In fact, it fails to give a satisfactory quantitative result for the absolute binding energy of heavy nuclei even with the considerable freedom allowed in the choice of the parameters L and L'. The quantitative deficiency regarding absolute values amounts to a disaster if one tries to calculate L and L' from nuclear forces. Nevertheless, the underlying principles of the statistical model are valid enough to give a general accounting of nuclear stability, or lack of it, as in the example of fission.

The objective of a nuclear model is to be more explicit in the definition of the individual particle orbits that have been alluded to in discussing the theory of the symmetry of nuclear wave-functions. There are two principal methods of approach to a more definite model which we shall call the independent particle model and the α-particle model. The former is the more intimately related to the theory of the symmetric Hamiltonian as it places no restrictions on the motion of a nucleon throughout the nucleus. It assumes that every nucleon moves in a constant potential field created by the other nucleons and extending to the radius of the boundary of the nucleus. The individual particle orbits are then essentially the quantum states of a particle moving freely in a spherical cavity with impenetrable walls. The wave-functions for this motion are well known to be the half-integral order Bessel functions times $r^{-\frac{1}{2}}$, and times the angular dependence appropriate to s-, p-, d-waves, etc. The energy of these states is then the average potential, which is the same for all, plus the kinetic energy, $\hbar^2k^2/2M$. Here k is the 'wave-number' for which the Bessel function has a node at the nuclear radius, R, $J_{l+\frac{1}{2}}(kR) = 0$. The three quantum numbers corresponding to the three degrees of translational freedom of the nucleon may then be taken, in the usual way, to be n, l, m_z. If k is determined by taking the rth root of $J_{l+\frac{1}{2}}$ the principal quantum number n will be defined by $n = l+r$. The kinetic energy then depends upon both n and l but not upon m_z so that there are $2l+1$ orbits of the same energy. The lowest eleven (degenerate) orbits, together with their values of kR and the maxi-

mum number of neutrons that can be accepted, i.e. the total neutron number if a closed shell is completed at each orbit, is shown in Table VIII.

TABLE VIII

Independent particle orbits

Orbit	$1s$	$2p$	$3d$	$2s$	$4f$	$3p$	$5g$	$4d$	$6h$	$3s$	$5f$
kR	3·14	4·48	5·85	6·28	6·97	7·70	8·17	9·08	9·34	9·41	10·42
Total N (or Z)	2	8	18	20	34	40	58	68	90	92	106

It is to be noticed that the model is successful in predicting closed shells at N and/or Z equal to 2, 8, and 20. The possibility of shells at 50 and 82, however, is not included. Many attempts have been made to modify the model so as to include the more obvious shells. The least objectionable prescription, perhaps, is to exchange the places of the pairs of orbits: $3p$ with $4f$, $4d$ with $5g$, and $5f$ with $6h$. This gives the shell-number sequence above 20: 26, 40, 50, 68, 82, etc., which has much in its favour. On the other hand, the correct physical grounds on which to modify the independent particle model so as to exchange the order of the levels as indicated has not been found. Modification of the shape of the well or penetrability of its walls, for example, has little effect.

The independent particle model (also called the Hartree model of nuclei) has been extensively investigated and used in attempts to calculate finer features of nuclei, mainly because of its great simplicity. But for nuclei heavier than He^5 it appears to be far from a good approximation. Numerous attempts to compute the average interaction between nucleons moving in $1s$ and $2p$ orbits have all led to the result that, if the two-body forces are the same as evidenced in scattering and in the deuteron, the predicted binding energy in the nuclei between He and O is only a small fraction of the observed binding energies. Or, in the terms used in § 1, attempts to calculate L and L' from the known forces between two nucleons, on the assumption that the nucleons are moving on independent orbits, lead to values that are much smaller than those found empirically. Higher-order perturbation calculations help but little. The origin of the discrepancy would appear to be in the possibility that the positions of the nucleons are appreciably correlated, e.g. the two extra neutrons in He^6 are much more likely to be found on the same side of the

α-particle core than would be computed on the basis of chance and assuming each neutron moves in an independent p-orbit.

The α-particle model goes to the opposite extreme in the matter of correlation of positions of the nucleons and assumes that each closed orbit of four nucleons forms a relatively stable sub-nucleus which occupies a certain position in the whole nucleus. The principal part of the binding in the nucleus is then automatically contained in the closed orbits. In addition, these alpha-particles are bound to each other by relatively weak forces of the van der Waals type, i.e. arising from a slight polarization of the saturated orbits. The model for Be^8 would then be a 'dumb-bell' with two alphas touching at one point, for C^{12} an equilateral triangle with three bonds, for O^{16} a tetrahedron with six bonds, and so on. Now the total mass of four helium atoms is 4 (4·00388) = 16·01552, and dividing the excess over the O^{16} atom by 6 we get 2·59 M.m.u. per bond. The masses of the 'α-particle' isotopes, the number of bonds, and the calculated and observed excess binding due to the bonds are shown in Table IX.

TABLE IX

Element	*Bonds*	*Mass*	*Excess obs. M.m.u.*	*Excess calc. M.m.u.*	*Diff. M.m.u.*
He^4	0	4·00388	—	—	—
Be^8	1	8·00777	−0·01	2·59	2·60
C^{12}	3	12·00398	7·66	7·77	0·11
O^{16}	6	16·00000	15·52	15·52	0·00
Ne^{20}	9	19·99881	20·59	23·31	2·72
Mg^{24}	12	23·9924	30·9	31·1	0·2
Si^{28}	16	27·9866	40·6	41·4	0·8
S^{32}	19	31·9823	47·8	49·2	1·4
A^{36}	22	35·9780	56·9	56·9	0·0
Ca^{40}	25	39·974	65	65	0·0

In general, above O^{16} the addition of an α-particle adds a tetrahedron to the structure already existing and so three bonds are allotted per addition. Somewhere in the neighbourhood of the seventh or eighth addition, however, a closed ring can be formed, hence the addition of an extra bond at Si^{28}. It is evident from Table IX that, except for Be^8 and Ne^{20}, the agreement with experimental values is extremely good; in fact, somewhat better than might be expected considering that the method of counting direct bonds does not allow for the long-range Coulomb repulsion between alphas not in direct contact. The agreement must therefore be fortuitous for the heavier isotopes in

Table IX, since beyond Ca the Coulomb effect is so important that the simple α-particle nuclei are no longer stable. The main significance of the success of the α-particle model in accounting for the absolute values of binding is the indication that the positions of the nucleons that are symmetrically coupled, i.e. in the same orbit, are probably highly correlated. Consequently, the assumption made in § 1 for the statistical model—that only the sign but not the magnitude of average interaction depends upon the symmetry of coupling—is not as close to the actual situation as could be desired.

Hafstad and Teller† have applied the α-particle model farther, by methods analogous to the theory of molecular structure, to describe nuclei having one more, and also to those having one less, nucleon than the α-particle nuclei. The simplest case is He^5, in which the extra neutron is placed in an orbit that has a plane node through the α-particle core, just as in the independent particle model. This neutron will be antisymmetrically coupled with the neutron of parallel spin in the core and will be repelled by it. In the α-particle model the average interaction is evaluated to be $b = 1{\cdot}3$ M.m.u. between the extra neutron and an α-particle. When there are several α-particles the extra neutron will interact directly with only one at a time so that b is the same. On the other hand, the alphas polarize each other and lead to indirect and exchange effects on the binding of the extra neutron which is designated by $R+Q$ per bond, in the notation of Hafstad and Teller, if the wave-function of the neutron has one plane node through all alphas. Hence this description applies directly to Be^9 and C^{13} in which a neutron wave with a plane node can be made orthogonal to the wave-functions in the alphas. The binding in these nuclei can then be represented by $R+Q = -3{\cdot}4$ M.m.u. For O^{17} the alphas are thought of as forming a tetrahedron, so it is impossible to pass a single node through the centres of all alphas simultaneously. One has either to choose a neutron wave with a single spherical node (s-wave), or one with two plane nodes (d-wave). In any case, the kinetic energy of such a wave will be higher than one with a single plane node and hence the smooth trend in binding energy will be broken at O^{16}. Thus the α-particle model predicts a bend in the binding-energy curve, as shown in Fig. 16, and previously interpreted on the independent particle model as evidence of the

† L. R. Hafstad and E. Teller, *Phys. Rev.* **54** (1938), 681.

closing of the $2p$ shell. Nuclei of the type $4n-1$ are also considered using the theory of 'holes' as applied to molecular wave-functions.

There is no satisfactory derivation of the constants b, R, and Q of this theory from the theory of nuclear forces. In fact, attempts to make such derivations have been as discouraging† as the corresponding results for the relation of nuclear forces to the average interaction in the independent particle model.

Even for the nuclei on which all models agree—H^3, He^3, and He^4—the derivation of the energies of binding from nuclear forces is not in a satisfactory state at the present time. The reason for this state of affairs is to be found in the dipole-dipole nature of the forces. The existence of such forces tells us that the wave-functions of H^3 and He^3 are not pure s-waves but, like H^2, will be s-waves mixed with waves of higher orbital angular momenta. The true wave-functions then become extremely complicated. Of course, one might replace the effect of coupling between s- and d-waves, say, by assuming different strengths of interaction in the triplet and singlet states of nucleon pairs, as was done rather successfully for the deuteron, and use only s-waves. The difficulty is that the relative strengths will be different in the different nuclei. A very careful calculation of the depth and range of forces that give the proper bindings in the two- and three-body nuclei has been made by Rarita and Present‡ assuming the space-dependence of forces between pairs of nucleons to be of the form $Ae^{-r/b}$. The result is

$$A = -242\ mc^2 \quad \text{and} \quad b = 1{\cdot}73 \times 10^{-13}\ \text{cm}.$$

with A' for the singlet interaction equal to $0{\cdot}6A$. But when these same constants are used in a variational calculation of the binding energy of He^4 they lead to 20 per cent. too much binding. Since a variational calculation should always lead to too little binding, this result shows that it is not admissible to approximate the dipole-dipole forces by an average spherical potential. In particular, the effects of coupling might be relatively less in the saturated orbit of He^4 than in the unsaturated nuclei, thus requiring a smaller value of A. The complexity introduced by the dipole-dipole interaction is so impressive that the hope of deriving the details of nuclear binding from our knowledge of nuclear forces seems rather remote.

† H. Margenau, *Phys. Rev.* **59** (1941), 37.
‡ W. Rarita and R. D. Present, ibid. **51** (1937), 788.

The two models discussed above have been studied most in their relation to the light nuclei because they are tractable. Except for quite general results, such as they are, which are mostly the same for both models, they are evidently too over-simplified to be acceptable nuclear models. In a notable effort to strike a compromise between tractability and acceptability, Wheeler† has proposed a quite general attack which he calls the method of resonating group structure. We shall discuss this model briefly as an improvement on the α-particle model, although the method is much more general. In the α-particle model the wave-function of the nucleus is, in first approximation, the product of wave-functions for He^4 each centred about a fixed point. Thus, in the wave-function written above for Be^8 (§ 1) a and b are essentially the same s-waves but with centres separated by R_{ab}. Wheeler's proposal is to elaborate on this function in two ways: the first is to include the wave-function describing the motion of the centres of the alphas, $F(R)$, as a factor; the second is to allow for the possibility that the composition of the alphas may change, i.e. the particles may dissolve their former partnerships and form a new set of alphas around different centres, in general. The new set may then dissolve into a third, or go back to the first, and so on. The lowest state of a nucleus is then described as a superposition of a number of complexions, all of the same symmetry and mostly of the same eigenvalues of energy, among which the nucleons 'resonate'. In the example of Be^8, the two neutrons in a may form a new α-wave a' with the protons in b, and the remaining nucleons form a new state b'. The new α-waves will have different centres, separated by the radius vector $R_{a'b'}$, and the total wave-function would look like:

$$\Psi = F(R_{ab}).a_1 a_2 a_3 a_4 b_1 b_2 b_3 b_4 + F(R_{a'b'}).a'_1 a'_2 a'_3 a'_4 b'_1 b'_2 b'_3 b_4.$$

The generalization of this method to groupings of all kinds (not only alphas), to more than two sub-groups at a time, and to cases including spin has been presented by Wheeler. The wave-equations for the various $F(R_{pq})$ are then obtained by minimizing the total energy (applying the variation principle) and assuming that the shapes of the sub-groups are not sensitive to their separation. This means that the polarization of one sub-group by another is neglected. If the nucleons did not change places with one another the method would amount to determining the motion of each centre in the average field of force

† J. A. Wheeler, ibid. **52** (1937), 1107.

created by the others. With the jumping about of the nucleons that characterizes the resonating group structure, the kinetic and potential energy of the centres will have integrals that contain the wave-functions F that describe different groupings, since it is, after all, the motion and forces in the nucleus that cause one grouping to change into another. Instead of a simple differential equation for the F's one gets an integro-differential equation, the integrals containing the overlap contributions to the energy. Since the value of the integral is sensitive to the relative velocity of the groups, the contribution to the forces represented in it are referred to as the 'velocity dependent forces'. This connotation is not to be confused with the possibility that the nuclear forces between pairs of elementary nucleons may depend upon velocity.

The main simplification assumed in the method of resonating group structure is the neglect of polarization of the sub-groups by the presence of others. Even so, it is probably somewhat more adequate than tractable. It is the natural method of solving problems in which exchange of partners has important consequences, as in the scattering of neutrons by deuterium. For more complex nuclei, however, further simplifications must be made for detailed application. On rather general lines, parallel to those followed in dealing with molecular structure, Wheeler† has shown how the model may be used to interpret excited states of light nuclei and also how it is related to the liquid drop model for heavy nuclei.

4. Spins and moments

The influence of the spins of the nucleons on the forces between them is reflected most sensitively in the resultant spins, magnetic dipole, and electrical quadrupole moments of the composite nuclei. All that can be expected from the rough nuclear models proposed hitherto is a qualitative description of these finer features, with more quantitative deductions concerning spin than concerning the moments. On the other hand, current experimental techniques in comparing magnetic moments make this property the most accurately known among the unquantized properties of nuclei. It is not known at present whether it is feasible to elaborate the nuclear models, or make new ones that can utilize the precise information given by these experimental results. The formidable nature of the

† J. A. Wheeler, *Phys. Rev.* **52** (1937), 1083.

general problem is sufficiently illustrated by one of the simplest examples, viz. the theory of the magnetic moment of the triton (nucleus of H^3). In order to account for the rather curious result that the magnetic moment of the triton is larger than that of the proton, Sachs† has found it necessary to mix the ^2S-wave with the ^2P-, ^4P-, *and* ^4D-wave of the three-nucleon system. This is indicative of the inherent complications that go with attempts to apply details of the theory to the complex nuclei.

So far as the nuclear models that have been discussed in this chapter are concerned, it is possible only to make some quite general statements about spins and moments. For example, one can account for the fact that all nuclei containing an even number of neutrons and an even number of protons have no spin, and hence no magnetic moment. Inasmuch as the state of maximum (negative) potential energy in such a nucleus is one in which the nucleons are completely paired off with opposite spins, the intrinsic moment of the nucleons certainly vanishes. This leaves the possibility that their orbital angular momenta might contribute to the nuclear spin. It turns out, however, that the state of lowest energy is also the state of lowest net orbital angular momentum. If the nucleon orbits are of the type postulated in the α-particle model, i.e. localized in space, this result is obvious since a net orbital angular momentum would mean that one of the alphas was circulating about and thus contributing more kinetic energy to the system than necessary. In the independent particle model, or the description of additional particle wave-functions in the α-particle model as proposed by Hafstad and Teller, where the orbits are labelled with an orbital quantum number the result holds, but in different form. Suppose the nucleons that are not in completely closed orbits are found in p-orbits. For a given number of such extra nucleons there are, in general, several ways of constructing a wave-function with the maximum number of symmetric couplings. As a simple example: He^6 has two extra neutrons that may be considered to be in p-orbits. From two p-waves one can construct symmetric S- and D-waves and one antisymmetric P-wave. In our first approximation, the S and D have the same energy. However, by calculating the average distance between the neutrons more exactly, it has been shown by Feenberg and Philips‡

† R. G. Sachs, ibid. **71** (1947), 457.
‡ E. Feenberg and M. Philips, ibid. **51** (1937), 597.

that the neutrons are a little more closely correlated in positions in the S-wave. The same authors calculated all the low terms in the series from He^4 to O^{16} and found that the lowest states have lowest total orbital angular momentum. This may be inferred to be a general rule and so account for the lack of spin in even-even nuclei on the basis of all models.

The second broad result on measurements of nuclear magnetic moments refers to nuclei with odd values of A. The results differ, however, depending upon whether it is the number of protons or the number of neutrons that is odd. Among the stable nuclei having an odd neutron the magnetic moments are known for about a score. These range from $+1$ nucl. magn. to -1 nucl. magn. with an average just about zero. About twice as many nuclei with odd proton have moments ranging from slightly negative values to 6·4 (In^{115}). The average of these is around 2·5 nucl. magn. Furthermore, the magnetic moment increases markedly with the value of the spin in case there is an odd proton, but there appears to be little correlation between total spin and magnetic moment in the case of an odd neutron. Of course, the number of data is not sufficient to provide good statistics, but reasoning from what data there is one would say that the net magnetic moment averages about the value that would be obtained if the spin of the nucleus were accounted for by the orbital motion of the odd particle, i.e. zero for an odd neutron and plus something (the average nuclear spin in the above data is also 2·5) for an odd proton. On the basis that the average value of the orbital angular momentum is the nuclear spin, Schmidt† has advanced a proposal for accounting for these regularities. The spread about the average value, according to this theory, is due to mixing of the individual particle orbits with $l = i \pm 1$. However, two individual particle orbits differing in orbital quantum number by unity differ in parity, so that either parity is not a good quantum 'number' in nuclei, or waves of the other nucleons are affected by the mixing. This point, together with the experimental finding that odd isobaric pairs, such as in Cu, Re, and In,‡ have nearly equal moments, are not by any means adequately elucidated by the present form of nuclear theory.§

† F. Schmidt, *Zs. f. Physik*, **106** (1937), 358.

‡ H. Schüler and H. Korsching, ibid., **105** (1937), 168.

§ For further discussion of the application of the theory of the symmetric Hamiltonian see H. Margenau and E. Wigner, *Phys. Rev.* **58** (1940), 103.

The potential complexity of interpreting spin effects in nuclear structure is illustrated in a convincing way by examples among the very light nuclei. There are two typical cases in point. The first of these concerns odd nuclei, say Li^7. According to the general argument, we should suppose that Li^7 is formed by adding a neutron in a p-wave to an S-state of Li^6. The question then arises whether the spin of the proton will be parallel to or opposite to the direction of its orbital angular momentum (the spins of the neutrons are presumed to form a singlet), i.e. whether the lowest state is a $^2P_{\frac{1}{2}}$ or $^2P_{\frac{3}{2}}$. If the splitting of these two states were due to electromagnetic forces, we should expect the $P_{\frac{1}{2}}$ state to lie lowest, just as in the atomic electron case. It has been pointed out by Inglis,† however, that the accelerations experienced by a nucleon are mostly induced by the nuclear forces, so that the extra energy due to the Thomas precession

$$\Delta E = \tfrac{1}{2}\hbar\boldsymbol{\sigma}.\boldsymbol{\omega}_T, \qquad \boldsymbol{\omega}_T = -\frac{\mathbf{v}X\mathbf{a}}{2c^3},$$

where a is the acceleration, v the velocity, of the nucleon, is proportionately much larger than in the atomic orbits of electrons (where it is just one-half of the electromagnetic effect). Since the Thomas effect is of the opposite sign to the electromagnetic effect, and overrides it, the order of states as split by spin-orbit coupling will be reversed for nuclei. According to this picture, therefore, the lowest state of Li^7 should have the angular momentum $\frac{3}{2}\hbar$, in agreement with the experimental value. As in the case of electronic orbits, if a shell is more than half-filled the order of the fine-structure levels is reversed. Thus, in the individual particle model the lowest state of C^{13} (nucleus) would be $^2P_{\frac{1}{2}}$. On the α-particle model, C^{13} is pictured as three α-particles with the extra neutron in a P-wave that has a plane node through all three alphas. According to Inglis's theory, therefore, the spin of the extra neutron should be parallel to its orbital angular momentum and the total spin of C^{13} should then be $\frac{3}{2}\hbar$. The experimental value is $\frac{1}{2}\hbar$, thus favouring the individual particle picture. This might be considered a conclusive indication that the individual particle picture is better than the α-particle one if it were not for the ubiquitous complication of the dipole-dipole forces between the nucleons.

The strongest evidence for the importance of the dipole-dipole

† D. R. Inglis, ibid. **50** (1936), 783.

forces is found in the stable odd-odd nuclei. Sachs† and others have made a study of the magnetic moments of H^2, Li^6, B^{10}, and N^{14}. The magnetic moments of these nuclei might be expected to be nearly the same. Instead, they decrease rapidly with increasing A, indicating that the mixture of S-waves with D-waves, and so on, becomes of major importance even in such light nuclei as B^{10} and N^{14}. It is estimated that in N^{14} there is at least a 50–50 mixture of S- and D-waves, as compared with a 20–1 mixture in H^2. The influence of this type of coupling among heavier nuclei is entirely unestimated, but may be imagined to be considerable; and the relative importance of such couplings and effects of the Thomas precession are not clear, at present, even for light nuclei. Thus, although calculations of the theoretical magnetic moments have been attempted for both the independent particle model‡ and the α-particle model,§ nothing like an adequate accounting for the several spin-effects can be inserted into the present theoretical approach.

The most striking evidence of the dipole-dipole forces is, of course, the development of quadrupole moments on the part of the stable nuclei. Wave-mechanically, the non-spherical charge distributions are formed by interference between waves of different orbital angular momentum, say S- and D-waves, as in the deuteron. Physically, this is interpreted as a predilection the nucleons have for lining up on an axis parallel to their spins. A simple description is possible in the case of the deuteron, as given in Chapter II, but for heavier nuclei only general statements can be made. The most general statement is that the nuclear spin must be at least $\hbar$ for the nucleus to have a quadrupole moment. This follows from elementary considerations to the effect that a quadrupole moment has the symmetry characteristics of an ellipsoid of revolution, i.e. rotation by 90° may change its value along a fixed axis from a maximum, say, to a minimum. If the nuclear spin is $\frac{1}{2}\hbar$, there are only two quantum states and these correspond to rotation by 180° under which the quadrupole moment does not change. The effect of a quadrupole moment can be noticed quantum mechanically, therefore, only if there are at least three polarizations of the nucleus, i.e. one polarization corresponding to quantization at 90° to the other two. Hence, the nuclear spin must

† R. G. Sachs, *Phys. Rev.* **72** (1947), 91; 312.
‡ M. E. Rose and H. A. Bethe, ibid. **51** (1937), 205, 993.
§ R. G. Sachs, ibid., **55** (1939), 825; D. Inglis, ibid. **60** (1941), 837.

be at least unity for a quadrupole moment to be evident. About two-dozen values of quadrupole moments are known, and these have been included in Table I of Chapter I. However, new methods are being brought to bear on the determination of quadrupole moments,† and these results will undoubtedly play an increasing role in our interpretation of the stable nuclei.

† W. A. Nierenberg, N. F. Ramsey, and S. Brody, ibid. **71** (1947), 466.

V

THE PROCESS OF β-TRANSFORMATION

1. General features of β-transformation

THE most characteristic feature of the spontaneous β-transformation of a nucleus is the continuous distribution in energy of the emitted electrons.† This is in sharp contrast to the line spectra observed for

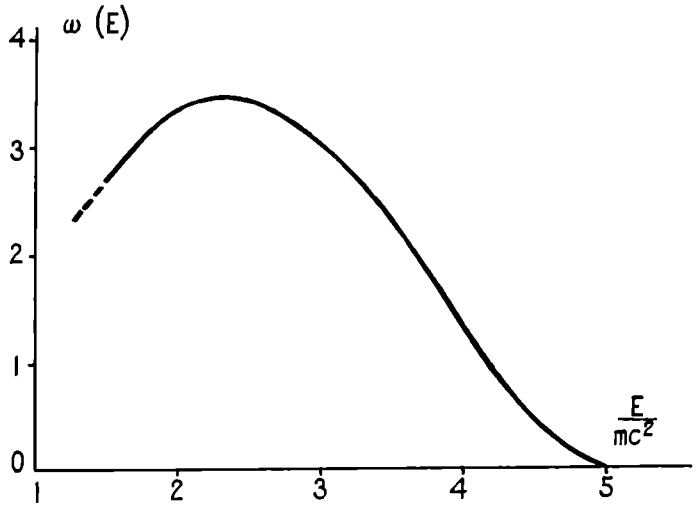


FIG. 17. β-spectrum of In^{114}.

α-rays, γ-rays, and the β-particles emitted by internal conversion of γ-rays. Similar to α-radiations, however, the β-radiations have characteristic half-lives, and all parts of the spectrum decay with the same period. If the disintegration leads to just one state of the daughter nucleus the spectrum is said to be simple. The shapes of the simple β-spectra are essentially the same for all nuclei (with a few possible exceptions), and in Fig. 17 is shown the experimentally determined spectrum of In^{114} which is probably the best known.‡ Other examples of the simple spectrum differ in the maximum energy of the spectrum and in the half-life. The very low energy region in Fig. 17 is shown dotted because the experimental errors due to scattering and absorption in the source are too great at low energies to give reliable measurements (comp. Appendix II).

The large amount of experimental work on β-disintegrations has firmly established that one electron (or positron) is emitted in the

† First demonstrated by J. Chadwick, *Verh. d. D. Phys. Ges.* **16** (1914), 383.
‡ Determined by J. L. Lawson and J. M. Cork, *Phys. Rev.* **57** (1940), 982.

disintegration, that there is a definite upper limit to the energy of the electron in a given spectrum,† and that the energy of the nucleus decreases by an amount just equal, or very nearly so, to the maximum energy of the emitted spectrum. This last property of the β-transitions is evidenced by the fact that γ-rays emitted when a nucleus may emit β-spectra of two distinct maximum energies are sharp lines which just make up the difference in maximum energy. Also, the mass differences between parent and daughter nucleus have been determined independently in several cases and found to agree with the maximum energy of the β-spectra. An interesting example of the latter type of experiment is afforded by the capture of a slow neutron by N^{14}. The resulting nucleus, N^{15}, emits, among other things, a proton of $0{\cdot}58 \pm 0{\cdot}03$ M.e.v. energy‡ resulting in C^{14}. Allowing for the difference between the mass of neutron and H atom, 0·75 M.e.v., the C^{14} atom lies $0{\cdot}17 \pm 0{\cdot}03$ M.e.v. above the N^{14} atom. The observed upper limit of the β-spectrum of C^{14} is 0·145 M.e.v.§ From this we should conclude that the disparity between the upper limit of the β-spectrum and the energy released by the transition lies between zero and 0·06 M.e.v.

The evidence cited in the last paragraph points strongly toward some form of conservation of energy in β-processes even though, most of the time, the β-particle carries away only part of that energy. For this reason, plus several others, the proposal of Bohr to disregard the principle of conservation of energy in making a theory of β-decay has been abandoned. The earlier suggestions that the continuous spectrum arises from secondary effects of scattering or absorption of the β-particles or that a light quantum is emitted simultaneously are ruled out by the calorimetric measurements of the energy of β-radiation.‖ In the experiment of Ellis and Wooster, the total heat produced by a known amount of RaE was determined. This element has a very weak γ-radiation, so that the rate of energy-production, divided by the number of atoms disintegrating per unit time, gives the average energy of disintegration per nucleus. The result was found to agree with the *average* energy of the RaE β-spectrum, whereas, if the continuous spectrum were produced by

† This was first shown by B. W. Sargent, *Proc. Roy. Soc.* A **109** (1925), 541.

‡ T. W. Bonner and W. M. Brubaker, *Phys. Rev.* **49** (1936), 778.

§ S. Ruben and M. D. Kamen, ibid. **59** (1941), 349.

‖ These measurements were first carried out by C. D. Ellis and W. A. Wooster, *Proc. Roy. Soc.* A **117** (1927), 109 and have been verified repeatedly.

secondary effects, the result should agree with the maximum energy. Thus it appears that the difference between the total energy released and that carried by the electron escapes detection in the calorimeter.

As pointed out in Chapter I, the accepted explanation of the continuous β-spectrum is due to Pauli and postulates that the energy released by the nucleus is the maximum energy of the β-spectrum but that this energy is shared between the electron and a neutrino which are emitted simultaneously. The neutrino is defined as an uncharged particle that is very hard to detect, i.e. it has a low ionizing power. There are many advantages to Pauli's hypothesis in addition to the conservation of energy. The neutron-proton model of atomic nuclei requires that nuclei with an odd number of heavy particles have a spin that is an odd multiple of $\frac{1}{2}\hbar$ and obey Fermi–Dirac statistics. Since β-decay does not change the number of heavy particles, the emission of a single electron from a nucleus would change the spin from an odd multiple to an even multiple of $\frac{1}{2}\hbar$ in contradiction to the model. It would also change the statistics of the nucleus from Fermi–Dirac to Bose–Einstein. This can be rectified by postulating that the neutrino also has an intrinsic spin of $\frac{1}{2}\hbar$ (or an odd multiple thereof), obeys Fermi–Dirac statistics, and is emitted simultaneously with the electron. The same argument holds for even nuclei, of course.

Certain properties of the neutrino can be bracketed experimentally. Attempts to detect neutrinos by ionization have been made by Nahmias, who reported that a neutrino produces less than one ion per 5×10^5 km. in air. Bethe has calculated† that this result requires the magnetic moment of the neutrino to be less than 10^{-4} that of the electron. The mass of the neutrino must also be small compared with that of the electron. This is evidenced in the shapes of β-spectra as predicted by the Fermi theory and also by the experimental agreement between energy released by nuclei and the maximum of the electron spectrum. The case of C^{14} cited above would allow a neutrino mass of $\frac{1}{10}$ an electron mass, at most. The scarcity of neighbouring isobars in the system of isotopes also speaks for a small mass for the neutrino. It is generally assumed that there is only one kind of neutrino and that its rest-mass is zero.

Some evidence for the momentum of the emitted neutrino has been obtained. Cloud chamber recoil tracks have been observed and

† H. A. Bethe, *Proc. Camb. Phil. Soc.* **31** (1935), 108.

evaluated by Crane and Halpern,† who conclude that there is evidence of the momentum of the neutrino in the recoil of the daughter nucleus. The most recent evidence of this general type is the detection of Li^7 recoils from the K-capture in Be^7.‡ Since γ-rays are emitted in this transition there will be recoils due to them as well as to neutrinos. However, the energy of the recoiling nuclei is 16 e.v. when the 0·45 M.e.v. gamma is emitted as compared with 58 e.v. when a 0·87 M.e.v. neutrino is emitted. By having the Li recoils take place from a Pt surface the atoms were ionized and could be detected in an electron multiplier tube. It was shown that the apparent energy of the recoils lay between 40 and 50 e.v. and that there was no appreciable coincidence between these recoils and the emission of γ-rays. Since there are no charged particles emitted by Be^7 the recoils must be due to neutrinos, and the fact that the energy of recoil is somewhat less than expected can be attributed to the extreme difficulty in preparing a non-absorbing source.

According to the Dirac theory of the electron, positrons and electrons have identical properties except for their different sign of electric charge. It is to be expected, therefore, that the β-radiations have the same characteristics, whether electron or positron is emitted, except in so far as the electric charge of the nucleus influences the process. The obvious effect of charge is that nuclei, especially heavy nuclei, repel positrons whereas they attract electrons. This may be expected to inhibit positron emission by nuclei whose potential barrier is larger than the β-energy. Positron emission by heavy nuclei is further made improbable by the ability of a radioactive nucleus to absorb one of its planetary electrons. This is the K-capture alluded to above and is so called because, in all known cases, it is an electron in the K-shell that is absorbed.§ That absorption of an electron is equivalent to the emission of a positron is a prediction of the Dirac theory of the electron and is emphasized by the fact that in some nuclei the two processes compete.

There are a few nuclei, for example Cl^{36} and As^{76}, that have an odd number of neutrons and an odd number of protons and are unstable against both electron and positron emission with K-capture

† H. R. Crane and J. Halpern, *Phys. Rev.* **53** (1938), 789.

‡ J. S. Allen, ibid. **61** (1942), 692.

§ This process was predicted by H. Yukawa and S. Sakata, ibid. **51** (1937), 677, and first observed by L. W. Alvarez, ibid. **52** (1937), 134.

competing with the latter. Furthermore, in As^{76}, as in easily half of the known β-activities, the various β-radiations are followed by γ-radiation. The latter come about either because a transition to the ground state of the daughter nucleus is forbidden by some selection rule so that the transition goes more readily to an excited state, or because several states of the daughter nucleus are equally available in the decay period.

Altogether over 150 β-active nuclei have been identified, their maximum energy or energies and lifetimes determined. Of these hardly a dozen are found among the natural radioactivities. The others are artificially radioactive nuclei that are formed by bombardment with neutrons, protons, deuterons, alphas, and gammas, and as products of fission. Many of the artificial nuclei are made in several different ways as a means of positive identification. Fermi and his collaborators† first established the radioactivity due to neutron irradiation. Slow neutrons are captured by nuclei to form either a stable isotope or one that has too few protons to be stable. In the latter event the number of protons is increased and the number of neutrons decreased by the emission of an electron in β-disintegration. Fast neutrons that eject protons or α-particles from the bombarded nucleus produce similar activities. Bombardment with deuterons often increases the neutron number, in a d, p reaction, and leads to subsequent negative electron emission. On the other hand, the use of protons or α-particles as projectiles in nuclear bombardment is more apt to produce overcharged nuclei that decay by positron emission. This phenomenon was first observed by Curie and Joliot‡ in the α-bombardment of light elements. Very fast neutrons can also produce positron emitters by knocking out two or more neutrons from the compound nucleus.

It was first pointed out by Sargent§ that there is a significant correlation between the characteristic lifetime of a β-disintegration and its maximum energy. This is demonstrated by plotting the logarithm of the decay-constant against the logarithm of the maximum energy. The relation between these quantities is of the same general type as the relation between energy and decay-constant of α-emitters;

† E. Amaldi, O. D'Agostino, E. Fermi, B. Pontecorvo, F. Rasetti, and E. Segre, *Proc. Roy. Soc.* **146** (1934), 483; **149** (1935), 522.

‡ I. Curie and F. Joliot, *C.R.* **198** (1934), 254.

§ B. W. Sargent, *Proc. Roy. Soc.* A **139** (1933), 659.

but the range of maximum energies found in β-emission is much greater than the range of energy of α-rays, whereas the range of observed lifetimes is generally much smaller than for α-activities.

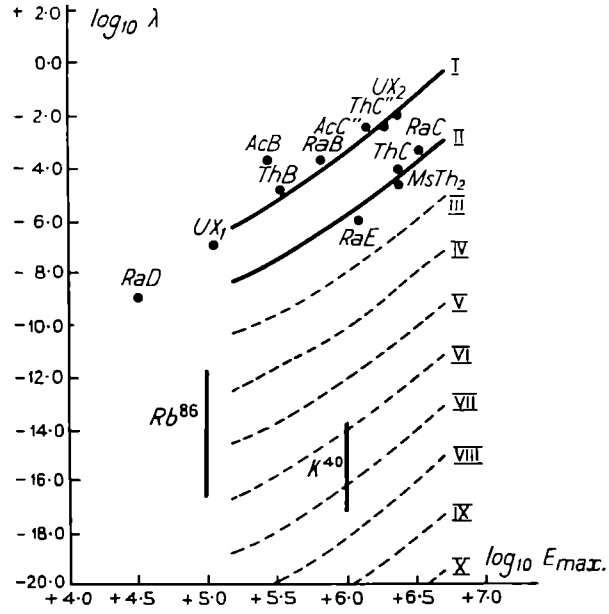


FIG. 18 (*a*).

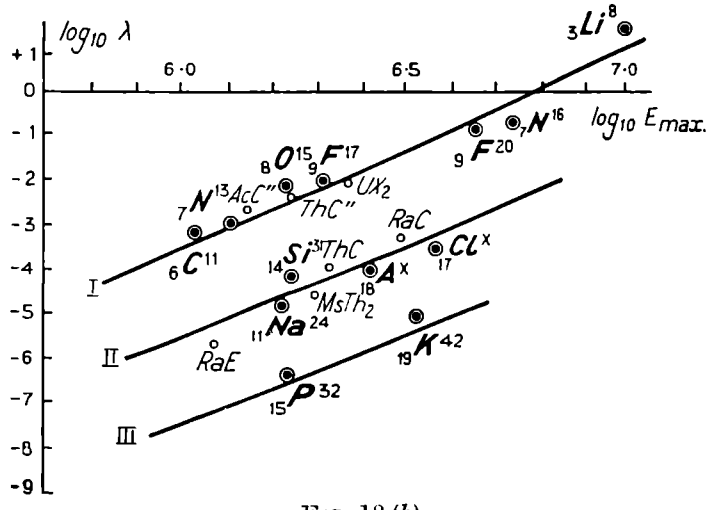


FIG. 18 (*b*).

The Sargent relation for the natural and artificial β-emitters is shown in Fig. 18 (*a*, *b*). In that figure there is included the correlation for a few of the positron emitters that have one more proton than neutron, i.e. N^{13}, O^{15}, and F^{17}. The reason for emphasizing this class of nuclei to the exclusion of the bulk of the data available will be more apparent in § 4; briefly it may be said that the similarity

between nuclear wave-functions of parent and daughter nucleus is expected to play a part in the correlation, and for the positron emitters represented in Fig. 18 (*b*) this factor should be particularly favourable since the daughter nucleus is obtained from the parent by interchanging neutrons and protons. The comparison is complicated, however, by the influence of the nuclear electric field which is much stronger for the natural emitters than for the light, positron emitters.

It is significant that the natural activities form two groups in Fig. 18 (*a*). The decay rate in the lower group appears to be slower by a factor 100 than in the higher group at the same energy. This effect in the natural β-activities is readily understood if it is supposed that the transitions in the lower group entail 'forbidden' spin or parity changes in the nuclear wave-function. Thus the β-radiations may be classed into various multipole radiations just as in the description of forbidden optical spectra and for the same reason, namely, that the wave-lengths of emitted particles are large compared with the emitting region. The principal difference between the two cases is that the wave-length of a photon is generally 1,000 times the radius of an atom but the wave-length of β-particles is only 10 times the nuclear radius as a rule. With these factors, nuclear wave-functions, Coulomb forces, and the possibility of forbidden transitions coming into the relation between lifetime and maximum energy it will be easily appreciated that a Sargent plot of all known β-disintegrations would be relatively useless.

2. Elementary theory of β-disintegration

The neutron-proton model of nuclear structure does not include electrons, positrons, and neutrinos as elements of the structure in the sense that it may include α-particles as such elements. It is much more natural to think of the β-particles being created at the instant of emission in the same way that a light quantum is created in an atomic transition. β-radiation then becomes analogous to electromagnetic radiation, but with some complications. In the first place, instead of the radiated energy consisting of a single uncharged quantum it is distributed in all possible proportions between the electron and the neutrino. Consequently the radiating particle in β-decay changes its electric charge as well as its energy in the process. Upon the discovery of the neutron, however, the way for a complete

analogy between the two emission processes was open and a satisfactory theory of β-disintegration was developed by Fermi.†

Fundamental to Fermi's theory is the hypothesis, originally due to Heisenberg, that neutron and proton are two states of the same particle. This hypothesis amounts to something more than a convenient way of describing β-disintegration, as we have seen in Chapter IV, where we considered the Pauli exclusion principle in its application to nuclear wave-functions. The Fermi theory then postulates that neutron and proton interact with the combined fields of electrons and neutrinos in such a way that an electron and a neutrino are created and radiated when neutron changes to proton. Conversely a positron and a neutrino are radiated when a proton changes to a neutron. Thus Fermi's theory by-passes any possible connexion between mesons and β-radiation. The modern view of the process analyses the radiation of the β-particles in two steps: a neutron changes into a proton, say, and emits a negatively charged meson into a virtual state, i.e. on borrowed energy, and the meson subsequently disintegrates into electron and neutrino, and returns the borrowed energy balance to the nucleon. The same picture applies with a positive meson when proton changes into neutron. This point of view will be elaborated in § 3. The net effect is the same as if nucleons interact directly with the electrons and neutrinos, and, in this section, we retain this point of view. Since the total energy is assumed to be conserved and since the wave-lengths of the emitted particles are large compared with nuclear dimensions, the number of β-disintegrations per second may be written as a generalization of radiation formula

$$w = (2\pi/\hbar)g^2|\mathbf{M}|^2\psi_e^2\psi_\nu^2\rho_e\rho_\nu, \tag{1}$$

where $\mathbf{M}$ is the matrix element calculated from the waves of neutron and proton. The interaction constant, g, takes the place of e in electromagnetic theory. ψ_e and ψ_ν are the amplitudes of electron and neutrino waves at the nucleus, so that $\psi_e^2\psi_\nu^2$ is equivalent in the formulism to E^2 in the electromagnetic case. ρ_e and ρ_ν are the numbers of available electron and neutrino states per unit energy.

The formula for w contains three factors: a numerical factor, one depending only on the nuclear particles, and one depending upon the strength of the emitted waves at the nucleus. The separation of the last two factors is an approximation that comes from assuming

† E. Fermi, *Zs. f. Phys.* **88** (1934), 161.

that the wave amplitudes of the emitted particles, electron and neutrino, do not vary appreciably over the region occupied by the heavy particles. In this section we shall consider the case in which both electron and neutrino can be emitted without orbital angular momentum, i.e. in S-waves, corresponding to the most favourable decay rate. At first we shall not differentiate between the contributions of the various spin components of the relativistic form of the electron and neutrino waves and we shall disregard the influence of the Coulomb field of the nucleus. For light nuclei, the S-waves of electrons are not greatly affected by the Coulomb field if the electron has several million volts energy and we are justified in approximating $\psi_e(r)$ as

$$\psi_e(r) = \sin(p_e r/\hbar)/r\sqrt{(2\pi R)}, \tag{2}$$

where we have normalized $\psi_e(r)$ in a sphere of radius R and p_e is the momentum of the electron.

Let the total energy liberated by the nucleus be W and let the energy of the electron be E. Then the normalized wave-function for the neutrino, of zero rest-mass, is

$$\psi_\nu(r) = \sin[(W-E)r/\hbar c]/r\sqrt{(2\pi R)}. \tag{3}$$

The numbers of S-states of electron and neutrino per unit energy are:

$$\rho_e = R/\pi\hbar v = RE/\pi\hbar c^2 p_e, \qquad \rho_\nu = R/\pi\hbar c. \tag{4}$$

Evaluating eq. (1) at $r = 0$ we get

$$w = \frac{g^2}{2\pi^3 c^5 \hbar^7} |\mathbf{M}|^2 p_e E(W-E)^2. \tag{5}$$

The dependence of w on the energy of the electron is the continuous β-spectrum. The procedure used in comparing the form of w with experimental results is to divide the experimental distribution in energy by $p_e E$, or its equivalent in the Coulomb field for heavy nuclei, and plot the square root of the resulting function against E†. Such a plot is known as a Kurie plot, after its inventor, and if the β-spectrum is simple the plot should be a straight line the intercept of which determines the maximum energy, W. It will be noted that W includes the rest-energy of the electron. Many such plots have been made, especially for light nuclei, and so far only two cases of convincing deviations from the predicted spectrum have been detected. The cleanest Kurie plot that has been obtained is that for

† More commonly the distribution in electron momentum is determined directly and the corresponding divisor is p_e^2.

In^{114}, the spectrum of which is shown in Fig. 17. Thus the experimental results confirm the assumed dependence of w upon the amplitudes of the light particle waves and also the assumed vanishing rest-mass of the neutrino.

Offhand it is a little surprising that practically all the observed spectra can be explained on the assumption that the electron and neutrino are emitted without orbital angular momentum. The wide variations in lifetimes would lead one to suspect that in many transitions the light particles are emitted in P-waves and D-waves which should introduce higher powers of the electron energy into the distribution function. Further consideration of the nature of forbidden transitions and of the more exact wave-functions for higher angular momenta leads to an understanding of this result and will be taken up in § 4.

Since w is the number of processes per second per unit energy range, the probability of disintegration per second is the integral of $w\,dE$ over E from mc^2 to W. This leads to

$$\lambda = \frac{g^2 m^5 c^4}{2\pi^3 \hbar^7} |\mathbf{M}|^2 I_0\left(\frac{W}{mc^2}\right), \tag{6}$$

with $$I_0(x) \equiv \tfrac{1}{60}(x^2-1)^{\frac{1}{2}}(2x^4-9x^2-8)+\tfrac{1}{4}x \ln[x+(x^2-1)^{\frac{1}{2}}]. \tag{7}$$

If W is large compared with mc^2, i.e. the maximum energy, including rest energy, is 2 M.e.v. or more, a very good approximation for λ is

$$\lambda \cong \frac{g^2}{60\pi^3 c^6 \hbar^7} |\mathbf{M}|^2 W^5. \tag{8}$$

Thus the Sargent plot should approach a straight line of slope 5, at least for the light elements. This law is very closely obeyed by the positron emitters shown in Fig. 18 (*b*).

An independent and ingenious check on the fifth-power law in the formula for the decay constant is provided in the energy distribution of α-particles emitted by the β-decay of Li^8:

$$Li^8 \rightarrow He^4 + He^4 + \beta + \nu + Q.$$

The α-particles are emitted with equal energies, but the sum of their energies varies from zero to the maximum energy of the reaction, $Q = 15{\cdot}8$ M.e.v. Let U be the energy of one of the alphas, then the energy in the β-radiation is $Q-2U$. The probability of emission of alphas with energy U will then be proportional to the product of $\lambda(Q-2U)$ and a statistical factor for the alphas which is $U^{\frac{1}{2}}$ if they

are emitted in an S-state and $U^{\frac{1}{2}}$ if they are in the D-state. The frequency distribution was measured† and found to agree closely with the factor $U^{\frac{1}{2}}(Q-2U)^5$, showing conclusively that the β-disintegration follows the law derived from Fermi's theory and also that the alphas are emitted in an S-state. This proof which was suggested by Gamow and Teller was the first indication that the original Fermi theory was correct and the modification introduced by Uhlenbeck and Konopinski was unnecessary.

The modification of the theory that was introduced by Uhlenbeck and Konopinski was to replace the wave-function of the neutrino, in the expression for w, eq. (1), by the first derivative of the neutrino wave-function. This introduces an additional factor proportional to the square of the energy of the neutrino and gives a better fit to the early experimental results on β-disintegration spectra. A critical analysis‡ of these early results indicates that the apparent agreement was due to the circumstance that most of the measured spectra were really composite, i.e. two activities were present with the same period.

Coulomb forces between nucleus and the β-particle influence the lifetime of the disintegration appreciably. Their effect on the spectrum itself is generally slight except in the region well below 1 M.e.v. where the measurements are also difficult to make. In the very heavy elements and in all forbidden spectra the Coulomb effect is characteristic of the relativistic motion of the electron, but in the allowed spectra of light elements the effect is simply to increase the amplitude of electron waves at the nucleus and to decrease the amplitude of positron waves. Negative electron emission is, therefore, more probable and positron emission less probable than given by the formulas above for w and λ. A very satisfactory method of correcting the expressions, eq. (5) and eq. (6), for elements in the lower half of the periodic system is to insert the factor

$$\frac{-2\pi\eta}{e^{-2\pi\eta}-1}, \qquad \eta \equiv \frac{Ze^2E}{\hbar c^2 p_e} \tag{9}$$

in eq. (5), where Z is the charge of the daughter nucleus in the case of electron emission§ and minus that charge for positron emission.

† L. H. Rumbaugh, R. B. Roberts, and L. R. Hafstad, *Phys. Rev.* **54** (1938), 657.

‡ H. A. Bethe, F. Hoyle, and R. Peierls, *Nature*, **143** (1939), 200.

§ This result may be derived from the solutions of the wave-equation for the motion in a Coulomb field as given in Chapter II, eq. (6), etc., by computing the ratio of amplitudes at large r and at $r = 0$.

In eq. (6) this factor should be included in the integrand, but it is such an insensitive function of E, except when p_e is small, that it may usually be considered a constant factor and taken out of the integral. The correction to eq. (6) then becomes the factor (9) with $\eta = \alpha Z$, where α is the fine structure constant.

The probability of K-capture admits of a similar elementary discussion. Here the electron is absorbed from a state of definite energy, namely the K-shell of the atom, so that the neutrino is emitted with a definite energy equal to the change in nuclear energy plus the rest-energy of one electron. The spectroscopic term is negligible. The same theory of transition probability then applies, but with the electron density of the K-shell in place of $\psi_e^2 \rho_e$ in eq. (1). The electron density at the nucleus, as calculated from the theory of the hydrogen atom, is

$$\psi^2 = Z^3/\pi a_0^3, \qquad a_0 = \hbar^2/me^2. \tag{10}$$

The Z in this formula is the 'effective' charge of the parent nucleus. Substituting in the transition probability we get, for K-capture:

$$\lambda_K = \frac{g^2 m^3}{\pi^2 \hbar^7} |\mathbf{M}|^2 (\alpha Z)^3 (W + mc^2)^2. \tag{11}$$

So far, our discussion of the Fermi theory has been primarily concerned with the dependence of spectral distributions and decay constants on the energy of the emitted particles. The evidence is fairly conclusive that the theory fits the experimental results and thus substantiates the form of eq. (1). The third factor in eq. (5) appears to be verified. Of the other two factors, the first is simply a number that should be the same for all spectra and which determines the value of the fundamental constant, g, which we shall interpret further in terms of meson theory in the following section. The second factor is the absolute square of the matrix element of the emitting system and, in general, may be expected to be different for different disintegrations.

According to the perturbation theory, on which eq. (1) is based, the matrix element, $\mathbf{M}$, depends upon the initial state of the nucleus Ψ_i, the final state Ψ_f, and the perturbation operator H', as

$$\mathbf{M} = \int \Psi_f^* H' \Psi_i \, d\mathbf{x}, \tag{12}$$

where the integral extends over all configuration space. This amounts to integrating the coordinates of every heavy particle in the nucleus

over the volume of the nucleus. The states Ψ_i and Ψ_f are normalized to the integer, A, the total number of nuclear particles. In the case of emission of electromagnetic waves, the operator H' is essentially the electric dipole, $e\mathbf{x}$. Although eq. (1) is a generalization of the electromagnetic case, it turns out that for the theory of β-decay the dipole operator is of importance only to forbidden transitions and allowed transitions involve the zero moment of position, i.e. they are *monopole* radiations. This circumstance is a result of the more general nature of the fields that describe β-radiation. On the other hand, the operator H' does contain operators, Q_k, for every nuclear particle such that Q_k changes the kth proton into a neutron in positron emission or changes the kth neutron into a proton for electron emission. If H' induces no other change in the wave-function we have

$$H' = Q_1 + Q_2 + \ldots + Q_A \equiv Q(1), \tag{13}$$

where $Q(1)$ is introduced for the sake of brevity.

The form of H' in eq. (13) is actually the form that applies in allowed transitions if Fermi's original theory is used. The fact that H' is invariant to rotations and inversions of the space coordinates is consistent with our assumption that the electron and neutrino are emitted in S-waves. Thus this form requires that the angular momentum of the nuclear states Ψ_i and Ψ_f be identical so that the spin of the emitting particle is unaffected by the disintegration and the electron and neutrino must be emitted in the singlet state.

In this theory, therefore, the value of M is just a sum over k of integrals of $\Psi_f^* \Psi_i^k$ in which the superscript k indicates that the identity of the kth particles has been changed by the operator Q. Thus, if the Pauli exclusion principle did not apply to nuclear particles, and all neutrons and protons were in the same state, the value of M would be Z for positron emission and $A-Z$ for electron emission. Since the exclusion principle does apply, however, changing the identity of a particle in Ψ_i usually leads to a state of very high energy that is not very similar to the ground state, Ψ_f, of the daughter nucleus or is completely forbidden by the exclusion principle if the particle is in a closed shell. Hence the contributions to the β-disintegration of heavy particles that are in closed shells, or nearly so, are negligible, and it is generally sufficient to consider only such heavy particles that may be in unsaturated orbits. In particular, the positron disintegration of nuclei having just one more proton than

neutron must be mostly contributed by the extra proton.† These nuclei have small charges, $Z < 21$, so that the wave-function of the extra neutron in Ψ_f should be very much the same as that of the extra proton in Ψ_i. The value of **M** will then be practically unity.

In the language of spectroscopy, the form of H' in eq. (13) leads to the selection rule $\Delta i = 0$ for allowed transitions, where i is the spin of the nucleus. This rule appears to be violated, however, for several of the nuclei discussed above, having one more proton than neutron, have composite spectra. In the particular case of Be^7, for example, the capture of the K-electron leads to the ground state of the Li^7-nucleus most of the time but to the 0·45 M.e.v. excited state part of the time. The most likely explanation of these two levels of the Li-nucleus is that they represent the spin-orbit splitting of the extra neutron, the spin $\frac{3}{2}$ belonging to the ground state and $\frac{1}{2}$ to the excited state. The ground state of Be^7 should have one or the other of these spins (probably $\frac{3}{2}$), and the fact that transitions to both states of Li^7 compete in an allowed K-capture violates the selection rule $\Delta i = 0$. The need for a change of spin in allowed β-disintegrations was first presented by Gamow and Teller‡ following a study of the branches of the ThC radioactivity. Another very convincing argument for change of nuclear spin in β-disintegration is found in the electron emission of He^6. This is an allowed transition and since He^6 has two neutrons more than He^4 its spin is almost certainly O, but it decays to the ground state of Li^6 which has unit spin. A similar analysis probably applies to the activity of F^{18}. In all these cases the change of spin can be carried away by emitting the electron and neutrino in the triplet S-state.

In order to permit the change of spin in allowed transitions, Gamow and Teller (loc. cit.) proposed that the Fermi interaction be modified to include the spin-operator $\boldsymbol{\sigma}$. Thus the perturbation, H', is replaced by the vector, H'', where

$$\mathbf{H}'' = Q_1\boldsymbol{\sigma}_1 + Q_2\boldsymbol{\sigma}_2 + \ldots + Q_A\boldsymbol{\sigma}_A \equiv Q(\boldsymbol{\sigma}) \qquad (14)$$

and $\boldsymbol{\sigma}_k$ is the vector spin-operator of the kth particle. The Gamow–Teller selection rules are then $\Delta i = 1$ or 0 ($0 \to 0$ not permitted). This revised selection rule is the most important consequence of the interaction $\mathbf{H}''$. In general, it is impossible to apply these operators

† These considerations were first put forward by L. W. Nordheim and F. L. Yost, *Phys. Rev.* **51** (1937), 942.

‡ G. Gamow and E. Teller, ibid. **49** (1936), 895.

to nuclear wave-functions and obtain a numerical value for $|\mathbf{M}|^2$ because the nuclear wave-functions are not known. For the very light nuclei, however, approximations are possible, and perhaps the most useful of these is Wigner's theory of the symmetric Hamiltonian discussed in Chapter IV. In this theory the forces between nuclear particles are assumed to be independent of ordinary spin and isotopic spin. It is then possible to classify the nuclear wave-functions into definite 'supermultiplets' the members of which have the same space-dependence and differ only in spin and character coordinates. In this approximation, therefore, β-transitions take place only between members of a given supermultiplet and the integral over coordinates in $\mathbf{M}$ is unity. The calculation of $|\mathbf{M}|^2$ is then a matter of summing over spin and isotopic spin-coordinates alone. These sums have been computed by Wigner† for certain simple cases and applied to nuclei for which one can be reasonably certain about the orbital angular momenta of the particles. Four of these values for $|\mathbf{M}|^2$ are shown in Table X.

TABLE X

Parent	*Term*	*Daughter*	*Term*	$\lvert M(1)\rvert^2$	$\lvert M(\boldsymbol{\sigma})\rvert^2$	ft'
H^3	$S_{\frac{1}{2}}$	He^3	$S_{\frac{1}{2}}$	1	3	4,230
He^6	S_0	Li^6	S_1	0	6	6,960
F^{18}	S_1	O^{18}	S_0	0	2	11,700
Ne^{19}	$S_{\frac{1}{2}}$	F^{19}	$S_{\frac{1}{2}}$	1	3	5,520

The values $|\mathbf{M}|^2$ for both the Fermi selection rules, $|M(1)|^2$, and the Gamow–Teller rules, $|M(\boldsymbol{\sigma})|^2$, are given; and the last column, labelled ft', is

$$ft' = |M(\boldsymbol{\sigma})|^2 . I_0(W/mc^2)\tau_{\frac{1}{2}}, \tag{15}$$

where $\tau_{\frac{1}{2}}$ is the observed half-life of the disintegration and I_0 is the function defined in eq. (7). The value of ft' should be the same for all transitions if $\mathbf{H}''$ is the correct operator and if the theory of the symmetric Hamiltonian were exact. Variations in ft' are probably representative of the crudeness of the approximations made.‡ The

† E. Wigner, *Phys. Rev.* **56** (1939), 519.

‡ It has been recently pointed out by E. J. Konopinski (ibid. **72** (1947), 518) that a possible difference between the theoretically predicted and the observed value of the β-decay constant of H^3 can be interpreted in terms of the finite mass of the neutrino. In fact, assuming that the rest-mass of neutrino is zero, and taking for the matrix element of the transition in question the theoretical plausible value $|\mathbf{M}|^2 = 3$, one calculates the mean life of H^3 to be about *200 years* as compared with the observed value of only about *30 years*. Konopinski points out that, because of the very small total energy of this β-decay ($E_0 = 15$ kv. $= 0{\cdot}021\ mc^2$), a comparatively small mass of the neutrino would change quite considerably the energy partition

variation can obviously be reduced in Table X if a combination of H' and $\mathbf{H}''$ were used in place of $\mathbf{H}''$ alone, and there is no *a priori* reason for not doing so, but the uncertainties do not justify the attempt. It is possible that with more accurate data and a better understanding of the wave-functions of light nuclei a quantitative analysis of this kind can be made and a decision reached as to whether the selection rules are purely Gamow–Teller or whether there is an admixture of the original Fermi interaction.

An analysis of the form of the true interaction would also be possible with accurate information on the complex spectra, such as F^{20} and Be^{7}, where the nucleus decays to states of two different spins. The information required is the relative strength of the two spectra, their maximum energies, and a knowledge of the nuclear wave-functions. An analysis of this type has been made with existing data by Gronblom,† the principal result of which is substantial evidence for including the operator $\mathbf{H}''$.

The values of ft' occurring in Table X determine four values of the interaction constant g, according to the formula,

$$g^2 = \frac{2\pi^3\hbar^7}{m^5c^4}\frac{\ln 2}{ft'}.$$

Just what the natural units should be for g is not known, but the four numerical values obtained in this way from the data in Table X are: 1·72, 1·34, 1·03, and 1·51 all times 10^{-49}. About the nearest nuclear quantity of the correct dimensions is the square of the nuclear magneton times the fine structure constant $\alpha\mu_n^2 = 1{\cdot}8\times10^{-49}$, but there is no theory to connect this quantity with g.

The difference between the elementary theory presented in this section and the more complete theory of the following section is merely that we have not used the correct relativistic forms for the waves of the electron and the neutrino. So far as allowed spectra are concerned the results are essentially the same for the light nuclei.

between the two emitted particles, thus leading to a different value of total λ for the same upper limit of β-spectrum. With the finite mass μ of the neutrino the integral standing in the expression for the decay constant takes the form:

$$f(E_0) = \int_0^{E_0} dE(1+E)(2E+E^2)^{\frac{1}{2}}(E_0-\mu-E)(E_0-E)^{\frac{1}{2}}(2\mu+E-E_0)^{\frac{1}{2}}.$$

In order to bring the theoretical value of the lifetime of H^3 to the observed value by using this formula, *it is necessary to assume the rest-mass of the neutrino to be about* $\frac{1}{45}$ *of the electron mass.* A more definite answer can be given, however, only by the direct measurement of the energy distribution in the β-spectrum of H^3, cf. Appendix III.

† B. Gronblom, *Phys. Rev.* **56** (1939), 508.

There are important differences, however, in the effect of the Coulomb field on allowed transitions in the heavy elements and in almost all forbidden transitions.

3. Field theories of β-decay

Fermi's† original theory of β-decay employs the major doctrines of the theory of interactions between fields, viz. the requirements of relativistic invariance and the quantization of the wave-equations for the electron and neutrino. The fundamentals of this theory are sketched in Chapter III, so that we can review the Fermi theory briefly in the language of that chapter. The interaction giving rise to β-decay takes place between the nucleon on the one hand and the combined electron neutrino field on the other. All the particles involved have spin $\frac{1}{2}\hbar$, so that the various covariant quantities that can be formed are the same as those presented in Table III of Chapter III. Fermi assumed that the interaction takes the form of the scalar product between the polar vector formed by neutron and proton waves with the polar vector formed by electron-neutrino waves. If we let Ψ be the wave-function of the proton, Φ that of the neutron, and ψ and ϕ those of electron and neutrino, as in the preceding section, the relativistic form of Fermi's proposed interaction may be written:

$$H' = g \int \{[\Psi(\mathbf{x})^+ \boldsymbol{\alpha} \Phi(\mathbf{x}), \psi(\mathbf{x})^+ \boldsymbol{\alpha} \phi(\mathbf{x})] - \Psi(\mathbf{x})^+ \Phi(\mathbf{x}) \psi(\mathbf{x})^+ \phi(\mathbf{x}) + \\ + [\Phi(\mathbf{x})^+ \boldsymbol{\alpha} \Psi(\mathbf{x}), \phi(\mathbf{x})^+ \boldsymbol{\alpha} \psi(\mathbf{x})] - \Phi(\mathbf{x})^+ \Psi(\mathbf{x}) \phi(\mathbf{x})^+ \psi(\mathbf{x})\} \, d\mathbf{x}. \quad (16)$$

This particular form was chosen in direct analogy to the form of interaction between an electron and the potential four-vector of an electromagnetic field, $(\mathbf{A}, A_0)$, as given by Dirac's theory:

$$H'_e = e \int \{[\psi(\mathbf{x})^+ \boldsymbol{\alpha} \psi(\mathbf{x}), \mathbf{A}] - \psi(\mathbf{x})^+ \psi(\mathbf{x}) A_0(\mathbf{x})\} \, d\mathbf{x}. \quad (17)$$

As is well known, the emission of light quanta is effected only by the term in $\mathbf{A}$. In this respect, the β-decay interaction is more general since, in the latter, the time components in the product of vectors also gives rise to particle emission.

We may interpret H' in eq. (16) in terms of the quantized wave-equation. The first set of terms corresponds to the simultaneous creation of a proton and an electron, through the appearance of the operators $\Psi(\mathbf{x})^+$ and $\psi(\mathbf{x})^+$, and at the same time that a neutron and a neutrino are annihilated, as expressed by the operators $\Phi(\mathbf{x})$ and

† E. Fermi, loc. cit.

$\phi(\mathbf{x})$. All these events take place at the same point in space, $\mathbf{x}$. Hence these terms give negative electron β-radiation. The latter half of the integrand in eq. (16) causes the disappearance of a proton and an electron at the same time and place that a neutron and a neutrino appear. The electron may disappear from the K-shell of the atom (K-capture) or from a negative energy state (positron emission). The 'hole theory' of negative energy states is extended to neutrinos also. If we had insisted that both electron and neutrino be created in the negative β-emission, both would be absorbed in positron radiation, and the components of the four-vector operator, $(\alpha_x, \alpha_y, \alpha_z, 1)$, as applied to the light particle waves, would be multiplied by the factor $-i\beta\alpha_y$. The components would then be $(-\beta\sigma_z, i\beta, \beta\sigma_x, -i\beta\alpha_y)$ in the same order.

It will be noted that, by quantizing the nucleon waves as well as the light particle waves, one has automatically expressed the operator Q which changes neutron into proton and vice versa.

Fermi's theory can now be generalized to include scalar products of any of the five covariant functions of two nucleon waves with the corresponding contravariant functions of electron-neutrino waves, as outlined in Chapter III. In principle, therefore, the theory admits of five independent values of g, one for each invariant form. Generally speaking, the components of the invariants can be divided into large and small components. The large components are independent of the velocity of the nucleons, whereas the small components contain $\boldsymbol{\alpha}$ in an odd power. The small components can be neglected, consequently, except when the large components do not contribute because of a 'forbidden' change in spin or parity. Among the five covariants, the pseudoscalar has no large component. The forms of the five possibilities and their large components are presented in Table XI. In that table we make use of the spin-vector operator:

$$\boldsymbol{\sigma} = -(\tfrac{1}{2}i)\boldsymbol{\alpha}\times\boldsymbol{\alpha}. \tag{18}$$

TABLE XI

Relativistically covariant operators

Interaction	*Matrix element*	*Large component*
Scalar	β	1
Polar vector	$\boldsymbol{\alpha}$, 1	1
Tensor	$+\beta\boldsymbol{\sigma}$, $+i\beta\boldsymbol{\alpha}$	$\boldsymbol{\sigma}$
Axial vector	$\boldsymbol{\sigma}$, $\frac{1}{3}(\boldsymbol{\sigma}, \boldsymbol{\alpha})$	$\boldsymbol{\sigma}$
Pseudoscalar	$\frac{1}{3}\beta(\boldsymbol{\sigma}, \boldsymbol{\alpha})$	0

It is evident that from the point of view of allowed transitions there are only two distinct types of matrix elements, 1 and σ, which are the forms discussed in § 2 as leading to Fermi and to Gamow–Teller selection rules, respectively.

We now turn to the interpretation of the β-process in terms of the meson theory. This will lead to two results in addition to those above: (1) a decomposition of the factor g into other interaction constants, and (2) a specialization of the forms of interaction, so that only certain covariants in Table XI are required. In addition one would like to be able to connect this interpretation with the observed mean life of the cosmic-ray meson (2·15 microseconds), thus fixing the ability of the meson to disintegrate into electron and neutrino by an independent measurement. We shall see that this leads to a contradiction, at least with the weak-coupling meson theories.

For simplicity, we consider first the case of the negatively charged *scalar* meson (spin 0), with the wave-function U. The invariant form of interaction between meson and electron-neutrino will then contain the product of the scalar U with the scalar function $\psi^+\beta\phi$:

$$H' = g' \int U(\mathbf{x})\psi(\mathbf{x})^+\beta\phi(\mathbf{x})\, d\mathbf{x} + g' \int U(\mathbf{x})^+\phi(\mathbf{x})^+\beta\psi(\mathbf{x})\, d\mathbf{x}. \quad (19)$$

The first term in H' gives the disappearance of the negative meson and the neutrino, coincident with the appearance of an electron. The second term gives the reverse process. The wave-function representing one free meson per unit volume (Chap. III) is, for wave-number k:

$$U(\mathbf{x}) = \left(\frac{4\pi\hbar c}{k_0}\right)^{\frac{1}{2}} e^{i(\mathbf{k}.\mathbf{r}-k_0 ct)}, \qquad k_0^2 = k^2 + æ^2, \qquad æ = \frac{mc}{\hbar}. \quad (20)$$

Hence, for a meson at rest ($k = 0$), the factor U in eq. (19) is simply equivalent to $(4\pi\hbar^2/m)^{\frac{1}{2}}$. The transition probability due to (19) is then

$$w = \frac{2\pi}{\hbar}|H'|^2\rho_{e\nu}, \quad (21)$$

where $\rho_{e\nu}$ is now the density, per unit energy, of final states representing an electron and a neutrino flying apart in opposite directions and with total momentum zero. The momentum of each is, therefore, $\frac{1}{2}mc$, and $\rho_{e\nu}$ becomes (we consider here all plane waves, rather than expand in states of angular momentum as before):

$$\rho_{e\nu} = \frac{4\pi p^2}{(2\pi\hbar)^3}\frac{dp}{dW} = \frac{1}{\hbar c}\left(\frac{mc}{4\pi\hbar}\right)^2, \qquad W = mc^2. \quad (22)$$

The plane-wave solutions of Dirac's equation, belonging to the energies $\pm E$ are four in number, comprising both signs of the energies and both orientations of the spin. The four components of the four solutions form the elements of the matrix

$$\frac{E+mc^2+c(\boldsymbol{\alpha},\mathbf{p})\beta}{\sqrt{\{2E(E+mc^2)\}}}. \tag{23}$$

If we let u_e be these components for the electron wave and u_ν for the neutrino, each belonging to the positive energy $\frac{1}{2}W$ and a definite spin-component, the expression for H' becomes (taking the integral over unit volume)

$$H' = g'\left(\frac{4\pi\hbar^2}{\mathfrak{m}}\right)^{\frac{1}{2}}\{u_e^+\beta u_\nu\},$$

and the expression for w becomes

$$w = \frac{2\pi}{\hbar}g'^2\frac{4\pi\hbar^2}{\mathfrak{m}}\frac{1}{\hbar c}\frac{\mathfrak{m}^2c^2}{16\pi^2\hbar^2}\sum_s |\{u_e^+\beta u_\nu\}|^2,$$

where $\sum_s$ means summation over the two spin orientations of the electron (or neutrino which must have the same orientation). In the high relativistic energy region for the electron its rest-mass is negligible, so that its wave-function, (23), is essentially the same as that of a neutrino, but for the reverse sign of $\mathbf{p}$. But, according to (23), one can obtain the wave-functions of positive energy and reversed momentum by simply multiplying u by β. Therefore, $u_e^+\beta u_\nu = u_e^+ u_e = 1$, and the sum over spins is just 2. The final value of w is then

$$w = \frac{\mathfrak{m}c^2}{\hbar}\frac{g'^2}{\hbar c}. \tag{24}$$

Substituting 200 electron masses for $\mathfrak{m}$ and 2·15 μsec. for w^{-1}, we find a value of g' in units of the electronic charge e

$$g' = 2\times 10^{-8}e. \tag{25}$$

In order to apply this theory to the β-disintegration of nucleons we use the wave-function $U(\mathbf{x})$ that is attached to the nucleon rather than the plane-wave (20). As seen in Chapter III, this wave-function is determined by the nucleon-density through

$$U(\mathbf{x}) = \mathfrak{g}\int \frac{e^{-\varkappa|r'-r|}}{|r'-r|}(\Psi^+\beta\Phi)_{r'}\,d\mathbf{r}'. \tag{26}$$

Substituting for $U(\mathbf{x})$ in the first member of eq. (19) we get

$$H' = \mathfrak{g}g' \int \frac{e^{-\varkappa|r'-r|}}{|r'-r|}(\Psi^+\beta\Phi)_{r'}(\psi^+\beta\phi)_r \, d\mathbf{r}'d\mathbf{r}$$

$$\cong g'\mathfrak{g} \int \frac{e^{-\varkappa x}}{x} 4\pi x^2 \, dx \int (\Psi^+\beta\Phi)_{r'}(\psi^+\beta\phi)_{r'} \, d\mathbf{r}'$$

$$= \frac{4\pi}{\varkappa^2} g'\mathfrak{g} \int (\Psi^+\beta\Phi)(\psi^+\beta\phi) \, d\tau, \tag{27}$$

where we have assumed that there is a negligible error in evaluating the light particle waves at r' instead of at r. This is a valid assumption for light particle waves that are long compared to $1/\varkappa$, i.e. light particle energies small compared with 100 M.e.v. Now eq. (27) is precisely the same form of interaction as in Fermi's theory, if the scalar form of the theory is assumed and if we write for Fermi's constant g:

$$g = \frac{4\pi}{\varkappa^2} g'\mathfrak{g}. \tag{28}$$

The values of g and g' are determined from the results on β-decay and the spontaneous disintegration of the meson, respectively, and thus serve to determine a value for $\mathfrak{g}$, the mesic source strength of a nucleon. Substituting $1{\cdot}8\times10^{-49}$ for g and $2\times10^{-8}e$ for g' we find $\mathfrak{g}$ to be

$$\mathfrak{g} = 80e. \tag{29}$$

The value of $\mathfrak{g}$ as determined from the β-activity is thus over 10 times larger than that required to account for the nuclear forces. In other words, to fit with the meson theory of forces and the observed mean life of the meson at rest, the ordinary β-decay would have to be 100 times less probable for a given energy than now observed. This represents a fundamental inconsistency in the meson theory of nuclear forces and β-decay, at least as originally proposed by Yukawa.

The same type of calculations as those just made have been carried through for the vector meson instead of the scalar.† This means that in place of the scalar wave-function $U(\mathbf{x})$ one introduces the polar vector ϕ^α and the antisymmetric tensor $\chi^{\alpha\beta}$ which describe the meson of spin unity. In this case, two independent forms of interaction may be chosen, one of which is formed by contracting ϕ^α on the polar vector of Table XI, the other by contracting $\chi^{\alpha\beta}$ on the tensor of that table. Through the latter interaction this theory will

† H. A. Bethe and L. W. Nordheim, *Phys. Rev.* **57** (1940), 998.

lead to the Gamow–Teller selection rules. Except for a factor of the order of unity, the numerical results for the vector meson theory are the same as for the scalar meson theory and the discrepancy in the two ways of estimating g persists.

4. Theory of the forbidden β-spectra

A general, relativistically invariant formula for the probability of β-radiation may now be obtained by considering the wave-functions of proton, neutron, electron, and neutrino as having four components each, as required by Dirac's theory and then selecting a particular set of covariants from Table XI. If we select a particular, bilinear covariant form with components M_k and designate the corresponding contravariant components by M'_k, the perturbation theory leads to the transition probability:

$$w(E) = \frac{2\pi}{\hbar} g^2 \sum_k \left| \int \Psi(\mathbf{x})^+ M_k \Phi(\mathbf{x}); \psi(\mathbf{x})^+ M'_k \phi(\mathbf{x})\, d\mathbf{x} \right|^2 \rho_e(E)\rho_\nu(W-E), \tag{30}$$

where $\rho_e(E)$ and $\rho_\nu(W-E)$ have the same meaning as in § 2. It is to be kept in mind, however, that the integral appearing in eq. (30) originates as an operator in the theory of the quantization of the wave-equation and thus really symbolizes a sum over all possible light particle states. Application of the formula is greatly simplified, therefore, when the wave-lengths of the electron and neutrino are long as compared with the radius of the nucleus. In this case, the infinitely many possibilities for the product $\psi(\mathbf{x})^+ M'_k \phi(\mathbf{x})$, which appears in the integrand, may be classified according to whether there are 0, 1, 2,..., etc., nodes at $r = 0$. In general, a product with n-nodes passing through the origin will have the angular dependence of a Legendre polynomial of order n and will approach the origin roughly as r^n. If we let R be the radius of the nucleus, the magnitude of a product with n-nodes will be of the order $(WR/\hbar c)^n \sim (R/\lambda\!\!\!^{-})^n$. Since we have postulated $R \ll \lambda\!\!\!^{-}$, this means that the higher the value of n the less probable is the transition.

We may now obtain an approximate formula for the transition probability by, (1) computing $w(E)$ only for the lowest value of n that gives a non-vanishing contribution, and (2) neglecting the variation of $\psi(\mathbf{x})^+ M'_k \phi(\mathbf{x})$ with radius. Let L_n^i be the ith possible product of the light particle waves (with M'_k) having the angular dependence

characteristic of an associated Legendre function, $P_n^{m_i}(\theta, \phi)$. Our approximate expression may then be written:

$$w_n(E) = \frac{2\pi}{\hbar} g^2 \sum_{i,k} |\psi(R)^* M'_k \phi(R)|_i^2 \rho_e(E) \rho_\nu(W-E) \times \times \left| \int \Psi(\mathbf{x})^* M_k L_n^i \Phi(\mathbf{x})\, d\mathbf{x} \right|^2, \quad (31)$$

where one sums over the (relatively few) possible combinations of the given n. $\psi(R)$ means the radial part of $\psi(\mathbf{x})$ evaluated at R.

Formula (31) is now of the same form as eq. (1), but generalized to the use of relativistic wave-functions and with a more explicit formulation of the 'matrix element'. If $n = 0$ the matrix element is of the same type as considered in § 2 and will be designated by M_0. This matrix element will not vanish (in general) if the orbital angular momentum of the nucleon does not change in the β-radiation, and if the spin does not change. In case the M_k are the tensor components or the axial vector components, the spin of the nucleon may change also. Under these conditions, we may use the approximate forms (large components) in Table XI; and substituting the normalized Dirac wave-functions for the electron in the field of a nucleus of charge Z in eq. (31) we obtain the formula for w_0 that was presented originally by Fermi:

$$w_0(E) = \frac{g^2}{2\pi^3} |M_0|^2 \frac{1+s_1}{2} p_e E (W-E)^2 (2p_e R)^{2s_1-2} \frac{e^{\pi\eta} |\Gamma(s_1+i\eta)|^2}{\frac{1}{4}[\Gamma(1+2s_1)]^2}, \quad (32)$$

as expressed in the 'rational system' of units such that

$$\hbar = c = m = 1$$

and with $\eta = \alpha Z(E/p_e), \qquad s_1 = \sqrt{(1-\alpha^2 Z^2)}.$

If we neglect $\alpha^2 Z^2$ compared with unity, in eq. (32), we get a working formula for the lighter nuclei:

$$w_0(E) = \frac{g^2}{2\pi^3} |M_0|^2 p_e E (W-E)^2 \frac{2\pi\eta}{1-e^{-2\pi\eta}} \quad (33)$$

in agreement with the result of the elementary theory of § 2. These formulae are derived on the assumption of zero rest-mass for the neutrino. In § 2 the formula was derived on the further assumption that electron and neutrino are emitted in S-waves, and this is essentially the significance of setting $n = 0$. Even in this case, however, the relativistic formula is a little more involved than the

non-relativistic one. Let us consider the relativistic form of the electron wave for which the 'large' components are S-waves, u_a/r. According to general considerations, Chapter III, the small components must then be P-waves with the radial function u_b/r, say. Except for a common, normalizing factor, the functions u_a and u_b belonging to the energy E may be written as the series:

$$\begin{aligned} u_a &= \sqrt{\left(\frac{E+1}{2E}\right)} r^{s_1-1}\left\{pr-\alpha Zp\frac{5E+1}{6}r^2-\tfrac{1}{6}p^3r^3-\ldots\right\}, \\ u_b &= \sqrt{\left(\frac{E-1}{2E}\right)} r^{s_1-1}\left\{\tfrac{1}{2}\alpha Z(E+1)r+\tfrac{1}{3}p^2r^2-\frac{\alpha Z}{16}p^2Er^3\ldots\right\}. \end{aligned} \tag{34}$$

Let us denote the electron wave-function just described, for which the functions u_a and u_b in eq. (34) give the radial dependence of the large and small components, respectively, by the symbol $(S;P)_e$. When $Z = 0$ the functions u_a and u_b become simple S- and P-wave radial functions. In fact, if we replace $E\pm 1$ by $W-E\pm\mu$, where μ is the ratio of the rest-mass of the neutrino to the rest-mass of the electron, and set $\alpha = 0$, we get the $(S;P)_\nu$ for the neutrino. In addition to these waves, however, the related waves, $(P;S)_e$ and $(P;S)_\nu$, in which the large and small components have been exchanged, contribute to $w_0(E)$. Hence, the sum over i in the general formula, eq. (31), extends over the two terms:

$$\begin{aligned} &|(S;P)_e M'_k(S;P)|_R^2+|(P;S)_e M'_k(P;S)|_R^2 \\ &\quad\cong |(S;0)_e M'_k(S;0)|_R^2+|(0;S)_e M'_k(0;S)|_R^2 \\ &\quad= |M'_k|^2 p_e^2(W-E)^2\frac{1}{2}\left\{\left(1+\frac{1}{E}\right)\left(1+\frac{\mu}{W-E}\right)+\left(1-\frac{1}{E}\right)\left(1-\frac{\mu}{W-E}\right)\right\} \\ &\quad\cong p_e^2(W-E)^2\left[1+\frac{\mu}{E(W-E)}\right]. \end{aligned} \tag{35}$$

If we set the mass of the neutrino equal to zero, $\mu = 0$, the result (35) is exactly the same as in the elementary theory for $\psi_e^2\phi_\nu^2$, except for numerical factors. It will be noted also u_a/r (cf. eq. (34)) cannot be evaluated at $r = 0$, as in the elementary theory, because this function diverges as r^{s_1-1} at the origin. Hence, even the S-waves have to be evaluated at the nuclear radius, R.

In the computations of eq. (35) it has been assumed that M'_k represents products of the 'large components' of the electron waves with the 'large components' of the neutrino waves, so that, in terms

of the approximate forms in Table XI, M'_k may be either 1 or $\boldsymbol{\sigma}$. Consequently M_k may be either 1 or $\boldsymbol{\sigma}$ and, since the L_0^i are unity, the 'matrix element' in eq. (31) is the same as that discussed in § 2. For this matrix element not to vanish, therefore, the orbital angular momenta must be the same for the neutron state and the proton state, and, in the case $M_k = 1$, the spins of neutron and proton must also be the same, giving the Fermi selection rules. If $M_k = \boldsymbol{\sigma}$ the spin of the nucleon may turn over during the transition and we get the Gamow–Teller selection rules.

If the orbital quantum number of the transforming nucleon *must* change during the β-radiation, i.e. if there is no quantum state of the daughter nucleus at lower energy and with the same orbital waves, the matrix element M_0 vanishes identically and the transition is said to be 'forbidden'. If the orbital quantum number changes by unity, the transition is said to be *first forbidden*. Consider the particular case that the initial neutron is in an S-state, i.e. the large components are S-waves and the small components are P-waves, which may be designated, $(S; P)_N$, and the proton into which it transforms is in the $P_{\frac{1}{2}}$-state, $(P; S)_P$. Then the matrix element, M_{sp}, vanishes

$$M_{sp} = \int (P; S)_P \, M_k \, L_n (S; P)_N \, d\mathbf{x} \tag{36}$$

if we set $L_n = L_0 = 1$ and $M_k = 1$ (or $\boldsymbol{\sigma}$), since then we get an integral over all directions of the product of an S-wave with a P-wave. There are two possibilities for obtaining a non-vanishing integral M_{sp}. One way is to retain $L_n = L_0 = 1$ and select a matrix element M_k that forms products between large and small components. Such matrix elements appear in Table II in the space-components of the polar vector, the space-time components of the tensor, the time component of the axial vector, and the pseudoscalar. Since, however, the amplitudes of the small components of the nucleonic waves are smaller by the factor v/c than the large components, where v is the average velocity of a nucleon in a nucleus, the transition probabilities of such transitions will be smaller than of allowed transitions by the factor $\beta^2 = v^2/c^2$. Of course, the large components of the electron waves will be multiplied by small components of the neutrino wave:

$$|(P; S)_e \, M'_k (S; P)|^2,$$

etc., but at the usual β-energies, large and small components are of the same order of magnitude, and we get essentially the same product

of S-waves ($L_n = L_0$). Hence, if the invariant form of interaction between nucleons and the electron-neutrino field contains components M_k, that are odd powers of the $\boldsymbol{\alpha}$-operators, there will be first forbidden transitions due to the finite velocity of the nucleons. The energy distribution of the electrons will be given by eq. (32), as before, only multiplied by a factor of the order β^2 (provided also that the rest-mass of the neutrino vanishes). Presumably the factor β^2 is relatively insensitive to the energy of the β-radiation. Referring to Table XI, it will be seen that the spin-selection rules for these transitions will be the Fermi rules, $\Delta i = 0$, for pseudoscalar and axial vector, Gamow–Teller rules, $\Delta i = \pm 1$ or $0\ 0 \to 0$ excluded, and completely forbidden for the scalar interaction.

The second possibility of obtaining a non-vanishing matrix element (36) is to go back to those M_k that do not mix large and small components but to choose $L_n = L_1 = P_1^{m_i}$. For these the effective wave of one of the emitted particles is a P-state, the other an S-state. This introduces the first surface harmonic into the calculation of the nuclear matrix element. The matrix element so obtained is, therefore, of the same type as those calculated in the emission of light quanta, and the radiation may be called *dipole* radiation. Let us represent the surface harmonic by the unit vector $\mathbf{x}_1$:

$$|M_1|^2 = \left| \int \Psi(\mathbf{x})^* M_k \mathbf{x}_1 \Phi(\mathbf{x})\, d\mathbf{x} \right|^2.$$

The approximate forms of M_k in Table XI may be applied, so that when $M_k = 1$ the matrix element is of the simple dipole type and the Gamow–Teller spin-selection rules are obeyed. When $M_k = \boldsymbol{\sigma}$, however, the symmetric tensor may be formed from $\boldsymbol{\sigma}$ and $\mathbf{x}_1$ (as well as the vector and scalar) and a total spin change of $2\hbar$ is possible.

The emission of one of the particles in a P-wave also changes the dependence of w on the electron energy, at least in principle. The square of the neutrino wave is proportional to $(W-E)^4R^2$ if it is in the P-state, and the square of the electron wave is a more complicated function. From eq. (34) we see that, owing to terms in αZ, the P-wave component, u_b, approaches the origin in the same power of r as the S-wave. At the nuclear radius, R, the ratio of the first term to the second, in the series for u_b, is $3\alpha Z/2(E-1)R$, and, in rational units, $2R = \alpha A^{\frac{1}{3}}$. Hence this ratio becomes of the order of unity for quite light nuclei, and much greater than unity for heavy nuclei. We therefore evaluate both terms in computing w_1. In the

approximation $\alpha^2 Z^2 \ll 1$, we may write the shape of the spectrum in dipole radiations in the general form:

$$w_1 = C_1(E, Z) \cdot R^2 |M_1|^2 \frac{g^2}{2\pi^3} p_e E (W-E)^2 \frac{2\pi\eta}{1-e^{-2\pi\eta}},$$
$$C_1(E, Z) = \frac{a'}{18}(p_e^2+p_\nu^2) + \frac{2b'}{3} p_\nu \left[\frac{\alpha Z}{2R} + \frac{p_e^2}{3E}\right] + c' \left[\frac{\alpha^2 Z^2}{4R^2} + \frac{\alpha Z}{3R}\frac{p_e^2}{E}\right], \tag{37}$$

with $p_e^2 = E^2-1$ and $p_\nu^2 = (W-E)^2$. The a', b', and c' are small integers, the values of which depend upon the form of invariant used.† Suffice it to remark that $c' = 1$, except when $\Delta i = 2$ and then $c' = b' = 0$. Since for most β-emitters the term $\alpha^2 Z^2/4R^2$ is much the largest term in C_1, the value of C_1 is a very insensitive function of the electron energy and the shapes of the dipole spectra will be essentially the same as for the allowed spectra. This is a consequence of the peculiar nature of the relativistic wave-function of the electron in a Coulomb field. The exceptions to this rule occur for $\Delta i = 2$, when the spectrum becomes modified by the factor $(p_e^2+p_\nu^2)$, and for nuclei of small Z and rather high β-energies so that all terms in C_1 are comparable in magnitude.

We may summarize the results on allowed and forbidden spectra as follows. In an allowed transition, the main light particle waves have no node at the origin, $n = 0$, and the nuclear matrix element contains the zero surface harmonic (1). Such transitions might then be called *monopole* transitions. The spectral distribution is of the normal type, eq. (32), and for certain nuclei, as discussed in § 2, the nuclear matrix element is of the order unity. These are the transitions between two nuclear states belonging to the same supermultiplet. Henceforth we shall refer to these transitions alone as *allowed*. More generally, the nuclear states for heavy nuclei will be mixtures of states belonging to many different supermultiplets. The probability of a transition (which *must* take place between states in the same supermultiplet since the β-interaction is symmetric in neutron and proton) is then reduced by a factor ϵ which represents the probability that a given supermultiplet is represented in the daughter nucleus. Thus, even though these are monopole transitions that are independent of the velocities of the nucleons we shall refer to them as *symmetry forbidden*. The term 'first forbidden' introduced above

† The values of these integers and similar ones for second forbidden transition are given by C. L. Critchfield, *Phys. Rev.* **61** (1942), 249.

(for historical connexion) then becomes inappropriate and we shall change this name to *parity forbidden*. Parity-forbidden transitions are also symmetry forbidden and, as we have just seen, may be divided into two classes: *monopole* and *dipole*. The parity-forbidden monopole transitions are then less probable than *allowed* transitions by the factor $\epsilon\beta^2$ and the dipole transitions by the factor $\epsilon(WR)^2$ or $\frac{1}{4}\epsilon\alpha^2Z^2$, whichever is larger. Generally, the shapes of all these spectra are expected to be of the Fermi type and, within a given class, the total transition probability will increase with W^5.

The case in which parent and daughter nucleus have the same parity but differ considerably in spin requires at least what is known historically as a 'second forbidden' transition. Since such transitions are forbidden by symmetry also, the name is, again, not too good and we shall simply call them *spin forbidden*. One way of accomplishing such a transition is by choosing $n = 1$ *and* mixing large with small components of the nuclear waves. The radiations are then of a dipole character, but without change of parity, and are therefore closely analogous to the magnetic dipole radiations in forbidden optical spectra. The general properties of these radiations are very similar to the ordinary dipole parity-forbidden emissions, but with a further factor β^2 in the square of the matrix element. The spectral shape is expected to be normal except when $\Delta i = 2$, or in very light nuclei.†

The other type of 'second forbidden' transition is by emission of $2\hbar$ orbital angular momentum into the field, i.e. with $n = 2$ and using the large components of the M_k. This is analogous to the electric quadrupole radiation of spectroscopy.‡

The half-lives of the various multipole radiations are obtained by integrating the correct expressions for the corresponding w_n over all electron energies from 1 to W.§ For changes of spin less than 2 these integrals lead to a W^5 dependence of the decay constant, i.e. to the Sargent law, even for the parity-forbidden transitions. This comes

† On the other hand, transitions of this type with $\Delta i = \pm 1$ are of interest only if Fermi selection rules apply to allowed spectra, since with Gamow–Teller rules $\Delta i = \pm 1$ can be obtained with $n = 0$. In the latter case, the dipole radiations without change of parity play a role only in $0 \to 0$ transitions or if $\Delta i = 2$.

‡ The shapes of the forbidden β-spectra have been presented first by E. J. Konopinski and G. E. Uhlenbeck, *Phys. Rev.* **60** (1941), 308. An explicit treatment in spherical coordinates is given by C. L. Critchfield, ibid. **63** (1943), 417.

§ Expressions for these integrals are to be found in ibid, **61** (1942) in independent articles by C. L. Critchfield, p. 249 (light nuclei only), R. E. Marshak, p. 431, and E. Greuling, p. 568.

about because of the curious effect of the Coulomb field on the P-waves for the electron, mentioned above. The half-lives for K-capture are subject to the same classification into forbidden spectra and are of special interest because the ratio of probability of K-capture to positron emission is independent of the value of the nuclear matrix element. The complete form for allowed K-capture, corresponding to eq. (32), is

$$\lambda_0^K = \frac{g^2}{\pi^2}(\alpha Z)^{2s_1+1}|M_0|^2\frac{1+s_1}{\Gamma(1+2s_1)}(W+s_1)^2. \tag{38}$$

5. Comparison of experimental results with theory

The lifetime of a β-active nucleus depends upon so many factors that it is impossible at the present stage of the investigation to give anything like a complete analysis of about 200 known β-activities. The formulas derived in § 4 give the dependence of the decay constant on maximum energy and nuclear charge only for a simple β-emission. Many, if not most, of the known spectra are complex, as is evidenced by the emission of γ-rays by the daughter nuclei. The lifetime of such radiations is determined by the co-operation of all the β-spectra that can be emitted. In addition to this, except for the very simple positron emitters presented in § 2, the nuclear matrix element, i.e. the integral over the nuclear wave-functions, is uncertain in its value. Furthermore, the exact form of the β-interaction is unknown, so that, even in those cases for which the matrix element can be estimated with reasonable accuracy, the magnitudes of the interaction constants applying to the five independent invariants are not determined. The fluctuations in lifetime from nucleus to nucleus arising from these unknowns may be expected to amount to something less than a factor 100, however, so that the influence of parity and spin-changes should be evident in the main.

By far the best method of analysing the lifetime of a β-disintegration is to compare the observed decay constant with the decay constant that would be predicted for an allowed transition with the same maximum energy. The customary procedure is to multiply the observed *half-life* by the function $I_0(W, Z)$, the form of which is essentially obtained from eq. (33) by integration

$$I_0(W,Z) = \int_1^W \frac{2\pi^3 w_0}{g^2|M_0|^2}\,dE \cong 2\pi\alpha Z I_0(W)/(1-e^{-2\pi\alpha Z}),$$

the approximate form holding only for the lighter nuclei. We shall denote this product by ft. The principal shortcoming of this method lies in the great sensitivity of ft to errors in the determination of W (since I varies as W^5). This variation introduces further uncertainties into the analysis. The product ft is then equal to

$$ft = \frac{\lambda_0}{\lambda_{obs}} \cdot \ln 2.$$

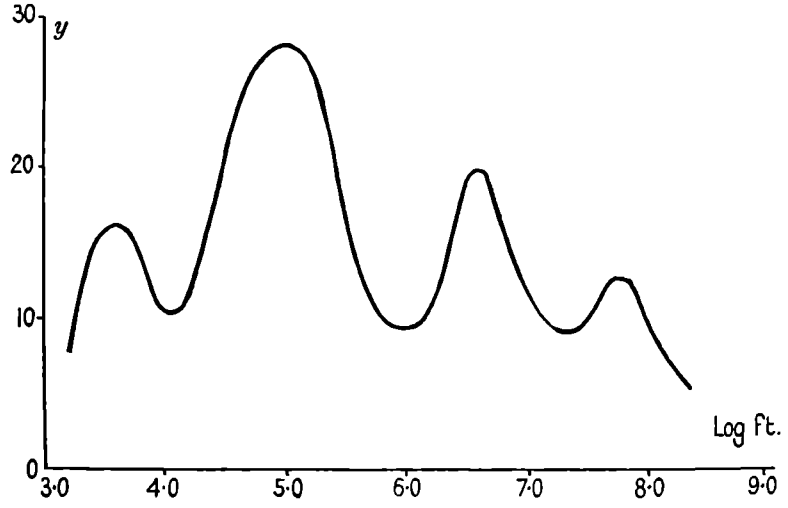


FIG. 19. Frequency distribution of all known values of log ft. The quantity y represents the number of activities per 0·5 in log ft.

Obviously, the larger the value of ft the more forbidden the radiation. A plot of the frequency distribution of all known radiations as a function of logft from logft = 3 to logft = 9 is shown in Fig. 19.† There are distinct maxima in the distribution at logft 3·6, 5·0, and 6·6; and a less convincing maximum at logft 7·8. The first maximum contains all the nuclei that are suspected of having matrix elements near unity, the really allowed transitions discussed above.

The highest maximum in the distribution appears for lives that are longer by a factor 25 than those of the really allowed transitions. Inasmuch as this peak is, by far, the largest and represents an average lifetime that is a little too short for parity-forbidden transitions it is natural to suppose that most of the transitions falling in this region of log ft are neither parity- nor spin-forbidden but simply symmetry-forbidden. This would demand an average value of $\epsilon = 0{\cdot}04$, which is a reasonable guess as to the amount of similarity that might be

† The values of these products are given in Konopinski's very good report on β-decay, E. J. Konopinski, *Rev. Mod. Phys.* **15** (1943), 209.

expected in complicated nuclei. The hypothesis that these are simply inter-multiplet transitions, without parity or forbidden spin-change, is substantiated by the fact that every known β-active nucleus having one more neutron than proton (except H^3), and also three more neutrons than protons, is under this hump. For the most part these nuclei would not be expected to be forced to make either parity or forbidden spin-changes, but they are expected to belong to different supermultiplets. The 'allowed' natural radioactivities are also in this hump.

Assuming that the nuclear matrix elements for all transitions, except the really allowed one under the first peak, are characterized by the average value of ϵ under the main peak of Fig. 19, the third peak should contain, for the most part, the transitions that are forbidden both by symmetry and by parity. The average lifetime in this peak is 40 times longer than for the simply symmetry-forbidden transitions. If the parity-forbidden transition is due to the small components of the nuclear waves, $\beta^2 = 1/40$, and the indicated value of v/c is 0·15, in good agreement with expectation. If the transition is of the dipole type, the ratio depends, in general, upon the form of the interaction. For all but a very few nuclei, however, the term in λ_n/λ_0 that depends upon Z^2 predominates and, as we saw in the preceding section, the ratio should be $\frac{1}{4}(\alpha Z)^2$, except when $\Delta i = 2$. The indicated average value of Z is therefore about 40, again in agreement with what is expected, especially when one considers that the hump embraces a factor 4 in ft, either way. It seems justifiable to conclude that this third maximum contains mostly those transitions that are forbidden by both symmetry and parity. The 'first forbidden' natural β-emissions fall at the high ft edge of this hump.

The fourth peak on the frequency diagram is so small that a quantitative analysis of its position is of doubtful value. The average lifetime in this hump is about 20 times longer than in the third maximum. This suggests that the fourth maximum may contain the dipole transitions without change of parity, and the factor 20 for $\frac{1}{4}(\alpha Z)^2$, instead of 40, may be due to the higher average Z for nuclei under this hump. The transitions in this region would then be forbidden by both symmetry and spin. RaE is the only natural activity definitely under this maximum.

The general conclusion to be drawn from this statistical analysis of the half-lives is that most radiations are not forbidden by either

parity or spin-changes but are slower than really allowed transitions because of other lack of similarity in the nuclear wave-functions. This result is easily understood when it is considered that β-energies are generally larger than the average spacing between energy-levels of the lower excitations of most nuclei, so that there is a good chance of there being an available level in the daughter nucleus of a desirable parity and spin. The large number of γ-radiations following β-transitions further supports this view, as also does the experimental fact that the highly forbidden radiations usually have very low energy.

The spectra of the β-radiations falling under the first and second maxima should thus be the normal Fermi type. There is also an appreciable number of transitions that are apparently forbidden by both symmetry and parity. Because of the dominance of the Z^2 term in the spectral modification factor, C_1 of eq. (37), these radiations should also exhibit a normal spectrum. Two types of exceptions may arise to this general rule. If the energy of emission is sufficiently high (or the charge of the daughter nucleus sufficiently low) the p^2R^2 terms will dominate. This also happens when $\Delta i = 2$ for which C_1 has no terms in Z or Z^2. In either case the nature of the deviation from a normal spectrum should be the same. The electron, if it carries the full energy of the transition, will have a shorter wave-length than a neutrino of maximum energy, because the rest-energy of the electron is subtracted from the latter. The factor in the squares of the momenta thus enhances the high-energy region of the spectrum, relative to the low-energy region, and the Kurie plot should be concave to the axes at its high-energy end.

Two of the more carefully determined β-spectra are those of P^{32} and RaE which fall under the fourth maximum. There are significant deviations of these spectra from the normal one, but in each case the high-energy end of the Kurie plot appears quite straight. If the transitions were of the quadrupole type, or dipole with spin change of 2, there should be deviations, and Konopinski and Uhlenbeck (loc. cit.) have analysed these spectra on that basis. They find that a judicious mixture of the two types of multipole can fit the spectra over a considerable range. The mixture of types is necessary because the relative weights of electron and neutrino momenta are different for the two, and a fit can be found by taking a linear combination of them. It is possible, in view of the straightness of the high-energy ends of the Kurie plots, that both these spectra are normal and the

deviations are instrumental. A transition falling under the fourth maximum in Fig. 19 can have a normal spectrum if it is a transition between nuclei of zero spin, for example, and such might be the case for RaE. It is also possible that P^{32} is a parity-forbidden transition that really belongs to the third maximum.

There are a few well-established β-activities of much longer lives than those included in the statistical analysis. Notable among these are the naturally occurring K^{40} and Rb^{87} and the artificially produced Be^{10} and C^{14}. A table of these four activities containing the spins of the parent and daughter nuclei is given in Table XII. The spin

TABLE XII

Z	*Parent*	*Spin*	*Daughter*	*Spin*	*W*	*log* $\tau_{\frac{1}{2}}$	*log ft*
19	K^{40}	4	Ca^{40}	0	3·6	16·65	17·88
37	Rb^{87}	3/2	Sr^{87}	9/2	1·26	18·78	16·68
4	Be^{10}	0?	B^{10}	1?	2·1	13·0	12·78
6	C^{14}	0	N^{14}	1	1·28	12	9·6

changes in K and Rb are known in so far as there is no detectable γ-radiation. Their values of 4 and 3, respectively, indicate rather highly forbidden transitions. A careful study of these transitions has been made by Marshak (loc. cit.) and by Greuling (loc. cit.), who conclude that both are 'third-forbidden', i.e. they are forbidden by symmetry, parity, *and* spin. The light nuclei, Be^{10} and C^{14}, are probably in the spin-forbidden class, the extraordinarily large value of log *ft* being due to the small charge and the low energy. But efforts to account for their forbidden character have been completely fruitless. The case of Be^{10} is very difficult to understand when one considers C^{10} which decays, by positron emission, to the same nucleus as Be^{10} does and which differs from Be^{10} only by interchange of neutrons and protons. The C^{10} activity is in the really allowed class. Evidently C^{10} can go to an excited state of B^{10} that is inaccessible to B^{10}. It is improbable that the Coulomb field of such a small nucleus could cause a great difference in the states of Be^{10} and C^{10}.†

In addition to the known, long-lived activities there are several pairs of neighbouring isobars among the apparently stable elements.

† Oppenheimer has suggested that the disparity in the predicted lifetimes of Be^{10} and C^{10} could be understood if the neutrino that is emitted when there is a change of spin has a finite rest-mass, so that the energy available to the electron in the low-energy spectra is much smaller than in the normal Fermi theory.

The pairs of this type are Sb^{123}—Te^{123}, In^{113}—Cd^{113}, Sn—In^{115}, and Os^{187}—Re^{187}. If the energy-difference between ground states of neighbouring isobars is too small to permit electron emission it should be possible for the nucleus of higher charge to absorb an orbital electron. In principle, therefore, neighbouring isobars are not both stable. Actually, the energy-difference may be so small that the half-life of the unstable nucleus is too long to be detected, especially if the transition is forbidden by spin or parity. Even for allowed transitions, the activity is probably undetectable if the maximum energy of the β-particle is 0·1 M.e.v. (plus rest energy); and since the usual energy-difference is 2 or 3 M.e.v., it is not surprising that 5 out of over 200 nuclei have neighbouring isobars.

VI

SPONTANEOUS DISINTEGRATION OF ATOMIC NUCLEI

1. Energy considerations

We shall now consider the processes in which an atomic nucleus splits spontaneously into two or more distinct fragments.

The possibility of such a spontaneous division of a nucleus is determined by the difference between the binding energy of that nucleus and the sum of the binding energies of the fragments into which it divides. A rough analysis of the energies of binding can be based on eq. (4) of Chapter IV which expresses the binding energy in terms of the mass-number A and atomic number Z (using also the isotopic spin $T_\zeta = \frac{1}{2}(A-2Z)$). In order to make use of that equation one should substitute the empirical expressions for L and L' that have been found in Chapter IV. This leads to a cumbersome formula for BE which we shall simplify, in the usual way, by retaining only those terms that are essentially proportional to A in higher than the zero power and by approximating L as it occurs in $\frac{1}{2}LT_\zeta(T_\zeta+4)$ by its value for the heaviest elements. Moreover, for the present discussion, it is convenient to convert BE to units of M.e.v., and we obtain finally:

$$BE = -14{\cdot}6A+14{\cdot}6A^{\frac{2}{3}}+0{\cdot}60\frac{Z^2}{A^{\frac{1}{3}}}+77\frac{T_\zeta^2}{A}, \qquad (1)$$

which is a simple generalization of eq. (10) of Chapter I.

Let us consider, as an example, the case in which the nucleus A splits into two parts with mass-numbers αA and $(1-\alpha)A$ and assume that the electric charge is distributed on the fragments in proportion to their masses. Then, the first and last terms of eq. (1) are simply additive and the difference in binding energy may be expressed:

$$\Delta BE = 14{\cdot}6A^{\frac{2}{3}}[1-\alpha^{\frac{2}{3}}-(1-\alpha)^{\frac{2}{3}}]+0{\cdot}60\frac{Z^2}{A^{\frac{1}{3}}}[1-\alpha^{\frac{5}{3}}-(1-\alpha)^{\frac{5}{3}}]. \qquad (2)$$

If $\Delta BE > 0$ a spontaneous division is possible, if $\Delta BE < 0$ it is not. Thus for each set of values of A and Z expression (2) determines the range of fragment sizes for which the process is possible. The results of such calculations are given in Fig. 20, from which we see that for the elements of the first half of the periodic system a spontaneous

break-up is impossible for any ratio of masses. For $Z > 42$ a division into two fragments becomes possible, first for $\alpha = 0{\cdot}5$ (where the expression for ΔBE has a maximum) and as Z increases for a wider range of fragment ratios. Thus for tin ($Z = 50$) the smaller fragment may contain as little as 15 per cent. of the original nuclear mass (atomic weight > 18); for Yb ($Z = 70$) this limit falls to only 6 per cent. (atomic weight $\geqslant 10$), whereas for the last few elements of the

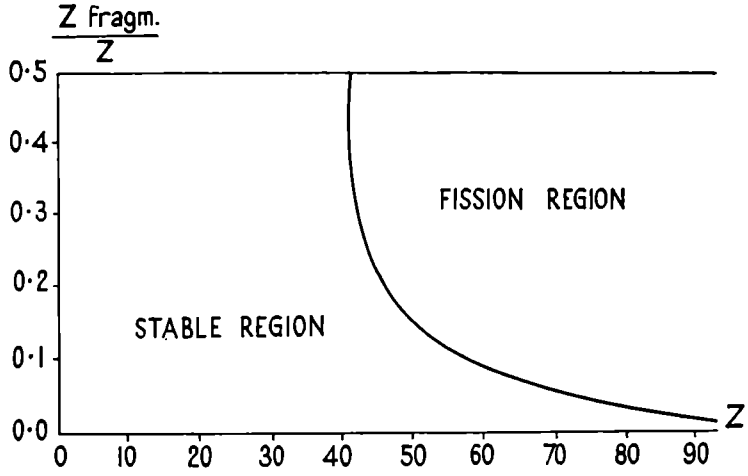


FIG. 20.

system the emission of fragments with the mass 4 (α-disintegration) becomes energetically possible.†

In a similar way we may investigate spontaneous nuclear splitting into three or more fragments, or the cases in which the electric charge is not distributed among the fragments in proportion to their masses. One finds, for example, that the heavy nuclei, like that of uranium, can split exoergically into as many as fourteen fragments of about equal size, and that the lower limit of Z necessary for such multiple splitting increases rapidly as the number of fragments increases.

There are two important cases of the spontaneous disintegration of atomic nuclei occurring in nature, viz. natural α-radiation and the spontaneous fission of U^{238} first observed by Petrzhak and Flerov.‡ The reason for these phenomena being the important ones is, of course, that their rates of reaction happen to be observable.

† The limit for the spontaneous emission of lighter nuclei (He^3, H^3, and H^2) lies at considerably higher atomic numbers because of unfavourable energy-balance resulting from abnormally low binding energies of these nuclei.

‡ J. S. Petrzhak and G. N. Flerov, *J. Phys. U.S.S.R.* **3** (1940), 275.

More details of the physics of the processes of fission and of α-emission will be discussed in the ensuing sections. For the present we shall illustrate some of the limitations of our rough energy-considerations by studying further some of the properties of the fission of the uranium nucleus.

According to the formula (2) the greatest amount of energy is released if the two fragments are of equal size, $\alpha = \frac{1}{2}$. Substituting this value into eq. (2) we get

$$\Delta BE_{\frac{1}{2}} = -3{\cdot}8A^{\frac{2}{3}} + 0{\cdot}222\frac{Z^2}{A^{\frac{1}{3}}}, \tag{3}$$

which for U^{238} yields 156 M.e.v. This energy-release is mostly contained in the kinetic energy of the fragments as they fly apart, but some is retained as vibrational or other excitation of the fragment nuclei (since eq. (1) is for the lowest state of a nucleus). Generally speaking reaction rates increase with the energy release, so that one should expect, on the basis of our rough analysis, that the most favourable mode of division of the uranium nucleus would be into two fragments of equal mass and that the statistical distribution of fragment sizes would have a maximum at $A = \frac{1}{2}\,238 = 119$. The observed distribution of the masses of fragments from U^{236} is shown in Fig. 21.† It will be noted that the most favoured mode of division is not into two fragments of mass 118 but rather into fragments of masses 96 and 140. The proportionate proton numbers are 38 and 54 and neutron numbers 58 and 86. According to eq. (2) such a ratio of masses would release only 143 M.e.v. From the nature of the neutron and proton numbers involved in the asymmetric fission it would appear that the 'shell-structure' in nuclei, particularly those shells characterized by the numbers 50 and 82, so distorts the dependence of binding energy on the number of particles that more energy is released by dividing asymmetrically. Such effects are not included in eqs. (1) and (2).

That possible neutron (and/or proton) shells influence the mass-ratio in fission, as suggested above, is mere conjecture at the present time. There is an interesting corroboration of the existence of neutron shells at 50 and 82 in the fission process, however, which illustrates

† From *Rev. Mod. Phys.* **18** (1946), 513. The curve given here corresponds to the fission of the excited compound nucleus formed by the absorption of incident neutron. However, one would expect that the distribution of fragments originating in spontaneous fission would be about the same.

also another type of spontaneous disintegration of a nucleus. Considering the above example of the most favoured division of U^{236} we found the proton numbers belonging to Sr and Xe. The corresponding neutron numbers, however, are 8 too many for the heaviest Sr isotope and 4 too many for the heaviest Xe isotope. In

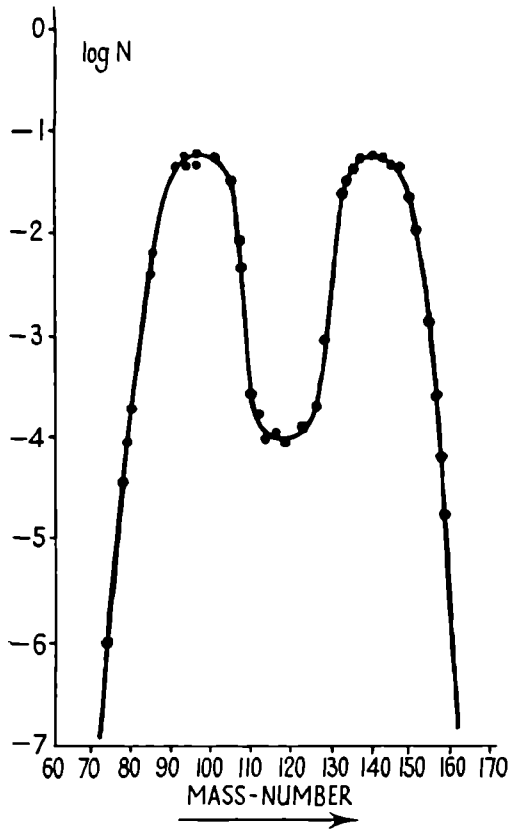


FIG. 21. Relative number of various fragments produced in the fission of U^{236} $(= U^{235}+n)$ nucleus.

general, the fission fragments will have too many neutrons, proportionately, to be stable, so that they will undergo several β-disintegrations in series so as to attain neutron and proton numbers that are stable for the given atomic weight. In a few of these β-decay series, however, the mass-number does not remain fixed but decreases by unity somewhere along the line. That is, a single neutron is emitted by one of the daughter nuclei and this represents a third type of spontaneous disintegration in which a single nucleon separates from the main body of nucleons. If one uses eq. (1) to estimate the neutron excess necessary to make it energetically possible to emit a

single neutron one finds values much larger than those common in fission fragments. It would appear, therefore, that neutron emission by fission fragments (the delayed neutrons that are so important to controlled chain-reactions, Chap. X) is made possible by the circumstance that certain neutrons are much more weakly bound to their nuclei than eq. (1) predicts. Two of the isotopes that emit delayed neutrons have been identified and these are:

$$
\begin{aligned}
{}_{35}\mathrm{Br}^{87} &\xrightarrow[\beta]{} {}_{36}\mathrm{Kr}^{87} \rightarrow {}_{36}\mathrm{Kr}^{86}+n \\
{}_{53}\mathrm{I}^{137} &\xrightarrow[\beta]{} {}_{54}\mathrm{Xe}^{137} \rightarrow {}_{54}\mathrm{Xe}^{136}+n,
\end{aligned}
$$

indicating that the 51st and 83rd neutrons are particularly weakly bound (or not at all). It should be pointed out, however, that disintegration of a nucleus A into one neutron and a nucleus $A-1$ is in a different class from α-emission or spontaneous fission since the neutron emission should take place almost instantaneously, whereas disintegrations involving charged fragments take many years. The finite half-lives of the delayed neutrons are due to the β-radiations in the series.

2. Nuclear potential barriers

In considering the relative probabilities of various modes of spontaneous nuclear disintegration it is important to remember that these probabilities depend not only on the energy liberated in the process but also on the masses and charges of the fragments formed. Thus the most probable way of splitting will not coincide, in general, with the most exoergic way and, in fact, we shall see that the splitting of the uranium nucleus into the two heavy fragments formed in spontaneous fission is a million times less probable than the splitting into an alpha-particle and the nucleus of UX I (natural α-decay) in spite of the fact that the former process liberates 50 times greater energy than the latter.

We shall discuss separately two extreme cases of nuclear break-up characterizing the two important categories cited in § 1: the case when one fragment is much smaller than the other, and the case in which there are two fragments of approximately equal size. In the former case one can assume to a good degree of approximation that the separation of the small fragment from the large one is not accompanied with any appreciable deformation of the original nucleus, so

that the problem can be treated in terms of the motion of a relatively light mass point in the field of the undisturbed residual nucleus. The potential energy of the system in this case will be represented by a curve of the type shown in Fig. 22, where r_0 is the radius of the resulting product-nucleus ($= 1{\cdot}22\times10^{-13}A^{\frac{1}{3}}$ cm.) plus the radius of emitted particle. Since in the ordinary α-emitters the height of the potential barrier thus formed is several times the energy of the α-particle, the principal part of the barrier, in its effect on the α-particle, is determined by the Coulomb interaction and the exact

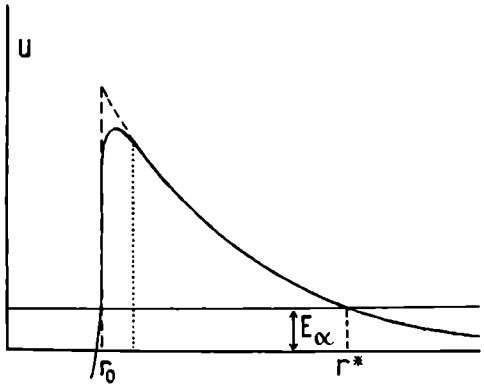


FIG. 22.

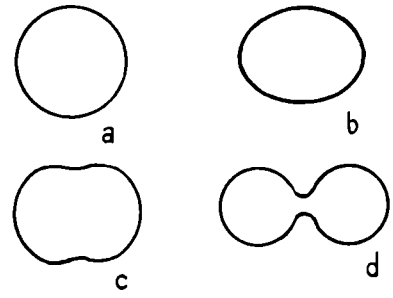


FIG. 23.

shape of the rise of potential at the surface of the nucleus is rather immaterial. For such a barrier we may assume:

$$\begin{aligned} U &= U_0 = \text{const.} \quad \text{for } r < r_0, \\ U &= \frac{Z'(Z-Z')e^2}{r} \quad \text{for } r > r_0, \end{aligned} \tag{4}$$

where $Z'(\ll Z)$ is the charge of the smaller fragment.

In the other extreme case of nuclear splitting into two almost equal fragments, i.e. the fission process proper, the situation is entirely different since, in this case, the splitting is preceded by a severe deformation of the original nucleus. This process of deformation has been studied in great detail by Bohr and Wheeler in their fundamental article on the theory of nuclear fission.† In Fig. 23 we give schematic pictures of various characteristic stages leading to the fission of a liquid droplet. For the lighter elements, in which the electric forces are small compared with the forces due to surface tension, the final separation into fragments would not occur until the deformation reaches the stage (*d*) in which the two parts are

† N. Bohr and J. A. Wheeler, *Phys. Rev.* **56** (1939), 426.

connected by a very narrow neck of liquid. Hence, in this case, the critical energy necessary for separation (i.e. the height of the potential barrier) can be calculated as the difference between the energy of the original nucleus and that of the two fragments just tangent to each other. One finds, writing the surface energy as $4\pi r_{00}^2\,\Omega A^{\frac{2}{3}}$.

$$\Delta E_{\text{crit}} = 2\cdot 4\pi r_{00}^2\,\Omega(\tfrac{1}{2}A)^{\frac{2}{3}} - 4\pi r_{00}^2\,\Omega A^{\frac{2}{3}} + 2\frac{3(\frac{1}{2}Ze)^2}{5r_{00}(\frac{1}{2}A)^{\frac{1}{3}}} - \frac{3(Ze)^2}{5r_{00}A^{\frac{1}{3}}} + \frac{(\frac{1}{2}Ze)^2}{2(\frac{1}{2}A)^{\frac{1}{3}}}, \tag{5}$$

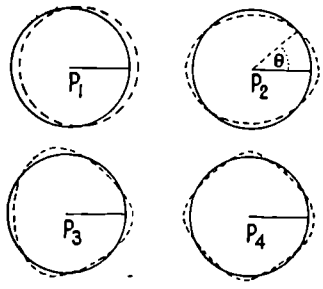


FIG. 24.

which can be written in the form:

$$\frac{\Delta E_{\text{crit}}}{4\pi r_{00}^2\,\Omega A^{\frac{2}{3}}} = 0\cdot 260 - 0\cdot 215x, \tag{6}$$

where $$x = \frac{Z^2/A}{(40\pi/3)\Omega(r_{00}^3/e^2)}. \tag{7}$$

Thus we see that for the lighter nuclei the critical energy of deformation is a linear function of the parameter Z^2/A.

In the opposite case of very heavy elements, in which electric forces play a predominant role, the complete separation of the nuclear droplet is expected to take place already at an early stage of the deformation since the repulsion between the charges carried by the two halves of a slightly elongated droplet will be strong enough to increase the elongation against the opposing tendency of the surface tension. To study this process in greater detail we shall represent an arbitrary deformation of the nucleus in terms of surface harmonics of various orders. If we write $r(\theta)$ for the distance from the centre of the nucleus to a point on its surface at the colatitude θ, an arbitrary deformation of the body, having axial symmetry, can be expressed in the form:

$$r(\theta) = r_0\{1+\alpha_0+\alpha_2 P_2(\cos\theta)+\alpha_3 P_3(\cos\theta)+\alpha_4 P_4(\cos\theta)\ldots\}, \tag{8}$$

where P_i are the Legendre polynomials of the order i.† For any set of vibration amplitudes, α_2, α_3, α_4, etc., we have to choose α_0 in such a way as to keep the volume of the nucleus the same. As can be seen from Fig. 24 representing the various modes of vibration corresponding to the various terms in (8), nuclear splitting into i fragments involves the action of the ith harmonic. Since we shall be interested

† The term corresponding to $P_1(\cos\theta)$ is missing from the expression because it represents a simple translation of the nucleus as a whole and is of no interest to the process.

here in the most important case of nuclear splitting into two fragments, we shall consider primarily the effect of the P_2-vibration. Straightforward calculations on this model show that the change in the combined surface- and electric-energies effected by the deformation (8) is given by the expression:

$$\Delta E_{S+E} = 4\pi r_{00}^2 A^{\frac{2}{3}} \Omega [\tfrac{2}{5}\alpha_2^2 + \tfrac{116}{105}\alpha_2^3 + \tfrac{101}{35}\alpha_2^4 + \tfrac{2}{35}\alpha_2^2 \alpha_4 + \alpha_4^2 \ldots] - \\ - \frac{3}{5}\frac{(Ze)^2}{r_{00} A^{\frac{1}{3}}} [\tfrac{1}{5}\alpha_2^2 + \tfrac{64}{105}\alpha_2^3 + \tfrac{58}{35}\alpha_2^4 + \tfrac{8}{35}\alpha_2^2 \alpha_4 + \tfrac{5}{27}\alpha_4^2 \ldots]. \quad (9)$$

This expression includes also the terms in α_4 which are necessary because of the strong coupling that sets in between the second and fourth modes of vibration for appreciable amplitudes. By minimizing (9) with respect to α_4 we find $\alpha_4 = -\frac{243}{595}\alpha_2^2$ which permits us to express the minimum energy of deformation in terms of α_2 alone. The negative sign in the relation between α_4 and α_2 emphasizes the fact that in the later stages of its elongation the nucleus must develop a cavity around its equatorial belt (cf. Fig. 23) in preparation for the separation.

Inspecting the coefficient of the main term in eq. (9), i.e. that proportional to α_2^2, we find that it becomes negative for sufficiently high values of Z; when this is the case the nucleus is intrinsically unstable since an infinitesimal deformation will lead directly to fission. Thus the condition for stability, ensuring the existence itself of a given nucleus, can be written in the form:

$$4\pi r_{00}^2 A^{\frac{2}{3}} \Omega \frac{2}{5} > \frac{3}{5}\frac{(Ze)^2}{r_{00} A^{\frac{1}{3}}}\frac{1}{5}, \quad (10)$$

or

$$\frac{Z^2}{A} < \frac{40\pi r_{00}^3 \Omega}{3e^2} = \left(\frac{Z^2}{A}\right)_{\text{lim}}.$$

Substituting numerical values, we find $(Z^2/A)_{\text{lim}} = 47{\cdot}8$, as compared with the value of only $92^2/238 = 35{\cdot}6$ characterizing U^{238}. Assuming that for transuranic elements A is approximately proportional to Z, we find that the limit of nuclear stability will be reached in the neighbourhood of $Z = 120$. It must be remembered, however, that the irregularities caused by nuclear shell-structure might shift this estimate quite considerably towards smaller Z's and better agreement with observation (cf. also App. IV).

In a system close to the limit of nuclear stability, so that a comparatively small deformation brings the nucleus 'to the upper rim'

of the potential barrier, formula (9) may be used for the calculation of the *critical deformation*. By direct calculation one finds that the maximum value of ΔE is attained for a certain value of α_2:

$$\alpha_2 = \alpha_{2\,\mathrm{crit}}, \tag{11}$$

and that at this point:

$$\Delta E_{\mathrm{crit}} = 4\pi r_{00}^2 A^{\frac{2}{3}}\Omega[\tfrac{98}{135}(1-x)^3 - \tfrac{11368}{34425}(1-x)^4 \ldots], \tag{12}$$

where

$$x = \frac{Z^2/A}{(Z^2/A)_{\mathrm{lim}}} \tag{13}$$

has the same meaning as in formulas (6) and (7). Thus, for example,

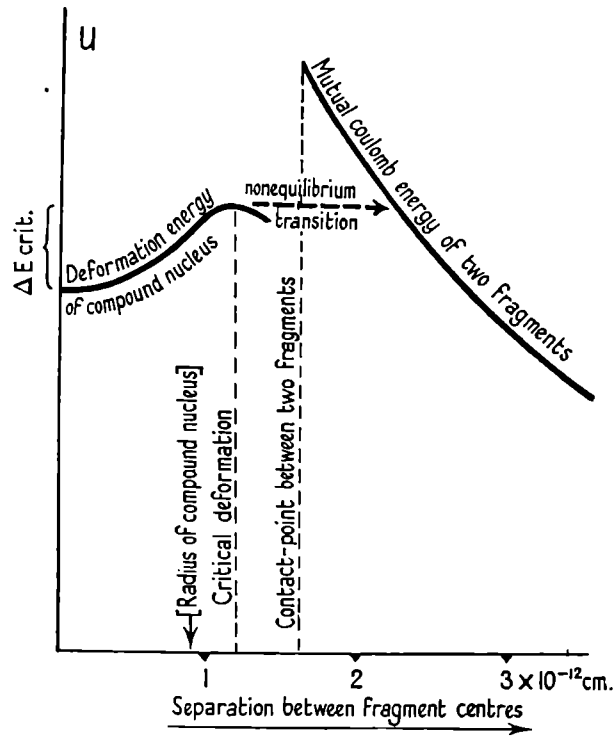


FIG. 25. Schematic presentation of non-equilibrium transition taking place during the fission process.

we obtain for the U^{238}-nucleus, $\Delta E_{\mathrm{crit}} \simeq 6$ M.e.v. The shape of the potential barrier corresponding to the general case of heavy nuclear fission is shown schematically in Fig. 25. Making a reasonable interpolation between the expression (6) which gives the critical deformation-energy for light nuclei, and the expression (12) which is applicable to the heavy ones, one can construct the curve, Fig. 26, representing the general dependence of ΔE_{crit} on the parameter: $x = (Z^2/A)/(Z^2/A)_{\mathrm{lim}}$.

The calculations above of the energy of a deformed nucleus are

based entirely on the classical picture of liquid droplet deformation, and it is important to see whether such an approach is valid, as an approximation, from the point of view of quantum mechanics. The possibility of using the classical picture depends essentially upon the smallness of the ratio between the zero-point amplitudes for the

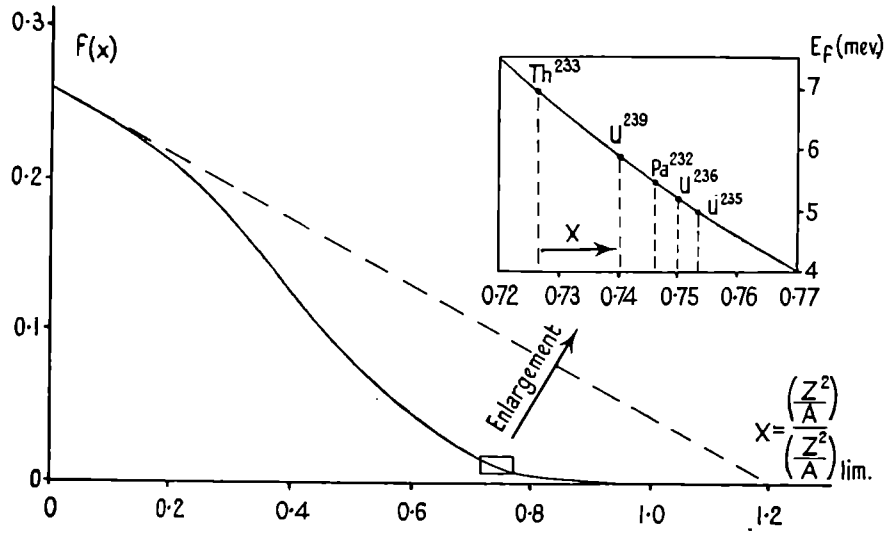


FIG. 26. Dependence of critical deformation energy on the parameter: Z^2/A, as calculated by Bohr and Wheeler. The region of radioactive elements represented separately in the upper right corner.

oscillations discussed above and the nuclear radius. A simple calculation gives for the square of the ratio in question the values:

$$(\overline{\alpha_n})^2_{\text{zero-point}} = A^{-\frac{1}{3}}\left[\frac{\hbar^2}{12Mr_{00}^2} : 4\pi r_{00}^2\,\Omega\right]^{\frac{1}{2}} n^{\frac{1}{2}}(2n+1)^{\frac{3}{2}} \times \times[(n-1)(n+2)(2n+1)-20(n-1)x]^{-\frac{1}{2}}. \quad (14)$$

Since
$$\left[\frac{\hbar^2}{12Mr_{00}^2} : 4\pi r_{00}^2\,\Omega\right]^{\frac{1}{2}} \simeq \frac{1}{3},$$

this ratio is indeed a small quantity, and it follows that deformations of magnitudes comparable with nuclear dimensions can be described approximately classically by suitable wave-packets constructed from the various quantum-mechanical solutions. In particular, it can be shown that the critical deformations leading to fission of heavy nuclei like uranium or thorium are large compared with the amplitude of the zero-point vibrations, so that the entire fission process of these nuclei may be treated in the classical manner as done above.

We may mention in conclusion that, from the practical point of

view, the fission process of atomic nuclei is a nearly irreversible process, and if we imagine the two fragments resulting from a fission to be reflected without loss of energy and to run directly towards each other, the electric repulsion between their charges would ordinarily prevent them from coming into contact. In fact, the difference in energy between two spherical fragments coming into contact, and the fused single nucleus, is given for all atomic weights by eq. (9), and we see from Fig. 25, in which this energy-difference is shown by the broken line, that it always lies considerably higher than the critical energy needed for fission. Thus the energy made available in the spontaneous fission of a heavy nucleus is considerably smaller than the energy that would be necessary to bring the two fragments back into contact as spherical droplets (cf. Fig. 25). This apparent paradox can be understood easily if we remember that the fission process actually takes place in a configuration in which the sum of surface- and electric-energies has a much smaller value than that corresponding to two rigid spheres in contact, or even to two globes distorted by the Coulomb forces. When the original nucleus breaks up, the two fragments ordinarily will possess shapes very different from spheres and must be considered, consequently, as in a very high state of internal excitation. Thus, if we wish to reverse the process of fission, we must take care that the fragments come together again sufficiently distorted, and indeed with the distortion so oriented that contact can be made between the bulges of the two surfaces. The probability that two atomic nuclei in any actual encounter will be suitably excited and possess the proper phase-relations so that fusion into a single system is possible will be extremely small. Hence the fusion of two fragments can be expected only if they collide with an energy much higher than that corresponding to the reverse, fission process.

3. Transparency of nuclear barriers

The phenomenon of *spontaneous nuclear decay*, in which the metastable nucleus breaks up into two (or more) parts without any external influence, is one of the most representative illustrations of the validity of the wave-mechanics. The wave-mechanical theory of processes of this type developed historically from an experiment by Rutherford† on the scattering of α-particles in uranium that led

† E. Rutherford, *Phil. Mag.* Ser. 7, **4** (1927), 580.

to an apparently paradoxical result. He found, in fact, that although the spontaneous decay of uranium produces α-particles with an energy of only 4 M.e.v., no deviation from the inverse square law of scattering was observed when a uranium target was bombarded by the α-particles of ThC′ which have an energy of 8·8 M.e.v. and which, because of this high energy, can approach the nucleus within a distance of 3×10^{-12} cm. from its centre. The absence of any deviation from the inverse-square scattering law indicates with certainty that the outer slope of the potential barrier surrounding the U-nucleus is formed by the pure Coulomb potential down to the radius 3×10^{-12} cm. where it reaches the height of 8·8 M.e.v. (14×10^{-6} erg). How is it possible that, in spite of such a high barrier, the nucleus of uranium emits α-particles with an energy of only 4 M.e.v. ? This fact, which is completely incomprehensible from the point of view of classical mechanics, can be understood easily on the basis of the wave-mechanical theory as shown independently by Gamow† and by Gurney and Condon.‡ In fact, the apparently paradoxical behaviour of the nuclear particle in this case has a full analogy in certain phenomena attending the reflection of light. It is well known that if a beam of light falls on the boundary between two media at an angle of incidence greater than the critical angle, then, according to geometrical optics, the phenomenon of total reflection will occur; all the light will be reflected at the surface and no disturbance will enter the second medium. According to the wave-theory of light, however, the process of total reflection is much more complicated. On this theory the disturbance in the second medium is not everywhere zero, but within the space of a few wave-lengths decreases exponentially to become entirely negligible at greater distances. This 'forbidden' penetration of the disturbance cannot be described in terms of rays of light at all, the lines representing the directions of energy-flow being curved and returning to the surface again. If, now, the second medium be confined to a thin sheet, of thickness less than the range of penetration of the disturbance already considered, and if it is backed on the far side by a further portion of the first medium, a small fraction of the disturbance which has penetrated the sheet will emerge from the far side. This transmission of energy is obviously

† G. Gamow, *Zs. f. Phys.* **51** (1928), 204; G. Gamow and F. Houtermans, ibid. **52** (1928), 496.

‡ R. W. Gurney and E. U. Condon, *Nature*, **122** (1928), 439, and *Phys. Rev.* **33** (1929), 127.

in contradiction to the predictions of purely geometrical optics: it is, however, established by experiment. The change from classical mechanics to the wave-mechanics introduces an exactly similar possibility for the transmission of a particle through a potential barrier which would otherwise be insurmountable. We describe the motion of the nuclear particle in the present case by a quasi-stationary de Broglie wave which gradually decreases in amplitude inside the nucleus on account of the leakage transmission through the finite potential

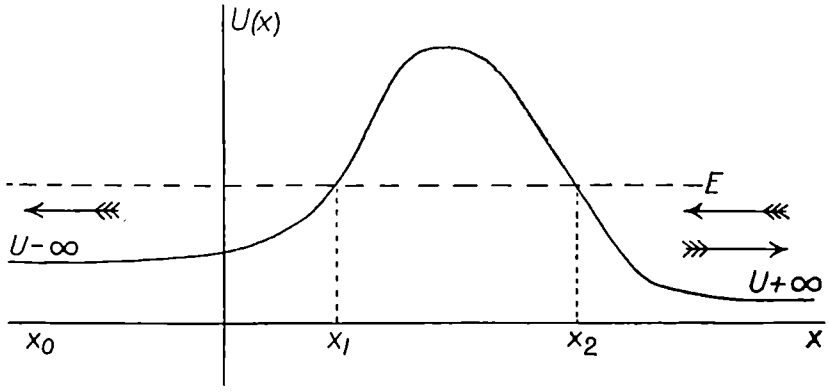


FIG. 27.

barrier. Obviously, this description predicts the possibility of observing the particle, eventually, outside the nucleus; the small transparency of potential barriers in general provides the formal explanation of the extremely long lifetimes exhibited by some α-ray bodies.

We proceed now to develop general formulae for the coefficient of transparency of potential barriers, considering, first of all, the one-dimensional case. Let a particle of total energy E fall (from the right) on a potential barrier represented by an arbitrary potential energy function $U(x)$, as shown in Fig. 27. We suppose that in a certain region ($x_1 < x < x_2$) the value of the function $U(x)$ is greater than E, and that for large distances from this region, on either side, constant values, $U_{-\infty}$ and $U_{+\infty}$, respectively, are reached. We write the Schrödinger wave-equation in the form

$$\frac{\partial^2\psi(x,t)}{\partial x^2}+\frac{2mi}{\hbar}\frac{\partial\psi(x,t)}{\partial t}-\frac{2m}{\hbar^2}U(x)\psi(x,t)=0, \tag{15}$$

where m is the mass of the particle and $\hbar$ ($= h/2\pi$) is the quantum constant. Since the potential energy $U(x)$ is assumed to be independent of the time, the solution may be written in the form

$$\psi(x,t)=\Psi(x)e^{-(i/\hbar)Et}, \tag{16}$$

in which Ψ satisfies the equation

$$\frac{d^2\Psi}{dx^2}+\frac{2m}{\hbar^2}[E-U(x)]\Psi(x) = 0. \tag{17}$$

Writing

$$\Psi(x) = e^{s(x)}, \tag{18}$$

eq. (5) reduces to

$$\frac{d^2s}{dx^2}+\left(\frac{ds}{dx}\right)^2+\frac{2m}{\hbar^2}[E-U(x)] = 0. \tag{19}$$

This equation can be solved by the method of successive approximations, so long as the first term is small compared with the second. As we shall see later, this condition, namely

$$\left|\frac{d^2s/dx^2}{(ds/dx)^2}\right| \equiv \left|\frac{d}{dx}\left(\frac{1}{ds/dx}\right)\right| \ll 1, \tag{20}$$

usually holds good for the potential barriers which occur in the theory of radioactive disintegration. We proceed, therefore, by the method of approximations, and obtain the first approximation from

$$\left(\frac{ds_1}{dx}\right)^2+\frac{2m}{\hbar^2}[E-U(x)] = 0. \tag{21}$$

Integrating, we have

$$s_1 = \pm\frac{\sqrt{(2m)}}{\hbar}\int_{x_0}^{x}[U(x)-E]^{\frac{1}{2}}\,dx+C_1. \tag{22}$$

For the second approximation we substitute from (22) in the first term of (19), writing s_1 for s. We have

$$\left(\frac{ds_2}{dx}\right)^2 = \frac{2m}{\hbar^2}[U(x)-E]\mp\frac{\sqrt{(2m)}}{\hbar}\frac{d}{dx}[U(x)-E]^{\frac{1}{2}}, \tag{23}$$

or

$$\frac{ds_2}{dx} = \pm\frac{\sqrt{(2m)}}{\hbar}[U(x)-E]^{\frac{1}{2}}-\frac{1}{2}\frac{(d/dx)[U(x)-E]^{\frac{1}{2}}}{[U(x)-E]^{\frac{1}{2}}}. \tag{24}$$

Integrating this expression, we obtain

$$s_2 = \pm\frac{\sqrt{(2m)}}{\hbar}\int_{x_0}^{x}[U(x)-E]^{\frac{1}{2}}\,dx-\tfrac{1}{2}\log[U(x)-E]^{\frac{1}{2}}+C_2. \tag{25}$$

We may proceed in exactly the same way to the third approximation, but already it is of no great importance. Thus we have, finally, with sufficient accuracy

$$\Psi(x) = e^{s_2(x)} = C_2[U(x)-E]^{-\frac{1}{4}}\exp\left(\pm\frac{\sqrt{(2m)}}{\hbar}\int_{x_0}^{x}[U(x)-E]^{\frac{1}{2}}\,dx\right). \tag{26}$$

This solution is valid only in those regions for which the condition (20) is satisfied. Using the first approximation for s, we write this condition in the form

$$\frac{\hbar}{\sqrt{(2m)}}\left|\frac{d}{dx}\left(\frac{1}{[U(x)-E]^{\frac{1}{2}}}\right)\right| \ll 1. \tag{27}$$

An estimate of the order of magnitude of the quantity occurring in (27) may be made in terms of its approximate equivalent,

$$\frac{\hbar}{\sqrt{(2m)}}\,\frac{1}{2}\,\frac{\Delta U(x)}{\Delta x[U(x)-E]^{\frac{3}{2}}}. \tag{28}$$

Remembering that, in nuclear potential barriers, we have to do with energies of the order of 10^{-5} erg, and with changes of potential energy of roughly this amount over distances of the order of 10^{-12} cm., we substitute in (28) $E \sim U(x) \sim \Delta U(x) \sim 10^{-5}$ erg, $\Delta x \sim 10^{-12}$ cm., and obtain (for $m \sim 6\times 10^{-24}$ gm.) the value $\sim \frac{1}{20}$. Near the points x_1, x_2 (Fig. 27) the left-hand side of (27) becomes infinite, but otherwise our numerical calculation shows that in practical cases the condition (20) is usually satisfied. In order to carry out the integration in the regions around x_1, x_2 we need to know the analytical form of $U(x)$, and, having substituted this in (19), to obtain the exact solution of the wave equation. If, however, these regions (in which $[U(x)-E]^{-\frac{1}{2}}$ varies too rapidly with x) are small enough, we can make the solutions for the neighbouring regions on the two sides satisfy the boundary conditions of continuity and thus deduce the complete solution by exclusively approximate methods.

Let us return, for the moment, to the more physical aspect of the problem in hand. We have postulated a beam of particles moving from right to left and falling on the right-hand side of the potential barrier of Fig. 27. Our mathematical solution must represent these particles as well as those reflected, and so moving from left to right, to the right of the barrier, and those transmitted (moving from right to left, to the left of the barrier). In particular, at large distances to the left of the barrier (where $U = U_{-\infty}$) the solution must represent an harmonic wave travelling from right to left. Thus in the general solution (26) we put $U = U_{-\infty}$ and choose the negative sign of the exponent to secure the correct direction of motion (compare with (16)). We obtain in this way

$$\Psi_{x\ll x_1} = C(E-U_{-\infty})^{-\frac{1}{4}}\exp\left\{-i\frac{\sqrt{(2m)}}{\hbar}(E-U_{-\infty})^{\frac{1}{2}}(x-x_0)\right\}, \tag{29}$$

representing a wave travelling from right to left, in a region where $E > U_{-\infty}$.†

In the region $x_1 < x < x_2$ the difference $[U(x)-E]$ becomes positive and both solutions, apart from the arbitrary constants, are real, one of them increasing and the other decreasing rapidly as x increases. To make the solutions continuous at $x = x_1$ we must take inside the barrier the general form with complex coefficients:

$$\Psi_{x_1<x<x_2} = C[E-U(x)]^{-\frac{1}{4}}\left\{b_+ \exp\left(\frac{\sqrt{(2m)}}{\hbar}\int_{x_1}^{x}[U(x)-E]^{\frac{1}{2}}\,dx\right)+\right.$$
$$\left.+b_- \exp\left(-\frac{\sqrt{(2m)}}{\hbar}\int_{x_1}^{x}[U(x)-E]^{\frac{1}{2}}\,dx\right)\right\}. \quad (30)$$

It is easy to see that the boundary conditions at $x = x_1$ (continuity of Ψ and $d\Psi/dx$) require

$$b_+ = \tfrac{1}{2}(1-i)\exp\left(-i\frac{\sqrt{(2m)}}{\hbar}\int_{x_0}^{x_1}[E-U(x)]^{\frac{1}{2}}\,dx\right);$$
$$b_- = \tfrac{1}{2}(1+i)\exp\left(-i\frac{\sqrt{(2m)}}{\hbar}\int_{x_0}^{x_1}[E-U(x)]^{\frac{1}{2}}\,dx\right). \quad (31)$$

At the other end of the barrier, where $x = x_2$, the second term in (30) is very small compared with the first. In that case the solution reduces to

$$\Psi_{x=x_2} = \tfrac{1}{2}C(1-i)[E-U(x_2)]^{-\frac{1}{4}}\exp\left(-i\frac{\sqrt{(2m)}}{\hbar}\int_{x_0}^{x_1}[E-U(x)]^{\frac{1}{2}}\,dx\right)\times$$
$$\times\exp\left(\frac{\sqrt{(2m)}}{\hbar}\int_{x_1}^{x_2}[U(x)-E]^{\frac{1}{2}}\,dx\right). \quad (32)$$

To join this function with the solution for $x > x_2$ we must again take a linear combination of the solutions representing oppositely directed waves,

$$\Psi_{x>x_2} = C[E-U(x)]^{-\frac{1}{4}}\left\{a_+ \exp\left(i\frac{\sqrt{(2m)}}{\hbar}\int_{x_2}^{x}[E-U(x)]^{\frac{1}{2}}\,dx\right)+\right.$$
$$\left.+a_- \exp\left(-i\frac{\sqrt{(2m)}}{\hbar}\int_{x_2}^{x}[E-U(x)]^{\frac{1}{2}}\,dx\right)\right\}, \quad (33)$$

† x_0 in (29) is a point at a great distance to the left of the barrier, i.e. $x_0 \ll x_1$.

and the boundary conditions give

$$a_+ = \tfrac{1}{2}(1-i)b_+ \exp\left(\frac{\sqrt{(2m)}}{\hbar}\int_{x_1}^{x_2}[U(x)-E]^{\frac{1}{2}}\,dx\right);$$
$$a_- = \tfrac{1}{2}(1+i)b_+ \exp\left(\frac{\sqrt{(2m)}}{\hbar}\int_{x_1}^{x_2}[U(x)-E]^{\frac{1}{2}}\,dx\right). \tag{34}$$

If, now, x_0' is some value of x lying at a great distance to the right of the barrier, the above solution may be written

$$\Psi_{x_2 \ll x} = A_+ \exp\left\{i\frac{\sqrt{(2m)}}{\hbar}(E-U_{+\infty})^{\frac{1}{2}}(x-x_0')\right\} + \\ + A_- \exp\left\{-i\frac{\sqrt{(2m)}}{\hbar}(E-U_{+\infty})^{\frac{1}{2}}(x-x_0')\right\}, \tag{35}$$

with

$$A_+ = -\tfrac{1}{2}iC[E-U(x)]^{-\frac{1}{4}} \exp\left\{\frac{\sqrt{(2m)}}{\hbar}\left(-i\int_{x_0}^{x_1}[E-U(x)]^{\frac{1}{2}}\,dx + \right.\right. \\ \left.\left. + \int_{x_1}^{x_2}[U(x)-E]^{\frac{1}{2}}\,dx + i\int_{x_2}^{x_0'}[E-U(x)]^{\frac{1}{2}}\,dx\right)\right\},$$
$$A_- = \tfrac{1}{2}C[E-U(x)]^{-\frac{1}{4}} \exp\left\{\frac{\sqrt{(2m)}}{\hbar}\left(-i\int_{x_0}^{x_1}[E-U(x)]^{\frac{1}{2}}\,dx + \right.\right. \\ \left.\left. + \int_{x_1}^{x_2}[U(x)-E]^{\frac{1}{2}}\,dx - i\int_{x_2}^{x_0'}[E-U(x)]^{\frac{1}{2}}\,dx\right)\right\}. \tag{36}$$

These two waves represent the incident and reflected beams of particles; they appear to have equal amplitude because we have neglected certain small-order terms (the second term in (30)). In actual fact a wave of small intensity has been transmitted by the barrier, and carrying out the calculations to the next order of approximation an expression representing the conservation of particles is obtained.

The coefficient of transparency of the barrier is given by the square of the ratio of the amplitudes of transmitted and incident waves, respectively. We have

$$G = \left|\frac{\Psi(-\infty)}{\Psi(+\infty)}\right|^2 = 4\left(\frac{E-U_{+\infty}}{E-U_{-\infty}}\right)^{\frac{1}{2}} \exp\left(-\frac{2\sqrt{(2m)}}{\hbar}\int_{x_1}^{x_2}[U(x)-E]^{\frac{1}{2}}\,dx\right). \tag{37}$$

This is the formula which we shall employ in future calculations of the transparency of nuclear potential barriers. When this transparency

is small ($2\sqrt{(2m)}\int_{x_1}^{x_2}[U(x)-E]^{\frac{1}{2}}\,dx \gg \hbar$), it is possible to show that (20), (27), the conditions of validity of the formula, are fulfilled.

We may remark, at this stage, that in certain special cases the solution of the wave-equation may be found directly in an analytical form. For the Coulomb potential barrier, for example, the solution may be expressed in the form of confluent hypergeometric functions. This type of barrier is of great importance in the theory of nuclear disintegration, but here the analytical form of solution does not lead to any greater accuracy in calculation and we shall not further be concerned with it.

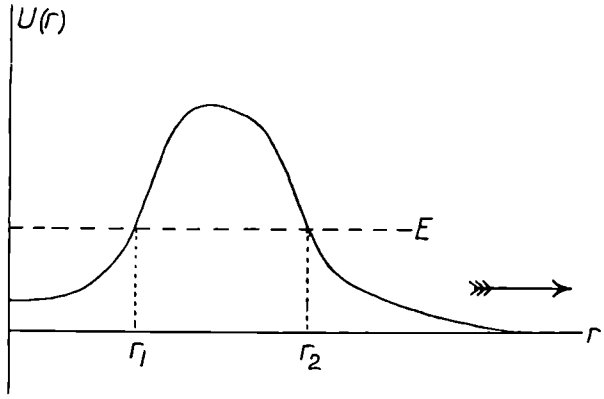


FIG. 28.

We are now in a position to develop the general theory of the escape of a particle from a region surrounded by a potential barrier—which is just the process involved in the emission of an α-particle by an atomic nucleus. We shall assume spherical symmetry for the distribution of potential energy around the centre of the nucleus and imagine that the appropriate function $U(r)$, starting from a certain finite value at the centre, increases to a large positive value at a relatively small distance and vanishes for very large distances, as indicated in the figure. The total energy of the particle considered is supposed to be smaller than the energy corresponding to the peak of the potential barrier, but larger than that corresponding to the bottom of the potential 'hole' in Fig. 28. The wave equation for the particle may be written in spherical polar coordinates as follows:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+$$
$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}+\frac{2im}{\hbar}\frac{\partial\psi}{\partial t}-\frac{2m}{\hbar^2}U(r)\psi=0. \quad (38)$$

If E is the energy of the particle, the solution of this equation will have the form

$$\psi = \Psi(r, \theta, \phi)\exp\left(-\frac{i}{\hbar}Et\right), \tag{39}$$

where Ψ satisfies the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)+$$

$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}+\frac{2m}{\hbar^2}[E-U(x)]\Psi = 0. \tag{40}$$

In so far as the potential energy of the α-particle in the neighbourhood of the nucleus is independent of the angular coordinates θ and ϕ, Ψ may be expressed in the form

$$\Psi = \frac{1}{r}P_j(\theta, \phi)\chi_j(r), \tag{41}$$

where P_j is a spherical harmonic of order j ($j = 0, 1, 2, \ldots$) and $\chi_j(r)$ satisfies the equation

$$\frac{\partial^2\chi_j(r)}{\partial r^2}+\frac{2m}{\hbar^2}\left[E-U(r)-\frac{\hbar^2}{2m}\frac{j(j+1)}{r^2}\right]\chi_j(r) = 0. \tag{42}$$

Here j represents the azimuthal quantum number and the last term within the brackets is to be interpreted as an additional potential energy corresponding to the centrifugal force. In order that Ψ shall remain finite at the centre of the nucleus, $\chi(0)$ must be zero, as may be seen by inspection of (41).

We may thus obtain a general solution to (38), but it should be noted that not every particular case of such a solution represents the process of spontaneous ejection of an α-particle from the nucleus—which process we wish to describe. In order that any solution shall do this it is necessary first to satisfy certain boundary conditions appropriate to the physical interpretation in question. We may specify these conditions, formally, as follows. Let us proceed to construct solutions of (42) for some arbitrary value of E, inserting, as above indicated, $\chi(0) = 0$. For large values of r we obtain in this way

$$\underset{r\to\infty}{\chi_{E,j}} = C_{+E,j}\exp\left(i\frac{\sqrt{(2m)}}{\hbar}E^{\frac{1}{2}}r\right)+C_{-E,j}\exp\left(-i\frac{\sqrt{(2m)}}{\hbar}E^{\frac{1}{2}}r\right), \tag{43}$$

where C_+ and C_- are the complex conjugate functions of E and j. Substituted in (39) and (41), the first and second terms of (43) represent, respectively, divergent and convergent waves passing

through the origin. Now the process of spontaneous disintegration must be described in terms of a diverging wave, only, thus

$$C_{+E,j} \neq 0; \qquad C_{-E,j} = 0. \tag{44}$$

At first sight it might seem that the conditions (44) were mutually contradictory, since C_+ and C_-, being complex conjugate functions, might be thought always to vanish simultaneously. However, it can easily be shown that this need not be so, if we allow complex values for E; complex conjugate functions of the same complex argument in general give quite different amplitudes. Postponing for a time the physical interpretation of this procedure, we write, therefore

$$E = E_0 - iE'. \tag{45}\dagger$$

We find now that the conditions (44) define, for each value of the azimuthal quantum number j, a discrete set of complex E values which we shall distinguish by different indices n ($n = 0, 1, 2, \ldots$). We may refer to these as the principal quantum numbers for this set of states. Here, as in the ordinary problem of electron motion in the outer atom, we come upon a set of discrete proper values of the energy, the only difference being that the values are now complex quantities which on interpretation correspond to positive, rather than to negative, total energy.

Proceeding now to the interpretation which is necessary to complete the investigation, we must turn our attention to the time-dependence of ψ. Substituting from (45) in (39), we obtain as the time-factor in the wave-function

$$\exp\left(-\frac{i}{\hbar}Et\right) = \exp\left(-\frac{i}{\hbar}(E_0 - iE')t\right) = \exp\left(-\frac{i}{\hbar}E_0 t\right)\exp\left(-\frac{E'}{\hbar}t\right). \tag{46}$$

Then the probability, $\psi\bar{\psi}$, that the α-particle will be found in any particular region of space, inside the nucleus or outside, becomes aperiodically dependent upon the time:

$$\psi\bar{\psi} \sim \left[\exp\left(-\frac{i}{\hbar}E_0 t\right)\exp\left(-\frac{E'}{\hbar}t\right)\right]\left[\exp\left(\frac{i}{\hbar}E_0 t\right)\exp\left(\frac{E'}{\hbar}t\right)\right]$$
$$= \exp\left(-\frac{2E't}{\hbar}\right) = \exp(-\lambda t). \tag{47}$$

In this expression the quantity λ ($= 2E'/h$) may be recognized as the disintegration constant which should appear as a result of our

† It may be shown that a divergent wave requires a negative sign in (45), a convergent wave a positive sign.

calculations. The complex values of the proper energies thus lead to exponentially decreasing probabilities of finding a particle, of given energy, in a finite region of space: they are characteristic of the solution of any problem which deals with decay processes, the imaginary part of the energy being immediately connected with the rate of decay. We may also regard the imaginary part of the energy from a slightly different point of view. The ψ-wave diverging from the nucleus will not be strictly harmonic, the aperiodic factor resulting in a damping of the wave. In optics such a damped disturbance would produce a broadened line in a spectrograph; with moving particles we employ a mass spectrograph, or its equivalent, and here the broadening of the line will be interpreted as due to an uncertainty in the energy of the particles. Developing the damped wave-function by Fourier expansion it may be shown that the uncertainty in the energy (half-breadth of the line) has the value

$$\Delta E \sim \hbar\lambda \sim E'. \tag{48}\dagger$$

From this point of view the average energy of the emitted particles is given by the real part of the proper energy, whilst the probable deviation from the average is given by the imaginary part.

Having carried the interpretation so far, it is necessary now to mention that the complexity of E introduces an exponential term involving the space-coordinate also. For large distances from the nucleus, where $U(r) \to 0$, we evidently have ($E' \ll E_0$)

$$\begin{aligned}\underset{r\to\infty}{\chi(r)} &= C_+ \exp\left(i\frac{\sqrt{(2m)}}{\hbar}E^{\frac{1}{2}}r\right) = C_+ \exp\left(i\frac{\sqrt{(2m)}}{\hbar}\left[E_0 - i\frac{\lambda\hbar}{2}\right]^{\frac{1}{2}}r\right)\\ &= C_+ \exp\left(i\frac{\sqrt{(2m)}}{\hbar}E_0^{\frac{1}{2}}r\right)\exp\left(\frac{\lambda}{2}\sqrt{\frac{m}{2E_0}}r\right).\end{aligned} \tag{49}$$

This implies that the amplitude of the divergent wave increases with increasing distance and tends to an infinite value at infinity; the same general result is obtained in all solutions of decay problems, for example in the classical solution for the radiation from a damped oscillator. It need occasion no surprise, since it merely expresses the fact that the disturbance at a great distance from the origin, at any moment, was emitted from the origin at a correspondingly distant past time, when the rate of radiation was greater. In the radioactive

† Remembering that the mean time, T, for an α-particle to remain in the nucleus is given by $T = 1/\lambda$, (48) may be written $T\Delta E \sim \hbar$, which is precisely Heisenberg's uncertainty relation, expressed in terms of time and energy.

decay problem the flux of particles at a large distance r, at a given instant, must obviously correspond to the strength of the source at a time r/v earlier. At that time the strength of the source was greater by the factor $\exp\left(\lambda\frac{r}{v}\right) = \exp\left(\lambda r\sqrt{\frac{m}{2E_0}}\right)$, which is precisely the factor to be explained in (49), when it is remembered that (49) gives the amplitude of the wave and the square of this amplitude enters into the calculation of the flux. In order to appreciate the relative magnitudes of the various quantities involved, we may take the case of radium C′, for which E'/E_0 is larger than for any other element concerning which full data are available. Here $E_0 = 1{\cdot}2\times10^{-5}$ erg, $\lambda = 10^4$ sec.$^{-1}$, thus $E = E_0 - i\frac{1}{2}\lambda\hbar = 1{\cdot}2\times10^{-5} - 5{\cdot}2\times10^{-24}i$. The uncertainty in the energy-value is only $4{\cdot}3\times10^{-17}$ per cent. of the average energy, and, according to (49), the amplitude of the ψ wave is not doubled until a distance of 2·8 km. from the nucleus is reached.

Although the method of complex E values, proposed by Gamow, gives perhaps the most satisfactory representation of disintegration processes, the necessary calculations are impossible except in respect of very simple nuclear models, such as that represented by the rectangular potential barrier discussed in detail by Kudar.† On the other hand, real nuclei must be treated on the basis of a potential-energy function for α-particles which at large distances is given by the Coulomb inverse-distance law and at small distances is modified by some unknown, rapidly varying, potential function which makes the resultant potential energy negative inside the nucleus. In this case it is possible to obtain a relation between the real and imaginary parts of the proper energy-values—that is between disintegration energy and decay-constant—without in the process determining the values of E_0 (energies of disintegration) to which the more complicated model leads. These values are very sensitive to the form of the potential barrier inside the nucleus, but the $E_0-\lambda$ relation may be obtained without exact knowledge of this form, as we shall presently discover. This relation may then very usefully be compared with the empirical relation which has already been discussed.

Suppose that we start with the expression, in terms of the wave-function ψ, for the conservation of particles; it is

$$\frac{\partial}{\partial t}\int \psi\bar{\psi}\, d\omega = \frac{\hbar}{2mi}\int\left[\psi\frac{\partial\bar{\psi}}{\partial n} - \bar{\psi}\frac{\partial\psi}{\partial n}\right]d\sigma, \tag{50}$$

† J. Kudar, *Zs. f. Phys.* **53** (1929), 95, 134.

the integral on the left being taken throughout a certain region of space and that on the right over its boundary surface. In the expression on the right, moreover, n is measured along the outward normal to the surface element $d\sigma$. In (50) ρ $(= \psi\bar{\psi})$ is the probability density relative to the particle in question and I_n $\left(= \frac{\hbar}{2mi}\left[\psi\frac{\partial\bar{\psi}}{\partial n} - \bar{\psi}\frac{\partial\psi}{\partial n}\right]\right)$ the normal component of the flux of probability across the surface. In the case of spherical symmetry, when we choose for the region to which (50) applies a sphere of radius R centred in the nucleus, we can, by using (39), (41), and (47), transform the conservation equation and obtain

$$\frac{\partial}{\partial t}\int_0^R \chi\bar{\chi}e^{-\lambda t}\,dr = \frac{\hbar}{2mi}e^{-\lambda t}\left[\chi\frac{\partial\bar{\chi}}{\partial r} - \bar{\chi}\frac{\partial\chi}{\partial r}\right]_{r=R} \tag{51}$$

or

$$\lambda = \frac{\hbar}{2mi}\left[\chi\frac{\partial\bar{\chi}}{\partial r} - \bar{\chi}\frac{\partial\chi}{\partial r}\right]_{r=R} \Big/ \int_0^R \chi\bar{\chi}\,dr. \tag{52}$$

We have thus obtained a formula for the disintegration constant which may be interpreted simply as showing that this constant is just the ratio of the flux of particles across any large sphere to the number of particles remaining within the sphere at the instant in question. A completely analogous relation is often employed for calculating the damping coefficient for any radiating oscillator in classical electrodynamics. In using (52) in the radioactive case it is quite unimportant what value of R we take, provided that it is large in comparison with r_1, the inner radius of the potential barrier. In fact we have seen that the rate of flow, as determined by

$$\frac{\hbar}{2mi}\left(\chi\frac{\partial\bar{\chi}}{\partial r} - \bar{\chi}\frac{\partial\chi}{\partial r}\right),$$

is fairly constant, increasing by a factor 2, only, over a distance which is almost always greater than 1 km. As concerns the denominator of (52), to a sufficient approximation we may take the integral here from $r = 0$ to $r = r_1$, since χ decreases so rapidly in the range $r_1 < r < r_2$ (within the barrier) that the portion of the integral corresponding to $r > r_1$ is negligibly small. For the radioactive nuclei $\chi\bar{\chi}$ outside the nucleus is less than 10^{-16} of its value inside, so that for $R = 1$ cm. the contribution of the extra-nuclear part of the integral will not be more than 0·01 per cent. of its total value. As we have seen above, the potential energy inside the nucleus may be regarded as approximately constant; consequently the solution for

χ (for $r < r_1$) will correspond very nearly to harmonic oscillations of amplitude A (say). Then, with sufficient accuracy,

$$\int_0^{r_1} \chi\bar{\chi}\, dr \sim \tfrac{1}{2}A^2 r_1. \tag{53}$$

Also, for large r values we have previously used

$$\chi_{r\to\infty} = C\exp\left(i\frac{\sqrt{(2m)}}{\hbar}E^{\frac{1}{2}}r\right), \tag{54}$$

so that the probability flux becomes

$$\frac{\hbar}{2mi}\left(\chi\frac{\partial\bar{\chi}}{\partial r} - \bar{\chi}\frac{\partial\chi}{\partial r}\right) = C^2\sqrt{\left(\frac{2E}{m}\right)} = C^2 v_e, \tag{55}$$

where v_e is the velocity of the particle outside the nucleus. Substituting from (53) and (55) in (52) we obtain

$$\lambda = \frac{2v_e}{r_1}\left(\frac{C}{A}\right)^2 = \frac{2v_e}{r_1}G, \tag{56}$$

G being the transparency of the barrier, which may be taken over from (37). Then

$$\begin{aligned}\lambda &= \frac{2v_e}{r_1}4\frac{v_i}{v_e}\exp\left(-\frac{2\sqrt{(2m)}}{\hbar}\int_{r_1}^{r_2}[U(r)-E]^{\frac{1}{2}}\,dr\right)\\ &= \frac{8v_i}{r_1}\exp\left(-\frac{2\sqrt{(2m)}}{\hbar}\int_{r_1}^{r_2}[U(r)-E]^{\frac{1}{2}}\,dr\right),\end{aligned} \tag{57}$$

in which expression v_i is written for the velocity of the particle inside the nucleus. Remembering that the de Broglie wave-length Λ_i corresponding to this velocity ($mv_i\Lambda_i = h = 2\pi\hbar$) must be an exact submultiple of the nuclear diameter ($k\Lambda_i = 2r_1$), where k is the radial quantum number), we have $v_i = \pi k\hbar/mr_1$, and may write the expression for the disintegration constant in the final form

$$\lambda = \frac{8\pi k\hbar}{mr_1^2}\exp\left(-\frac{2\sqrt{(2m)}}{\hbar}\int_{r_1}^{r_2}[U(r)-E]^{\frac{1}{2}}\,dr\right). \tag{58}$$

4. Comparison with observation

We shall now compare the results of the wave-mechanical theory of the transparency of nuclear potential barriers with the known facts concerning the rates of various types of spontaneous nuclear disintegration.

Spontaneous α-decay of various naturally radioactive substances presents us with abundant observational material for comparison with the theory. The three known radioactive families, that of thorium, uranium, and actinium, contain altogether twenty-four α-decaying

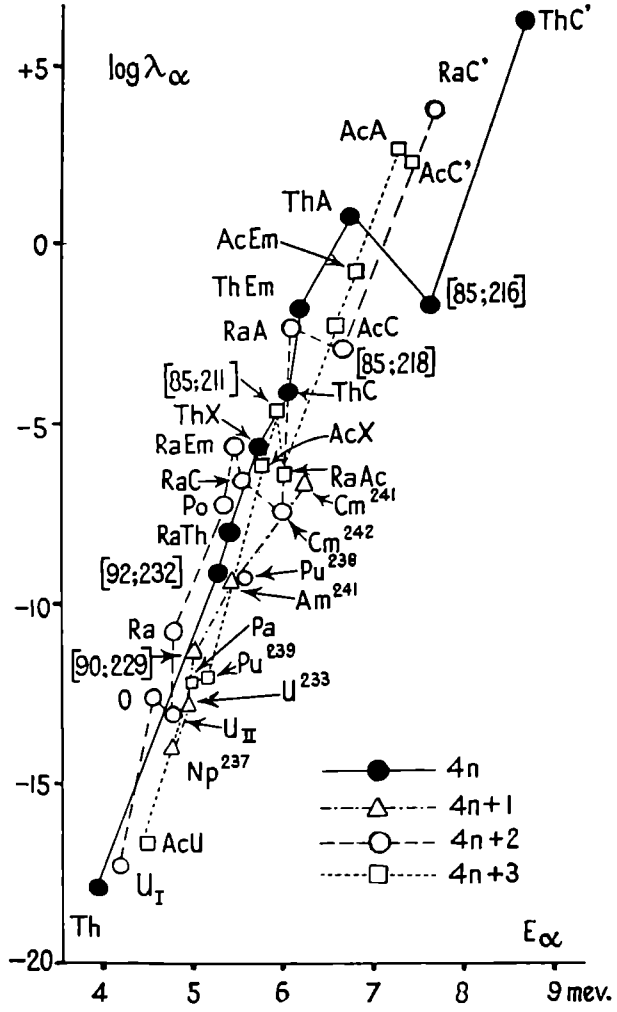


FIG. 29. Geiger–Nuttall diagram of radioactive families (including the newly discovered α-emitters).

bodies for which the energies of emitted α-particles and the corresponding decay-constants have been measured with great accuracy.

These data are shown in Table XIII, which also includes the information concerning a number of α-unstable nuclei which do not normally exist in nature but can be produced artificially by the methods of modern alchemy.

The observed values of $E\alpha$ and λ were known to be connected through the empirical law of *Geiger and Nuttall* (Fig. 29) according

TABLE XIII†

Properties of known α-emitters

Z	*Name*	*A*	*Eα (M.e.v.)*	*Half-life*	λ_α *sec.*$^{-1}$
96	Curium	242	6·0	0·7 y.	$3{\cdot}1\times10^{-8}$
	,,	241	6·25	30 d.	$2{\cdot}7\times10^{-7}$
95	Americium	241	5·47	50 y.	$4{\cdot}4\times10^{-10}$
94	Plutonium	239	5·16	$2{\cdot}4\times10^{4}$ y.	$9{\cdot}15\times10^{-13}$
	,,	238	5·52	60 y.	$3{\cdot}65\times10^{-10}$
93	Neptunium	237	4·77	$2{\cdot}2\times10^{6}$ y.	$1{\cdot}0\times10^{-14}$
92	Uranium I	238	4·18	$4{\cdot}5\times10^{9}$ y.	$4{\cdot}88\times10^{-18}$
	Actinouranium	235	4·52	$8{\cdot}8\times10^{8}$ y.	$2{\cdot}5\times10^{-17}$
	Uranium II	234	4·76	$2{\cdot}5\times10^{5}$ y.	$8{\cdot}8\times10^{-14}$
		233	4·82	$1{\cdot}6\times10^{5}$ y.	$1{\cdot}3\times10^{-13}$
		232	5·31	30 y.	$7{\cdot}3\times10^{-10}$
91	Protactinium	231	5·05	$3{\cdot}2\times10^{4}$ y.	$6{\cdot}85\times10^{-13}$
90	Thorium	232	4·23	$1{\cdot}3\times10^{10}$ y.	$1{\cdot}3\times10^{-18}$
	Ionium	230	4·54	$7{\cdot}6\times10^{4}$ y.	$2{\cdot}9\times10^{-13}$
		229	5·0	5×10^{3} y.	$4{\cdot}4\times10^{-12}$
	Radiothorium	238	5·42	1·9 y.	$1{\cdot}16\times10^{-8}$
	Radioactinium	237	6·05	18·9 d.	$4{\cdot}24\times10^{-7}$
88	Radium	226	4·793	1,590 y.	$1{\cdot}373\times10^{-11}$
	Thorium X	224	5·682	3·64 d.	$2{\cdot}20\times10^{-6}$
	Actinium X	223	5·719	11·2 d.	$7{\cdot}16\times10^{-7}$
86	Radium Em.	222	5·488	3·852 d.	$2{\cdot}097\times10^{-6}$
	Thorium Em.	220	6·283	54·5 s.	$1{\cdot}27\times10^{-2}$
	Actinium Em.	219	6·826	3·92 s.	0·117
85		218	6·63	710 s.	$9{\cdot}75\times10^{-4}$
		216	7·63	50 s.	$1{\cdot}38\times10^{-2}$
		211	5·94	7·5 h.	$2{\cdot}56\times10^{-5}$
84	Radium A	218	6·112	3·05 m.	$3{\cdot}79\times10^{-3}$
	Thorium A	216	6·776	0·145 s.	4·78
	Actinium A	215	7·368	2×10^{-3} s.	350
	Radium C^1	214	7·68	$1{\cdot}5\times10^{-4}$ s.	$4{\cdot}6\times10^{+3}$
	Thorium C^1	212	8·78	3×10^{-7} s.	$2{\cdot}3\times10^{+6}$
	Actinium C^1	211	7·43	5×10^{-3} s.	$1{\cdot}38\times10^{+2}$
	Polonium	210	5·300	136·3 s.	$5{\cdot}88\times10^{-8}$
83	Radium C	214	5·507	..‡	$1{\cdot}78\times10^{-7}$
	Thorium C	212	6·084	..‡	$6{\cdot}65\times10^{-5}$
	Actinium C	211	6·611	..‡	$5{\cdot}35\times10^{-3}$
62	Samarium	148	2·0	$1{\cdot}6\times10^{11}$ y.	$1{\cdot}38\times10^{-19}$

† The data for this table are taken principally from W. B. Lewis and B. V. Bowden, *Proc. Roy. Soc.* **145** (1934), 235. The data on new α-emitters are from I. M. Cork, *Rad. and Nucl. Phys.*, Ann Arbor, Mich., 1946.

‡ In the case of C-bodies λ_α are estimated from total λ_α and the branching ratios. The 'partial half-lives' are not given as having no physical sense.

to which the decay-constants of different elements within the same family increase exponentially with increasing energies of emission. Since, according to the wave-mechanical treatment, the decay-constants for α-transformations are essentially the same as the corresponding transparencies of the potential barriers involved, the empirical relation cited above becomes immediately understandable from the point of view of the theory.

In order to derive an exact expression for the decay-constant as a function of the energy of the emitted α-particle and of other characteristics of the decaying nucleus, we shall use the approximate shape of the potential barrier discussed in § 2 of this chapter and represented graphically in Fig. 22.

We may notice here that the approximation which is to be adopted will scarcely affect the value of the coefficient of transparency of the barrier, since the chief part of the integral in formula (37) is taken over the region of the barrier where deviations from the Coulomb law are very small. On the other hand, our model will give only very rough results concerning the energy-levels for α-particles within the nucleus, as the positions of these depend essentially upon the actual shape of the potential-energy curve. With this warning, however, we may proceed to calculate the proper-energies involved. The solution for $r < r_0$ is

$$\chi_{E,j} = A r^{\frac{1}{2}} J_{j+\frac{1}{2}}\left\{\frac{\sqrt{(2m)}}{\hbar}(E-U_0)^{\frac{1}{2}} r\right\}, \tag{59}$$

where $J_{j+\frac{1}{2}}$ is a Bessel function of order $j+\frac{1}{2}$. If $k_{n,j+\frac{1}{2}}$ are the roots of these functions, we have

$$E_{n,j} = \frac{k^2\hbar^2}{2mr_0^2} + U_0. \tag{60}$$

In the case of radial oscillations

$$\chi_{E,0} = A \sin \frac{\sqrt{(2m)}}{\hbar}(E-U_0)^{\frac{1}{2}} r, \tag{61}$$

and we have, simply,

$$E_{n,0} = \frac{n^2\hbar^2\pi^2}{2mr_0^2} + U_0. \tag{62}$$

In calculating the decay constant for our model we shall first examine the case of radial motion; this is precisely what has already been done in using (37) to obtain the expressions (57) and (58) of § 3.

We may rewrite (58) with the new assumptions regarding $U(r)$ as follows:

$$\lambda_{E,0} = \frac{8\pi k\hbar}{mr_0^2}\exp\left(-\frac{2\sqrt{(2m)}}{\hbar}\int_{r_0}^{2(Z-2)e^2/E}[2(Z-2)e^2/r-E]^{\frac{1}{2}}\,dr\right). \quad (63)$$

The integral in the exponent of (63) may be simply evaluated by means of the substitution†

$$\cos^2 u = \frac{rE}{2(Z-2)e^2} = \frac{r}{r^*}. \quad (64)$$

We have $$\lambda_{E,0} = \frac{8\pi k\hbar}{mr_0^2}\exp\left\{-\frac{4e^2}{\hbar}\frac{(Z-2)}{v_e}(2u_0-\sin 2u_0)\right\}, \quad (65)$$

where $$\cos^2 u_0 = \frac{r_0}{r^*} = \frac{r_0 E}{2(Z-2)e^2}. \quad (65')$$

In radioactive nuclei the value of r_0/r^* is small and the exponent of (65) may be developed in powers of this small quantity. Taking the first two terms only we obtain a simple formula which is very convenient for the calculation of decay constants,

$$\lambda_{E,0} = \frac{8\pi k\hbar}{mr_0^2}\exp\left(-\frac{4\pi e^2(Z-2)}{\hbar v_e}+\frac{8e\sqrt{m}}{\hbar}[(Z-2)r_0]^{\frac{1}{2}}\right). \quad (66)$$

If the azimuthal quantum number differs from zero we must take count of the additional potential energy corresponding to the centrifugal force, and obtain the expression for the decay constant in the form

$$\lambda_{E,j} = \frac{8\pi k\hbar}{mr_0^2}\exp\left(-\frac{2\sqrt{(2m)}}{\hbar}\int_{r_0}^{r^*}\left[\frac{2(Z-2)e^2}{r}+\frac{\hbar^2}{2m}\frac{j(j+1)}{r^2}-E\right]^{\frac{1}{2}}dr\right). \quad (67)$$

In radioactive nuclei ($Z = 80$ to 90; $r_0 \sim 10^{-12}$ cm.) the potential energy corresponding to centrifugal forces will, in general, be very small compared with the Coulomb potential energy, the ratio σ being given by

$$\sigma = \frac{\hbar^2}{2m}\frac{j(j+1)}{r_0^2}:\frac{2(Z-2)e^2}{r_0} \sim 0{\cdot}002j(j+1). \quad (68)$$

Thus, for calculation of the integral (67) we may develop the square root in powers of the small quantity σ. We get, finally,

$$\lambda_{E,j} = \frac{8\pi k\hbar}{m_0 r^2}\exp\left\{-\frac{4\pi e^2}{\hbar}\frac{(Z-2)}{v_e}+\frac{8e\sqrt{m}}{\hbar}[(Z-2)r_0]^{\frac{1}{2}}(1-\tfrac{1}{2}\sigma)\right\}. \quad (69)‡$$

† r^* is the so-called classical radius of the nucleus, i.e. the closest distance of approach of the α-particle according to classical mechanics.

‡ In subsequent calculations we shall always take $k = 1$.

This shows that the azimuthal quantum number of the emitted particle must be assumed 'responsible' for a decrease (for the same energy of disintegration) of the disintegration probability; if this angular momentum amounts to several units the decrease may be quite appreciable. As we shall see later, we may obtain interesting information concerning the changes of spin of decaying nuclei on this basis.

Formula (69), which can be written in the form

$$\log_{10}\lambda = \log_{10}\frac{8\pi\hbar}{mr_0^2} - \frac{4\pi e^2}{\hbar l}\frac{Z-2}{v} + \frac{8e\sqrt{m}}{\hbar l}\sqrt{(Z-2)}\sqrt{r_0}\left(1-\frac{\sigma}{2}\right), \quad (70)\dagger$$

or, putting in numerical values, as

$$\log_{10}\lambda = 21{\cdot}6693 - 1{\cdot}191\times 10^9\frac{Z-2}{v} + \\ + 4{\cdot}084\times 10^6\sqrt{(Z-2)}\sqrt{r_0}[1-0{\cdot}001j(j+1)], \quad (71)$$

corresponds to the empirical relation found between the decay constants and the energies (or velocities) of α-disintegration for different radioactive bodies by Geiger and Nuttall (p. 170). We see from (70) that $\log_{10}\lambda$ depends actually not only on the velocity of the ejected α-particle but also on the charge-number Z and the radius r_0 of the nucleus, and that therefore it cannot be represented on a two-dimensional graph. The reason why Geiger and Nuttall could get a smooth curve by plotting simply $\log_{10}\lambda$ against E (or v) is due to the fact that the variation of Z, E, and r_0 in a radioactive series is practically monotonic as we go down the series. We should expect certain anomalies in the Geiger–Nuttall graph at the points where this regularity breaks down. Fig. 29 shows that such deviations actually occur; for example, for AcX the point does not lie on the curve—and this is due to the fact that the velocity of the α-particles ejected by this element is smaller than for the previous element RaAc, whereas, generally, the velocity increases as we go down the series. The values of $\log_{10}\lambda$ as calculated from (70) for a chosen constant value of r_0 give already a good representation of the Geiger–Nuttall graph because the small changes in radius from element to element (entering only in the last term) only slightly affect the results; a much better approximation can be obtained, however, if we assume that the nuclear radius varies proportionately to the cube root of the mass number of the nucleus (see Chap. I).

† Here $l = 2{\cdot}303$ is the factor for converting natural logarithms to the base 10.

Using experimental data for decay constants and energies of disintegration we can calculate from (71) the values of

$$r_{\text{eff}} = r_0[1-0{\cdot}001j(j+1)]^2 \sim r_0[1-0{\cdot}002j(j+1)], \qquad (72)$$

which we may call the effective radius of the nucleus. The effective radius is the same as the true radius when the emitted α-particle

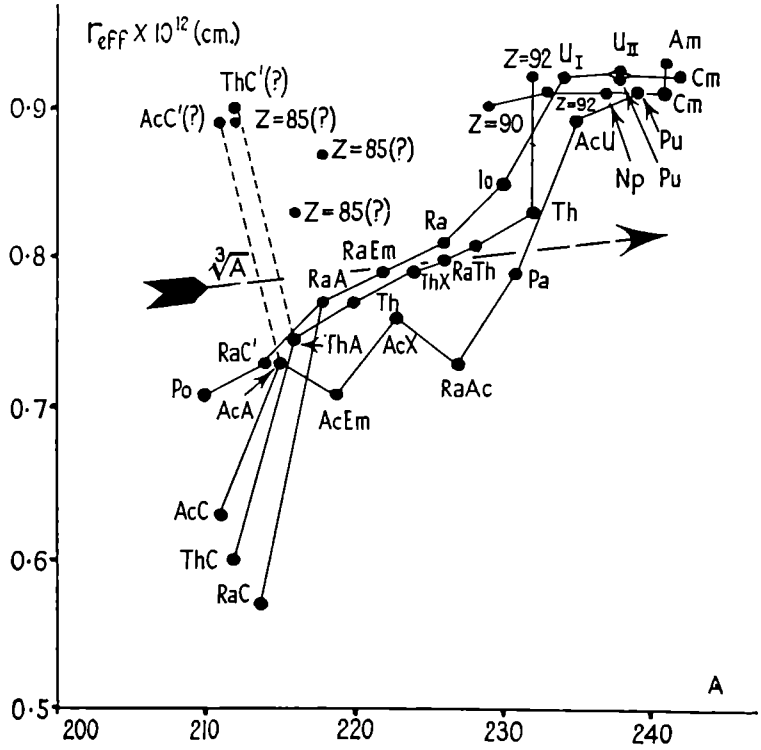


FIG. 30. Calculated nuclear radii of various α-emitters plotted as the function of atomic weight. The points for ThC′, AcC′ and the new element $Z = 85$ fall considerably off the general curve which may be due to unreliability of measurements. (It may be also possible that in these cases the values of λ_α or E_α were simply estimated by means of old Geiger–Nuttall curves.)

possesses no angular momentum; (72) shows that in other cases it is smaller than r_0. The values of the effective radii for all known α-emitters are given in Table XIV and are also plotted against the mass number in Fig. 30. We see from this figure that effective nuclear radii vary in general smoothly, increasing with the mass-number. However, the ratio is not strictly constant but increases by 15 per cent. from RaF to U, showing that in the region of radioactive elements we have a slight decrease of nuclear density with increasing mass; this effect may be connected with the formation of a new shell at this stage.

TABLE XIV

Calculated effective radii of α-decaying nuclei

4n-type		*4n+1 type*		*4n+2 type*		*4n+3 type*	
Nucleus	$r_{\text{eff}} \times 10^{12}$ *cm.*	*Nucleus*	$r_{\text{eff}} \times 10^{12}$ *cm.*	*Nucleus*	$r_{\text{eff}} \times 10^{12}$ *cm.*	*Nucleus*	$r_{\text{eff}} \times 10^{12}$ *cm.*
$_{92}U^{232}$	0·92	$_{96}Cm^{241}$	0·91	$_{96}Cm^{242}$	0·92	$_{94}Pu^{239}$	0·91
$_{90}Th^{232}$	0·83	$_{95}Am^{241}$	0·93	$_{94}Pu^{238}$	0·92	$_{92}AcU^{235}$	0·89
$_{90}RaTh^{228}$	0·81	$_{93}Np^{237}$	0·91	$_{92}U_{I}^{238}$	0·92	$_{91}Pa^{231}$	0·89
$_{88}ThX^{224}$	0·79	$_{92}U^{233}$	0·91	$_{92}U_{II}^{234}$	0·92	$_{90}RaAc^{227}$	0·73
$_{86}ThEm^{220}$	0·77	$_{90}Th^{229}$	0·90	$_{90}Io^{230}$	0·85	$_{88}AcX^{223}$	0·76
85^{216}	0·83			$_{88}Ra^{226}$	0·81	$_{86}AcEm^{219}$	0·71
$_{84}ThA^{216}$	0·75			$_{86}RaEm^{222}$	0·79	85^{211}	0·89
$_{84}ThC'^{212}$	0·90			85^{218}	0·87	$_{84}AcA^{215}$	0·73
$_{83}ThC^{212}$	0·60			$_{84}RaA^{218}$	0·77	$_{84}AcC'^{211}$	0·89
$_{62}Sm^{148}$	0·95			$_{84}RaC'^{214}$	0·73	$_{83}AcC^{211}$	0·63
				$_{84}Po^{210}$	0·71		
				$_{83}RaC^{214}$	0·57		

It is very important to notice that for all three C-products, as well as for most members of the actinium family, the values of effective radii are anomalously small. This, as was indicated by Gamow,† may be due to the fact that in these cases α-particles are emitted with angular momentum, different from zero. In order to explain the deviations observed (~ 5 per cent.) we should, according to formula (72), suppose that in those cases the angular momentum of the emitted particles is about 4 or 5 quantum units, or, in other words, that the spins of the disintegrating and product nuclei differ by about as much. We will see in the next chapter that the effect of the spin-change is of particular importance for the understanding of the phenomenon of the so-called 'fine structure' which is typical for C-products of all radioactive families.

In conclusion we may remark that there is known one α-decaying element which does not belong to regular radioactive families, and is located in the region of considerably smaller atomic numbers. This is the samarium isotope $_{62}Sm^{148}$ which forms about 14 per cent. of natural samarium mixture. The α-activity of Sm, which was first discovered by Hevesy and later studied in some detail by Wilkins and Dempster,‡ represents an interesting example of irregularities in

† G. Gamow, *Nature*, **129** (1932), 470.

‡ T. Wilkins and A. Dempster, *Phys. Rev.* **54** (1938), 315.

energy-balance caused by the formation of nuclear shells. The radius of Sm-nucleus, as calculated from the known values of λ and $E\alpha$, comes out to be $0{\cdot}95\times10^{-12}$ cm. as compared with $0{\cdot}70\times10^{-12}$ cm. which should follow from the assumption of constant nuclear density.

Spontaneous emission of nuclear fragments heavier than α by various radioactive elements is also an energetically possible process (cf. Fig. 20), and the failure to observe the emission of C-particles (Carbon nuclei) or O-particles in spontaneous decay of heavy elements must be ascribed entirely to the low probability of such processes. To illustrate this statement let us calculate, for example, the probabilities of the following spontaneous disintegrations:

$$_{90}\mathrm{Io}^{230} \rightarrow {}_{84}\mathrm{RaA}^{218} + {}_{6}\mathrm{C}^{12} \tag{73}$$

$$_{90}\mathrm{Io}^{230} \rightarrow {}_{82}\mathrm{RaB}^{214} + {}_{8}\mathrm{O}^{16}. \tag{74}$$

The energy of emission in these two examples can be calculated as the difference between the sum of α-decay-energies between the initial and the final nucleus, and the binding energy of the emitted particles relative to He atoms (i.e. a negative energy). In the first case we have:

$$E_{\mathrm{C}} = [4{\cdot}54+4{\cdot}79+5{\cdot}49+7{\cdot}44]\ \text{M.e.v.} = 22{\cdot}3\ \text{M.e.v.}$$

and in the second:

$$E_{\mathrm{O}} = [4{\cdot}54+4{\cdot}79+5{\cdot}49+6{\cdot}11+14{\cdot}04]\ \text{M.e.v.} = 35{\cdot}0\ \text{M.e.v.}$$

Using these values for the energies, along with the corresponding values of masses and charges of the emitted particles, we easily obtain the pertinent decay-constants:

$$\lambda_{\mathrm{C}}(\mathrm{Io}) \cong 10^{-125}\ \text{sec.}^{-1} (T \cong 10^{118}\ \text{years})$$

$$\lambda_{\mathrm{O}}(\mathrm{Io}) \cong 10^{-153}\ \text{sec.}^{-1} (T \cong 10^{146}\ \text{years}).$$

The fact that there are only 10^{-112} carbon and 10^{-140} oxygen nuclei emitted per one α-particle easily explains the failure to observe the emission of the heavier nuclei.

Spontaneous fission of the U-nucleus has been mentioned already in § 1 as first observed by Petrzhak and Flerov, who estimate that there is one fission process per million α-decays; thus the value of λ_{fls} is about 5×10^{-24} sec.$^{-1}$ Theoretical calculation of the fission process was first attempted by Bohr and Wheeler (loc. cit.) and was later elaborated by Present, Reines, and Knipp.† The fission process is so complex dynamically, however, that the theories are not adequate to make quantitative predictions (cf. App. IV).

† R. D. Present, F. Reines, and J. K. Knipp, ibid. **70** (1946), 557.

VII

ELECTROMAGNETIC RADIATION OF NUCLEI

1. γ-spectra and internal conversion

THE atomic nucleus, being a quantized system, is in general capable of existence in any one of a number of states of different energy. In its various excited states it may be characterized by definite probabilities of transition to the various states which have smaller energy than the state in question: when such a transition occurs it will usually be with the emission of the excess energy in the form of a quantum of electromagnetic radiation. This is the formal description of those nuclear processes which are involved in the emission, in spontaneous and artificial transformations of nuclei, of a quantum radiation to which the name γ-radiation is generally given.

In so far as the energy-differences between nuclear levels are in general much larger than the corresponding differences for the extra-nuclear system, the wave-lengths of γ-rays are much shorter than the wave-lengths of atomic radiations and the ordinary methods of spectroscopy using ruled gratings or crystals are hardly suited to them. When these methods have been used—as originally by Rutherford and Andrade—they have been successful only for the less energetic components of the radiation. The usual method of γ-ray spectroscopy, worked out independently by Ellis and Meitner, is based on the phenomenon of 'internal conversion' of γ-rays. This phenomenon can be regarded, to some extent, as a special case of the photoelectric effect, in that the γ-ray quantum is absorbed in the same atom from the nucleus of which it has just been emitted. As in the external photoelectric effect an electron is thereby ejected from one of the atomic levels. That this description of the phenomenon is imperfect need not concern us here, but we shall see later that there is also another possibility—the direct mechanical transmission of energy from the excited nucleus to one of the atomic electrons. Because the internal photoelectric effect may happen for any electronic shell in the atom, there will arise, for every line in the γ-ray spectrum of the nucleus, a number of secondary electronic groups having energies $h\nu - E_K$, $h\nu - E_{L_I}$, $h\nu - E_{L_{II}}$, $h\nu - E_{L_{III}}$, $h\nu - E_{M_I}$,..., where E_K, E_{L_I}, $E_{L_{II}}$, $E_{L_{III}}$, E_{M_I},... are the binding energies corresponding to the different electron levels and ν is the

frequency of the γ-ray line. Now it has been shown that the γ-radiation associated with any particle-disintegration of a radioactive nucleus is in fact characteristic of the excited product of the disintegration; the binding energies above used must thus be understood to refer to this nucleus and not to its immediate parent. The method of Ellis and Meitner, therefore, consists in determining the energies of the secondary electron groups† by magnetic analysis and applying the above relations to deduce the energies and so the frequencies in terms of which the primary γ-ray spectra are to be specified. As an example of the numerical procedure, the table below contains the results of the work of Meitner on the natural β-ray spectrum of actinium X and its analysis in terms of the γ-rays which are emitted from the nuclei left excited by the previous α-disintegration, $AcX \xrightarrow{\alpha} An$.

TABLE XV

Analysis of secondary β-ray spectrum emitted by AcX ($Z = 88$)

Energy of sec. β-group $\times 10^6$ *erg*	*Intensity*	*Level of origin for* $Z = 86$	*Energy of the level* $\times 10^6$ *erg*	*Energy of the γ-ray* $\times 10^6$ *erg*	*Deviations per cent.*
0·0723	80	K	0·1554	0·2277	0·6
0·2003	50	L_I	0·0286	0·2289	
0·2204	25	M_I	0·0071	0·2275	
0·0889	100	K	0·1554	0·2443	0·6
0·2159	60	L_I	0·0286	0·2445	
0·2358	15	M_I	0·0071	0·2429	
0·0950	40	K	0·1554	0·2504	1·0
0·2204	25	L_I	0·0286	0·2490	
0·2407	15	M	0·0071	0·2478	
0·1658	40	K	0·1554	0·3212	2·0
0·2860	15	L_I	0·0286	0·3146	
0·2717	100	K	0·1554	0·4271	0·2
0·3990	30	L_I	0·0286	0·4276	

As has been stated already, the phenomenon of internal conversion of γ-rays consists in the emission of an extranuclear electron in place of a nuclear γ-quantum. It would be erroneous, however, to consider this phenomenon entirely as a two-step process consisting of the emission of a γ-quantum by the nucleus and the subsequent 'internal' photo-effect in the electronic envelope of the same atom. In fact, one must not forget that, due to the finite values of electronic wave-functions in

† These may easily be distinguished from the distribution of primary β-particles, which have all energies between wide limits (see Chap. V).

the region occupied by the nucleus, one must also take into account the possibility of a direct transfer of energy from the excited nucleus to an atomic electron. Transitions of that kind are analogous to collisions of the second kind investigated theoretically by Klein and Rosseland. Since in practice there is no means of separating the effects of these two processes of absorption, it is more reasonable to consider the phenomenon of internal conversion as a single process of transmission of the energy of nuclear excitation to the atomic electrons through the action of extra- and intranuclear fields of force. Only in the special case ($i = 0 \to i = 0$) is this action confined to one only of these fields; in this case the periodic components of the extranuclear electromagnetic fields vanish absolutely, γ-radiation is completely forbidden, and internal conversion is due entirely to the interaction of atomic electrons and the excited nucleus in the region 'within the nucleus'.

In the general case the total nuclear transition probability may be written as the sum of two terms, for dipole radiation, in the above notation, as $\kappa(\omega)+\mu(\omega)$, where $\mu(\omega)$ denotes the probability of the ejection of an atomic electron simultaneously with the nuclear energy-change. In so far as the secondary electron may originate in any one of the extranuclear levels of the atom, $\mu(\omega)$ may be written in the form $\mu(\omega) = \mu_K(\omega)+\mu_{L_I}(\omega)+\ldots$, where successive terms on the right-hand side refer to successive levels in order of decreasing energy of binding. The ratio of the number of emitted electrons to the total number of nuclear transitions involving the energy-quantum $\hbar\omega$ is defined as the coefficient of internal conversion for nuclear radiation of the corresponding frequency. We have

$$\alpha(\omega) = \mu(\omega)/\{\kappa(\omega)+\mu(\omega)\} \quad \text{or} \quad \mu(\omega)/\kappa(\omega) = \alpha(\omega)/\{1-\alpha(\omega)\}. \quad (1)$$

Again, $\alpha(\omega)$ is the sum of a number of terms, $\alpha_K(\omega)$, $\alpha_{L_I}(\omega),\ldots$, corresponding to the various μ's; these terms may be regarded as giving the internal-conversion coefficients relative to specified electron levels in the atom. In any case the value of the internal-conversion coefficient depends upon the atomic electron level involved and upon the symmetry characteristics of the nuclear transition in question.

Applying standard methods of wave-mechanical perturbation theory the probability $\mu(\omega)$ may be calculated from the equation

$$\mu(\omega) = \frac{2\pi}{\hbar}\sum\left|\int \psi_f^*[eA_0+e\rho_3\boldsymbol{\sigma}\mathbf{A}]\psi_0\, d\tau\right|^2, \quad (2)$$

where ψ_0 and ψ_f are the wave-functions describing the orbital and

free motion of the electron, A_0 and $\mathbf{A}$ the scalar and vector potentials of the radiative field of the nucleus, and ρ_3 and $\boldsymbol{\sigma}$ are the matrices in Dirac's theory. For the purpose in hand the relativistic form of the equation for the electron must be used. Calculations on the basis of (2) were carried out by Hulme† on the assumption that the

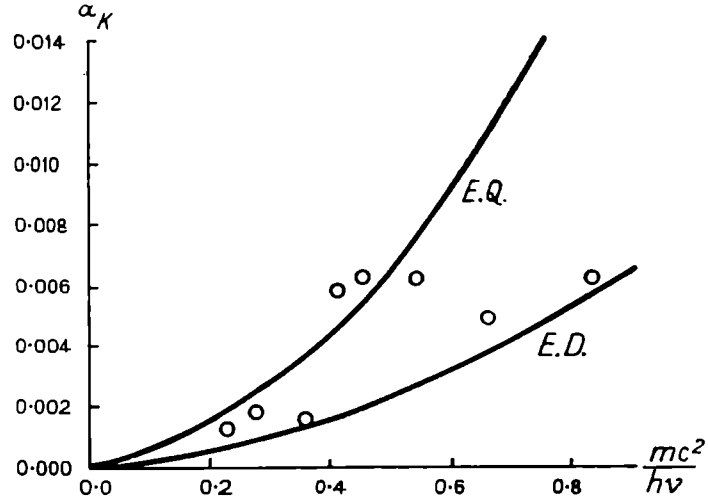


FIG. 31.

electromagnetic field of the nucleus is that of a dipole situated at its centre. This field may be represented by the potentials

$$\begin{aligned} A_0 &= B\frac{i}{r}\exp\left\{2\pi i\left(\frac{r}{\Lambda}-\nu t\right)\right\}\cos\theta\left(1+\frac{1}{qr}\right)+\text{complex conjugate},\\ A_z &= B\frac{i}{r}\exp\left\{2\pi i\left(\frac{r}{\Lambda}-\nu t\right)\right\}, \qquad (3)\\ A_x &= A_y = 0, \end{aligned}$$

with $$q = \frac{2\pi\nu}{c} = \frac{\omega}{c}.$$

Having obtained $\mu(\omega)$ as above described, $\kappa(\omega)$ is given by dividing the rate of radiation of energy by the field by $\hbar\omega$, the quantum energy of the radiation. As is well known, the rate of radiation of energy by the field specified by (3) is $\frac{4}{3}\frac{B^2\omega^2}{c}$, thus $\kappa(\omega) = \frac{4}{3}\frac{B^2\omega}{\hbar c}$. From $\mu(\omega)$ and $\kappa(\omega)$ the internal-conversion coefficient $\alpha(\omega)$ is derived by equation (1). The values obtained by Hulme in this way for the internal-conversion coefficients relative to K-shell absorption in the RaC′ atom ($Z = 84$) are plotted as a function of $mc^2/\hbar\omega$ in Fig. 31

† H. R. Hulme, *Proc. Roy. Soc.* **138** (1932), 643. Previous calculations by Swirles and Casimir were based on much less exact approximations; for that reason they will not be further discussed at this stage.

(curve *ED*). The experimental values, also given in the following table, are those of Ellis and Aston† for the eight most intense γ-rays

TABLE XVI

Internal conversion of γ-rays of RaCC′ *and* RaBC

(RaCC′)			(RaBC)		
$h\nu$ $\times 10^6$ *e.v.*	α_K	*Number of γ-quanta per disintegration*	$h\nu$ $\times 10^6$ *e.v.*	α_K	*Number of γ-quanta per disintegration*
0·612	0·0061	0·658	0·243	0·364	0·115
0·773	0·0048	0·065	0·297	0·186	0·258
0·941	0·0061	0·067	0·354	0·117	0·450
1·130	0·0062	0·206			
1·248	0·0057	0·063			
1·390	0·0014	0·064			
1·426	> 0·1 (= 1?)	.. (= 0?)			
1·778	0·0016	0·258			
2·219	0·0013	0·074			

emitted by this nucleus. Five of these points fit with fair accuracy on the theoretical curve for dipole radiation. The three points which do not fit represent considerably larger internal-conversion coefficients: as we shall see later they must be explained on the basis of nuclear quadrupole radiation. Internal-conversion coefficients relative to the electronic L levels were also evaluated by Hulme, with the following result. In the limiting case, $\hbar\omega \to \infty$, for RaC′,

$$\alpha_K : \alpha_{L_I} : \alpha_{L_{II}} : \alpha_{L_{III}} = 1 : 0{\cdot}149 : 0{\cdot}0013 : 0{\cdot}0066.$$

This is in satisfactory agreement with the values experimentally obtained.

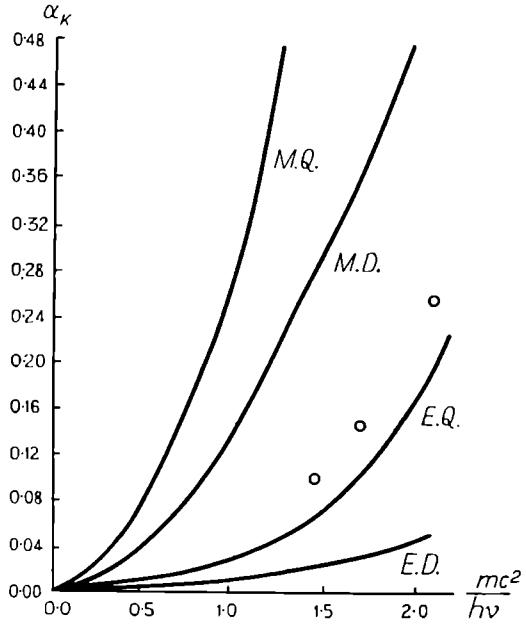


FIG. 32.

Fig. 32 gives the internal conversion coefficients for somewhat softer γ-rays. They have been calculated for $Z = 83$ in order to be compared with the values of

† E. D. Ellis and G. H. Aston, *Proc. Roy. Soc.* **129** (1930), 180.

Ellis and Aston for the γ-rays emitted after the β-particle change RaB $\rightarrow$ RaC. The experimental values from the above table are to be considered in relation to curve *ED*. It will be seen that experimental values are all considerably greater than those given by this curve, which may be taken to prove that the nuclear transitions in question cannot be described by a dipole field.

In order to account for the large values found for the internal-conversion coefficient in such cases, Taylor and Mott† carried out the calculations for quadrupole radiation, in terms of the potentials

$$A_0 = -C\frac{1}{r}\exp\left\{2\pi i\left(\frac{r}{\Lambda}-\nu t\right)\right\}\left\{2P_2(\cos\theta)\left[1+\frac{3i}{qr}-\frac{3}{q^2r^2}\right]+1\right\}+$$

$$+\text{complex conjugate},$$

$$A_z = -3C\frac{1}{r}\exp\left\{2\pi i\left(\frac{r}{\Lambda}-\nu t\right)\right\}\cos\theta\left(1+\frac{i}{qr}\right)+\text{complex conjugate},$$

$$A_x = A_y = 0. \qquad (4)$$

In this case $\kappa(\omega) = \frac{12}{5}\frac{C^2\omega}{\hbar c}$ γ-quanta per sec. The curves *EQ* of Figs. 31 and 32 give the results of these calculations. It is evident that these curves go some way towards explaining the experimental results, although the agreement for the soft γ-rays of RaB–C is still far from exact. This may be due in part to the lower accuracy of the calculations for γ-rays of small quantum energy and to the neglect of the screening effect of the atomic electrons. It may be noticed here that apart from the electric dipole and quadrupole radiation discussed above there is also the possibility of *magnetic* dipole and quadrupole radiation due to oscillations of the magnetic moments of the system. Taylor‡ has shown that if some of the nuclear γ-rays have a magnetic origin the coefficients of internal conversion must be considerably larger than is the case for ordinary electric radiation. The theoretical values of these coefficients, in the case of magnetic radiation, are shown in Fig. 32 by curves *MD* and *MQ*. We must notice, however, that since the magnetic radiation possesses, in general, very small probability, it has to be taken into account only if electric dipole and quadrupole transitions are not permitted.

The calculations of Hulme and of Taylor and Mott are based on the hypothesis of dipole and quadrupole moments localized in

† H. M. Taylor and N. F. Mott, ibid. **138** (1932), 665.
‡ H. M. Taylor, ibid. **146** (1934), 178.

extremely minute regions in the centre of the nucleus. By considering the effect of the finite size of the nucleus and the possibility that the expressions for the various potentials, (3) and (4), may require modification for values of r less than the nuclear radius, Fowler† was able to show, however, that the value of the integral (14) changes but slightly if these deviations take place inside the region $r \sim 10^{-12}$ cm. Only in the case of the prohibited radiative transition ($i = 0 \rightarrow i = 0$) do such considerations become all-important. In this case the direct interaction of an atomic electron with the excited nucleus represents the one method of the release of energy. The probability of this process may be evaluated as to order of magnitude from the perturbation formula

$$\mu' \sim \frac{2\pi}{\hbar}[\tfrac{4}{3}\pi r_0^3 \overline{V}\psi_0(0)\psi_f(0)]^2, \tag{5}$$

where $\overline{V}$ is the average interaction potential between electron and nucleus for values of the separation of the order of r_0. Using the relativistic expressions for the wave-functions, we obtain, for a transition energy of 10^6 e.v. communicated to a K-electron in an atom of RaC′, $\mu' \sim 10^{11}$ sec.$^{-1}$ If the interaction is greater than has been supposed much larger values of μ' become possible. These values are of the same order as those calculated above for quadrupole radiative transitions of the nucleus; they indicate that, whenever the mean life of an excited nucleus is of the order of 10^{-12} to 10^{-11} sec., internal conversion by direct interaction is an important mode of transference of energy. At present the only well-established case of non-radiative nuclear transition is that associated with the quantum energy $1{\cdot}414 \times 10^6$ e.v. in RaC–C′. Secondary electron groups corresponding to this quantum are particularly intense, but no evidence of the corresponding γ-ray has been found in emission. Before the above simple explanation of the results was advanced it was necessary to assume that for this particular radiation an internal-conversion coefficient of unity was applicable—one formed the strange picture of the K-electrons absorbing a large fraction of the outgoing quanta and of the remaining electrons being able completely to absorb the rest. Secondary electron groups were observed corresponding to absorption in the K, L, and M levels, respectively. Assuming the correctness of the present explanation, that we are dealing with a

† R. H. Fowler, *Proc. Roy. Soc.* **129** (1930), 1; see also M. Delbrück and G. Gamow, *Zs. f. Phys.* **72** (1931), 492.

non-radiative nuclear transition, very important conclusions may be drawn. We conclude that the two nuclear levels involved are both S levels, in other words that these two states of excitation possess zero spin. Moreover, as will appear later, the observed correlation between γ-ray energies and the energies of the long-range α-particles from RaC′ indicates that the transition in question takes place between one of the excited states of this nucleus and the ground state. We conclude, therefore, that in its ground state the nucleus RaC′ has no spin: $i = 0$.

In calculations of the internal conversion coefficients for the γ-rays emitted by natural radioactive elements we must use the exact expression for the motion of the orbital electron, and the final results (such as those shown in Figs. 31, 32) can be obtained only by means of rather lengthy numerical computations. In the case of lighter elements (internal conversion of γ-rays produced in artificial transformations) the problem can be solved purely analytically, and permits a more general study of the dependence of the internal conversion coefficient on the multipole nature of the transition. Calculations of that kind have been performed by Dancoff and Morrison† (for K-shell conversion) and by Hebb and Nelson‡ (for L-shell conversion). For the elements with $Z < 30$ (non-relativistic K-electron) and for γ-ray energies much higher than the binding energy of the K-shell, one obtains for the electric and magnetic 2^l-pole transitions the following expressions for the K-conversion coefficients:

$$\alpha_{\substack{K,l \\ \text{(electric)}}} = \frac{2Z^3\alpha}{\gamma^3}\left(\frac{\gamma+2}{\gamma}\right)^{l-\frac{1}{2}}\left[\frac{(l+1)\gamma^2+4l}{l+1}\right], \tag{6}$$

and

$$\alpha_{\substack{K,l \\ \text{(magnetic)}}} = \frac{2Z^3\alpha^4}{\gamma}\left(\frac{\gamma+2}{\gamma}\right)^{l+\frac{1}{2}}, \tag{7}$$

where $\gamma = h\nu/mc^2$ and α is the fine-structure constant. Similar, though somewhat longer, expressions are obtained for the conversion of the electric and magnetic multipole radiations in the L-shell.

Comparison of the observed conversion coefficients with the theoretical formulae affords a very convenient method for determining the changes in nuclear angular momentum that are associated with the emission of different γ-lines and for assigning azimuthal quantum numbers to the corresponding energy-levels.

† S. M. Dancoff and P. Morrison, *Phys. Rev.* **55** (1939), 122.
‡ M. H. Hebb and E. Nelson, ibid. **58** (1940), 486.

It was pointed out by Oppenheimer and Nedelskij† that a nucleus that is excited by more than 1 M.e.v. can dispose of its energy of excitation not only by direct γ-ray emission, or by the ejection of one of the atomic electrons, but also by creating a positron-electron pair in the nuclear electric field. Calculation of the probability of this process, known as *internal pair-formation*, can be reduced to the evaluation of the matrix element of the perturbation due to radiation. One can write Dirac's relativistic equation for an electron in a Coulomb field with the nuclear charge $\alpha^{\frac{1}{2}}Z$ (α being the fine-structure constant) in the form

$$i\frac{\partial\psi}{\partial t} = \left\{(\boldsymbol{\alpha}.\mathbf{p})+\beta+\frac{\alpha Z}{r}-\alpha^{\frac{1}{2}}\mathfrak{H}\right\}\psi, \tag{8}$$

where $\mathfrak{H}$ is the perturbation of the radiation field determined by the scalar and vector potentials through the relations

$$\begin{aligned}\mathfrak{H} &= He^{-ikt}+\text{complex conjugate}\\ H &= V+(\boldsymbol{\alpha},\mathbf{A}).\end{aligned} \tag{9}$$

The potentials V and A are given by:

dipole
$$\begin{cases} V = \dfrac{1}{r}e^{ikr}\left(i-\dfrac{1}{kr}\right)\cos\theta,\\[2ex] A_z = \dfrac{i}{r}e^{ikr}; \qquad A_x = A_y = 0, \end{cases} \tag{10}$$

quadrupole
$$\begin{cases} V = \dfrac{1}{r}e^{ikr}\left\{2P_2(\cos\theta)\left(1+\dfrac{3i}{kr}-\dfrac{3}{k^2r^2}\right)+1\right\},\\[2ex] A_z = \dfrac{3}{r}e^{ikr}\left(1+\dfrac{i}{kr}\right)\cos\theta; \qquad A_x = A_y = 0, \end{cases} \tag{11}$$

where all lengths are expressed in terms of the Compton wavelength $\hbar/mc$.

Applying the usual method of the variation of constants, the solutions of eq. (8) can be expressed in terms of probability amplitudes, $a_n(t)$, of the various states. At the time $t = 0$ the electron is in the negative energy-state, W_0, so that $a_n(0) = \delta_{n0}$. One finds from (8) the probability $|a_n|^2$ that the electron will be in the state W_n at the time t:

$$|a_n|^2 = 4\alpha|(n|H|0)|^2\frac{\sin^2(W_n-W_0-k)\frac{1}{2}t}{(W_n-W_0-k)^2}, \tag{12}$$

† J. R. Oppenheimer and L. Nedelskij, *Phys. Rev.* **44** (1933), 986.

where all energies are expressed in units of mc^2 and Dirac's notation for matrix elements is used. It remains to sum $|a_n|^2$ over the spins, energies, and directions of the positive and negative electrons formed in the process.

If one uses the exact solutions of Dirac's equation for an electron the computations can be performed only by means of numerical integration. Exact calculations of that kind have been carried out by Jaeger and Hulme,† whose results for the total conversion-coefficient, α_p, for pair-formation in the case of dipole and quadrupole transitions

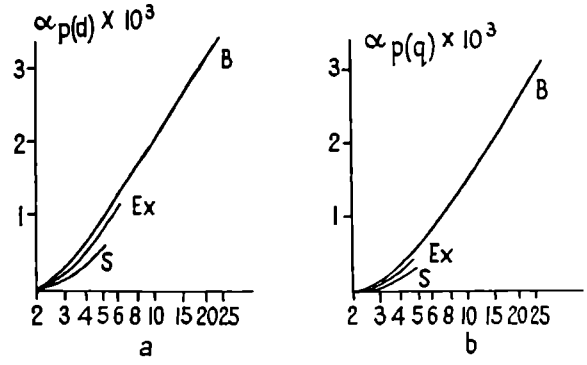


FIG. 33. The coefficient of internal pair formation calculated for dipole (*a*) and quadrupole (*b*) transitions.

are shown graphically in Fig. 33 *a*, *b* (curves marked *Ex*). One can obtain reasonably close analytical expressions for the conversion probabilities also by using one of the standard approximation methods of the perturbation theory, as was done by Rose and Uhlenbeck‡ using two different methods of approximation (Born's and Schrödinger's). These authors obtained the following expressions for the conversion coefficients:

$$\text{Born's method} \quad \begin{cases} \alpha_{p(\alpha)}(W_+) = \dfrac{\alpha}{\pi\kappa^3}\{2p_+p_- + (W_+^2 + W_-^2)\ln b\}, \\ \alpha_{p(q)}(W_+) = \dfrac{\alpha}{3\pi\kappa^5}\{8p_+p_-(W_+W_- - 1) + \\ \qquad\qquad + 3\kappa^2(p_+^2 + p_-^2)\ln b\}, \\ b = \dfrac{1 + W_+W_- + p_+p_-}{\kappa}; \end{cases} \tag{13}$$

† J. C. Jaeger and H. R. Hulme, *Proc. Roy. Soc.* **148** (1935), 708.
‡ M. E. Rose and G. E. Uhlenbeck, *Phys. Rev.* **48** (1935), 211.

$$\text{Schrödinger's method} \begin{cases} \alpha_{p(\alpha)}(W_+) = \dfrac{48\pi\alpha^3 Z^2 W_+ W_-}{\kappa^5(e^{2\pi\alpha Z/p_+}-1)(1-e^{-2\pi\alpha Z/p_-})}, \\ \alpha_{p(q)}(W_+) = \dfrac{20\pi\alpha^3 Z^2}{3\kappa^5}\left(1-\dfrac{4}{\kappa}+\dfrac{12}{\kappa^2}\right)\times \\ \qquad\qquad \times \dfrac{W_+ W_-(p_+^2+p_-^2+2\alpha^2 Z^2)}{(e^{2\pi\alpha Z/p_+}-1)(1-e^{-2\pi\alpha Z/p_-})}, \end{cases} \tag{14}$$

where W_+, W_-, p_+, p_- stand for energy (in units of mc^2) and momentum (in units of mc) of the positive and negative electrons. Integrating graphically for all possible partitions of energy between the two electrons one obtains the values of total conversion coefficients shown by the curves marked *B* and *Sh* in Fig. 33. We see that the analytical expressions (13) and (14) lead to fairly good agreement with the results of the exact numerical calculations.

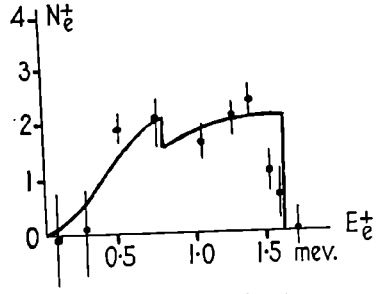


FIG. 34. Energy distribution of positive electrons from Th(B+C+C″) source. Points —observed values: solid line— theory.

The phenomenon of 'internal pair-formation' was first observed by Chadwick, Blackett, and Occhialini† in the case of a Th(B+C+C″) source, and was reported later by many authors for other radioactive sources possessing strong γ-radiation.‡ The most detailed study of this process was carried out by Bradt, Halter, Heine, and Scherrer,§ who have measured the total number as well as the energy distribution of positive electrons emitted from a Th(B+C+C″)-source. This was done with a magnetic β-spectrograph. Their measurements lead to the value of $2{\cdot}3\times10^{-4}$ pairs per decaying ThB-nucleus; the observed energy-distribution of the emitted positive electrons being shown graphically in Fig. 34. In order to find the theoretically expected number of positive electrons, one must remember that the source used in these experiments possesses two strong γ-lines capable of producing electron pairs. These are the quadrupole line at 2·62 M.e.v. belonging to the transition ThC″ → ThD and the weaker dipole line at 1·802 M.e.v. belonging to ThC → ThC′. Using the theory

† J. Chadwick, P. M. S. Blackett, and G. P. S. Occhialini, *Proc. Roy. Soc.* A. **144** (1934), 235.

‡ For a complete list of references comparo H. Bradt, H. G. Heine, and P. Scherrer, *Helvetica Physica Acta*, **14** (1943), 492.

§ H. Bradt, J. Halter, H. G. Heine, and P. Scherrer, ibid. **19** (1946), 431.

for internal pair-formation described above, we calculate the expected number of electron pairs produced by these two γ-rays to be 2×10^{-4} pairs per ThB disintegration, in excellent agreement with the experiment results. The energy distribution expected theoretically is shown by the solid line in Fig. 34 and is also in excellent agreement with the results of measurement.

It may be mentioned in conclusion that the phenomenon of internal pair-formation by an excited nucleus may be closely related to some of the theories of the nature of nuclear forces discussed in Chapter III. In fact, the experiments of Conversi, Pancini, and Piccioni† on the nuclear capture of cosmic-ray mesons and the theory of this process as given by Fermi, Teller, and Weisskopf‡ and by Wheeler,§ seem to indicate that the interaction between charged mesons and the nucleus is too small to make these particles responsible for the nuclear forces. Alternative possibilities are to ascribe nuclear forces to an exchange of neutral mesons or, to the similar process, the exchange of electron pairs. It was pointed out by Gamow‖ that if nuclear forces involve the exchange of electron pairs one should expect that an excited nucleus will emit pairs through this mechanism also and that the probability of such an emission may be comparable to, if not larger than, the probability of ordinary pair-formation.

2. Emission probabilities and selection rules

If we attempt to treat electromagnetic radiation emitted by a nucleus in a way similar to that used in the case of an atom, i.e. by ascribing it to the motion of a single, charged particle (proton, α-particle, . . .), the emission probability of the γ-quantum can be calculated in the conventional, simple way. The dipole transition probability from a state m to a state n of the radiating system will be given by the familiar formula:

$$\alpha_{\gamma(d)} = \frac{\Gamma_{\gamma(d)}}{\hbar} = \frac{4}{3}\frac{\omega^3}{3\hbar c^3}|M_{mn}|^2, \tag{15}$$

where Γ_γ is the so-called *radiation-width* for a transition from one level to one other energy-level, ω the 'barred' frequency ($\hbar\omega = h\nu$)

† M. Conversi, E. Pancini, and O. Piccioni, *Phys. Rev.* **71** (1947), 29; see also T. Sigurgeirson and A. Yamakawa, ibid. p. 319.

‡ E. Fermi, E. Teller, and V. Weisskopf, ibid. p. 314.

§ J. A. Wheeler, ibid. p. 320.

‖ G. Gamow, ibid. p. 550.

of the emitted radiation, and M_{mn} is the matrix element of the electric dipole moment for the transition. For dipole radiation due to the motion of a single proton one can write

$$|M_{mn}|^2 = \frac{e^2\hbar}{2M\omega} f_{mn}, \tag{16}$$

where f_{mn} is the so-called 'oscillator strength' which is expected to be of the order of unity for strong transitions, as in the case of atomic radiations. From (15) and (16) we obtain:

$$\alpha_{\gamma(d)} = \frac{\Gamma_{\gamma(d)}}{\hbar} = \frac{2e^2\omega^2}{3Mc^3} f_{mn}, \tag{17}$$

or numerically, expressing $\hbar\omega$ in M.e.v.:

$$\alpha_{\gamma(d)} = \frac{\Gamma_{\gamma(d)}}{\hbar} = 8{\cdot}0 \times 10^{15} (\hbar\omega)^2 f_{mn} \text{ sec.}^{-1} \tag{18}$$

In the case of quadrupole emission, the probability is expected to be reduced by a factor $(r_0/\lambda\!\!\!^{-})^2$, where r_0 is the radius of the emitting system (nuclear radius) and $\lambda\!\!\!^{-}$ the 'barred' wave-length defined by $\lambda\!\!\!^{-} = \lambda/2\pi$.

In general, for a 2^l-pole radiation, the emission probability will be given roughly by the expression (calculated for the liquid droplet model by Lowen†)

$$\alpha_{\gamma(l)} = \frac{3}{4} \frac{Z^2e^2}{c} \frac{l+1}{AMr_0^2} \frac{(r_0\,\omega/c)^{2l}}{1^2 3^2 \ldots (2l+1)^2}. \tag{19}$$

In relating the order l of the emitted radiation to the change Δi of the nuclear angular momentum, one must take into account also the parity of the nuclear states in question. From the reflection properties of the vector potential, we can see that the electric 2^l-poles have parity $(-1)^l$ the magnetic $(-1)^{l+1}$. Thus, to a given parity change and Δi there corresponds a minimum electric and minimum magnetic multipole, as shown in Table XVII.

TABLE XVII

Minimum allowed multipoles for given parity change and Δi

Parity change	*Minimum allowed multipole*	
	for Δi even	for Δi odd
Even	electric $2^{\Delta i}$-pole magnetic $2^{\Delta i+1}$-pole	electric $2^{\Delta i+1}$-pole magnetic $2^{\Delta i}$-pole
Odd	electric $2^{\Delta i+1}$-pole magnetic $2^{\Delta i}$-pole	electric $2^{\Delta i}$-pole magnetic $2^{\Delta i+1}$-pole

† I. S. Lowen, *Phys. Rev.* **59** (1941), 835.

Turning now to the observed emission probabilities, we notice first of all that in the case of nuclear γ-spectra the dipole and quadrupole transitions seem to possess comparable intensities. In fact, looking through Table XVI of the preceding section, we fail to notice any systematic difference between the intensities of γ-lines that, according to their conversion coefficients, have sometimes to be classed with dipole transitions and sometimes with quadrupole transitions. Since the theoretically predicted ratio of intensities for the two types of transitions, in this region of energy ($h\nu \simeq 1$ M.e.v.; $\lambda\!\!\!^{-} \simeq 2\times 10^{-11}$ cm.) is $(10^{-12}/2\times 10^{-11})^2 = \frac{1}{400}$, we conclude that for some reason the probability of nuclear dipole radiation is reduced so drastically as to be brought down to the same order as the probability of nuclear quadrupole radiation.

We come to the same conclusion on the basis of studies of the experimental results on the absolute probability of nuclear γ-emission. As first indicated by Delbrück and Gamow,† these absolute magnitudes for transition probabilities can be found readily from a study of the so-called 'long-range' α-particles which are emitted by certain radioactive elements. When the nucleus of a natural α-emitter is excited as the result of the transformation in which it was produced, the excess energy may be released either in the form of a γ-quantum, or else it can be given to the α-particle which then leaves the nucleus with a higher speed ('long-range'). The relative frequencies of the two processes are governed obviously by the relative probabilities of γ-emission and of α-particle emission with the surplus energy. In the case of α-emitters which already possess very short lives for normal α-decay, the probabilities of long-range α-emission may be expected to be extremely short and to compete successfully with the probabilities of γ-ray emission. Such long-range groups of α-particles have been actually observed in the case of the shortlived α-emitters, RaC′ ($\lambda \simeq 10^3$ sec.$^{-1}$) and ThC′ ($\lambda \simeq 10^6$ sec.$^{-1}$);‡ their energies and relative numbers are listed in Tables XVIII and XIX. We see, for example, that in the case of the first long-range group (α_{I}) of RaC′ the probability of α-emission is $0{\cdot}43\times 10^{-6}$ per disintegration. On the other hand, the study of the absolute intensities of the γ-rays emitted by this element indicates that the number of γ-quanta of

† M. Delbrück and G. Gamow, *Zs. f. Phys.* **72** (1931), 492.

‡ E. Rutherford, F. Ward, and W. Lewis, *Proc. Roy. Soc.* A **131** (1931), 684; **142** (1933), 347.

TABLE XVIII

'Long-range' particles of RaC'

Name of the group	*Energy of disintegration* $\times 10^{-6}$ *e.v.*	*Energy-difference* $\times 10^{-6}$ *e.v.*	*Number of particles per million disintegrations*
α_0	7·829	..	10^6
α^{I}	8·437	0·608	0·43
α^{II}	9·112	1·283	0·45 (?)
α^{III}	9·242	1·412	22·0
α^{IV}	9·493	1·663	0·38
α^{V}	9·673	1·844	1·35
α^{VI}	9·844	2·015	0·35
α^{VII}	9·968	2·139	1·06
α^{VIII}	10·097	2·268	0·36
α^{IX}	10·269	2·440	1·67
α^{X}	10·342	2·513	0·38
α^{XI}	10·526	2·697	1·12
α^{XII}	10·709	2·880	0·23

TABLE XIX

'Long-range' particles of ThC'

Name of the group	*Energy of disintegration* $\times 10^{-6}$ *e.v.*	*Energy-difference* $\times 10^{-6}$ *e.v.*	*Number of particles per million disintegrations*
Normal α_0	8·948	..	10^6
α^{I}	9·674	0·726	34
α^{II}	10·745	1·797	190

energy 0·608 M.e.v. (corresponding to the selected level) is 0·66 per disintegration. Thus we have:

$$\frac{\lambda_{\alpha_I}}{\Gamma_\gamma/\hbar} = \frac{0{\cdot}43\times 10^{-6}}{0{\cdot}66} = 0{\cdot}65\times 10^{-6}.$$

Using the formula (71) of Chapter VI we can calculate that, for the α-particles in question, $\lambda_{\alpha_I} \simeq 2\times 10^5$ sec.$^{-1}$ Thus we get:

$$\frac{\Gamma_\gamma}{\hbar} = \frac{2\times 10^5}{0{\cdot}67\times 10^{-6}} = 3\times 10^{11} \text{ sec.}^{-1},$$

which is 1,000 times smaller than the value, 3×10^{14} sec.$^{-1}$, predicted by the dipole formula (17).

Since the line $h\nu = 0{\cdot}603$ is a typical strong dipole line of the γ-ray spectra, the result just obtained shows definitely that the absolute probabilities of γ-ray emission are closer to those that would be predicted by the quadrupole formula. On the basis of the above

considerations it is also easy to understand why one has failed to observe any γ-radiation corresponding to long-range particles from ThC'. In fact, one can calculate that for the first long-range group of ThC' ($E_{\alpha_I} = 9{\cdot}67$ M.e.v., $h\nu = 0{\cdot}73$ M.e.v.):

$$\lambda_{\alpha_I} = 3\times10^{10}\ \text{sec.}^{-1} \quad \text{and} \quad \frac{\Gamma_{\gamma q}}{\hbar} = 2\times10^{11}\ \text{sec.}^{-1}$$

Thus, the expected intensity of the γ-radiation must be only

$$\frac{2\times10^{11}}{3\times10^{10}}\times 34\times10^{-6} \simeq 2\times10^{-4}$$

per disintegration, in this case, and this would escape detection.

This drastic reduction of the probability of dipole transitions can be understood easily if we assume that the various 'moving parts' of the excited nucleus possess the same specific charge; in fact, in this case the centre of gravity of the vibrating system would coincide with its centre of charge and the dipole moment will be identically zero. Such an assumption would seem reasonable for those light nuclei (for which $A = 2Z$) which may be considered as built up entirely from α-particles, but it is by no means self-evident for the heavier elements which contain a large excess of neutrons. One can assume, however, that also in this case the motion of nuclear particles takes place in such a symmetrical fashion that the centre of gravity of the protons always coincides with the centre of gravity of the neutrons. Such motions would obtain, for example, if we consider the excited states of the nucleus as semi-classical vibrations of a charged droplet of nuclear fluid in which the protons and neutrons are rather tightly bound to one another.

3. Selection rules and metastable states

As was first indicated by Weizsäcker,† there may exist nuclei in which the angular momentum of the first excited state is very different from that of the ground state. Since in this event the emission of a γ-quantum, as well as the ejection of an atomic electron (internal conversion), will be prohibited to a very high degree, the excited state of the nucleus will be able to exist for a quite considerable length of time. If the transformation which led to the formation of this 'metastable state' of the nucleus, or *nuclear isomer*, is to be followed by further transformations, there is also a possibility that

† C. V. Weizsäcker, *Naturwiss.* **24** (1936), 813.

in many cases the next transformation will take place *before* the excess energy will be released through the emission of a γ-quantum or a conversion electron. A typical situation of that kind seems to exist in the nucleus formed as the result of β-decay of UX_1. In fact, as was first observed by Hahn,† the β-transformation of UX_1 appears to lead to two different kinds of nuclei, known as UX_2 and UZ with decay constants $1{\cdot}01 \times 10^{-2}$ sec.$^{-1}$ and $2{\cdot}87 \times 10^{-5}$ sec.$^{-1}$, respectively (half-lives 1·14 min. and 6·7 hrs.). Through the emission of another β-particle, with a maximum energy of 2·32 M.e.v. in the case of UX_2 and of 0·45 M.e.v. in the case of UZ, both substances are transformed into the nucleus of U_{II} which is a normal α-emitter. Since, according to these data, the nuclei of UX_2 and UZ possess the same atomic and mass-numbers and differ only in their energy content, this case must be interpreted as a typical example of nuclear metastability in which the state of higher energy is prevented from losing its energy in the form of a γ-quantum or conversion electron by the existence of a large difference between the spin in this state and the spin in the ground state.‡ The scheme of transformations pertaining to this part of the uranium family has been investigated in great detail by Feather and Bretscher and Dunworth§ and particularly by Bradt and Scherrer,‖ whose results are shown in Fig. 35. One finds that, whereas the emission of the 2·32 M.e.v. β-particle transforms the UX_2 nucleus directly into the ground state of U_{II}, the 0·45 M.e.v. β-emission by UZ leads to the formation of an excited state of the U_{II} nucleus which later goes over into the normal state by the cascade emission of two γ-quanta with a combined energy of 1·5 M.e.v.†† From these data we find that the energy-difference between UX_2 and UZ nuclei is $2{\cdot}32-[0{\cdot}45+1{\cdot}50] \simeq 0{\cdot}37$ M.e.v.

And, in fact, observation shows that UX_2 possesses a strong γ-line with an energy of 0·394 M.e.v., which is strongly converted in K-, L-, and M-shells of that atom. The observed fact that the conversion of

† O. Hahn, *Ber. Dtsh. Chem. Ges.* **54** (1921), 1131.

‡ In the second edition of this book an attempt was made to explain nuclear isomerism as due to the presence within the nuclear system of an hypothetical particle known as the negative proton. However, in view of the much simpler explanation given below, such an assumption does not appear to be necessary any more.

§ N. Feather and E. Bretscher, *Proc. Roy. Soc.* A. **165** (1938), 530; N. Feather and J. V. Dunworth, ibid. **168** (1938), 566.

‖ H. Bradt and P. Scherrer, *Helv. Phys. Acta*, **18** (1945), 405.

†† One actually observes, in this case, a γ-radiation with $h\nu \sim 0{\cdot}75$ M.e.v., and the intensity of about two γ-quanta per transformation.

this line in L- and M-shells is much stronger than the conversion in the K-shell indicates that we are dealing with radiation of a high-order multipole. The absolute probability of this particular γ-transition can be estimated from the known decay-constant of 2·32 M.e.v. β-transformations and the observed branching ratio of the forking $UX_2 \xrightarrow{\gamma} UZ$ and $UX_2 \xrightarrow{\beta} U_{II}$. We find

$$\text{æ}_\gamma = 0{\cdot}12 \times 1{\cdot}01 \times 10^{-2}\,\text{sec.}^{-1} = 1{\cdot}2 \times 10^{-3}\,\text{sec.}^{-1},$$

which corresponds to a *γ-half-life of 13 hours*! In order to explain

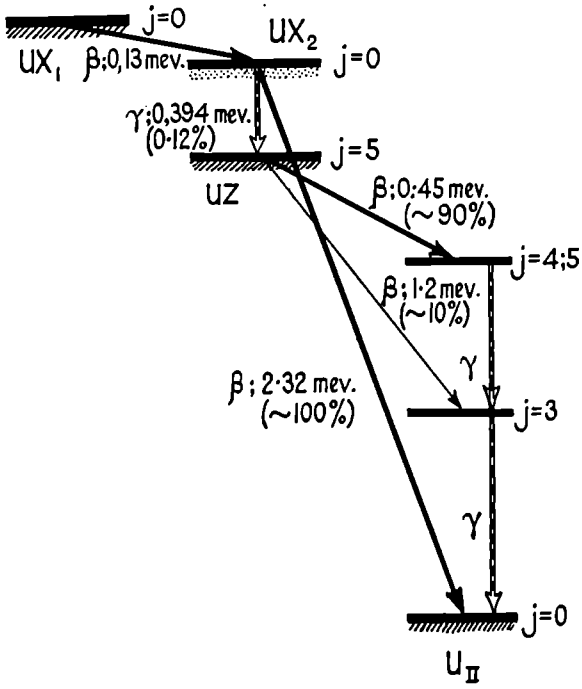


FIG. 35. Transformation scheme of UX_2 and UZ.

this extremely small probability of γ-ray emission we have to assume that the spin difference between UX_2 and UZ-states is equal to five units; in fact, with $l = 5$ and $\hbar\omega = 0{\cdot}4$ M.e.v. the formula (19) leads to the half-life of 15 hours. Since UX_1, having even atomic number and even atomic weight, must be expected with great certainty to have zero spin, and since the transition $UX_1 \rightarrow UX_2$ is a permitted β-transformation, this large value of the spin must be ascribed to the UZ-nucleus. This explains why no direct transition $UX_1 \rightarrow UZ$ was ever observed and also why the β-decay of UZ always leads to an excited state of the U_{II}-nucleus (which must also have spin zero in its

ground state). Fig. 35 gives also other spin quantum numbers which can be assigned to the various states involved.

Although the UX_2 nucleus affords an exceptional example of nuclear metastability, a number of other cases have been established in the study of nuclei produced in various artificial transformations. A 50-hour emission in In^{114} has been alluded to already in Chapter V as a useful circumstance for the careful determination of the shape of the beta-spectrum that follows the isomeric transition. Long-lived isomers in Rh, Ag, Cd, In, and Au have proved useful also for the detection of excitation of these nuclei under bombardment by X-rays.† Such excitation is the reverse process to γ-radiation and is carried out by varying the quantum energy of a beam of X-rays, which is falling on a specific target, until an energy is found for which the target nuclei absorb the X-rays and jump to an excited state (line-absorption). The energy of excitation is then re-emitted, but the emission does not lead necessarily to the ground state at once. There is a certain chance that on re-emission the nucleus will find itself in a low, metastable state instead of the ground state. The subsequent radioactivity of the metastable state then provides a record of the original absorption of the X-rays and this is the way the energies of the line-absorptions are determined. The whole process is reminiscent of that in the production of the Aurorae. In Table XX we give the levels of the elements cited above as determined in this way by Wiedenbeck (loc. cit.).

TABLE XX

Low-lying levels of excitation (M.e.v.)

Rh^{103}	Ag	Cd	In^{115}	Au^{197}
3·05	3·13	..	..	
				2·97
2·71	2·76			
		2·56	2·63	2·56
2·37	2·32			
2·02	1·95	2·08	2·13	2·15
1·64	1·59	1·68	1·55	1·68
1·26	1·18	1·25	1·12	1·22
0·040	0·093	0·020	0·338	0·025 (metastable)
45 min.	40·4 sec.	48·8 min.	4·42 hr.	7·5 sec. half-life

Another interesting example of nuclear isomerism is observed in the case of iridium subjected to the bombardment by slow neutrons.

† M. L. Wiedenbeck, *Phys. Rev.* **68** (1945), 1, 237.

One observes in this case a delayed activity with the half-life of 1·5 min. which was originally ascribed to the β-decaying product formed by neutron capture in one of Ir-isotopes. It was, however, shown by Goldhaber, Muehlhause, and Turkel† that the electron-emission associated with that period represents actually the result of L-shell conversion of the primary γ-radiation emitted by the nucleus. They were able, in fact, to observe this primary γ-radiation, and measured its energy to be about 60 k.e.v. (leading to 47 k.e.v. electrons in L-shell conversion). Thus it appears likely that we deal here with a comparatively long-living isomeric state of Ir^{192} nucleus which is formed as the result of neutron capture by the normal Ir-isotope 191.

We shall mention here also the experiments of de Benedetti and McGowan‡ who applied delayed-coincidence counters to look for extremely shortlived metastable states of certain artificially produced isotopes. They have found that the β-transformation of Hf^{181} leads to a metastable state of Ta^{181} which has a half-life of 22 microseconds, and that the β-decay of W^{187} leads to a metastable Re^{187} nucleus with a half-life of about 1 microsecond. Both these cases were demonstrated by observing the β-ray and then, a certain number of microseconds later, counting the γ-rays. The plot of the logarithm of the number of coincidences obtained against the delay time gives the half-life of the metastable product. Both these examples can be understood as forbidden γ-spectra if the difference in spin entailed is three units.

4. Nuclear excitation by α-decay

In discussing processes of β-transformation we often encounter situations (as, for example, in the case of $UZ \xrightarrow{\beta} U_{II}$, Fig. 35) in which the emission of the electron corresponds to transitions to excited states of the product nucleus because the transition to the ground state is forbidden by spin (or parity) selection rules. This process is essentially the same as that just discussed in connexion with metastable states, the difference being that, in general, the excited state will emit a γ-ray in a normal time, comparable to 10^{-11} sec. and the β-ray and γ-ray will appear to be in exact coincidence. In fact, such coincidence selection of the γ-rays permits a thorough study of the level systems

† M. Goldhaber, C. O. Muehlhause, S. H. Turkel, *Phys. Rev.* **71** (1947), 372.
‡ S. de Benedetti and F. K. McGowan, ibid. **70** (1946), 569; **71** (1947), 380.

in daughter nuclei, and in this way Deutsch and co-workers have established a number of low-lying levels of excitation in the lighter nuclei. An example is given by Fe^{56},† for which combining levels were found at 0·845, 2·11, 2·65, and 2·98 M.e.v.

A situation similar to that for β-transformations may also exist in the case of α-transformations with the difference that we may expect this phenomenon to be limited to small excitation energies because of the rapid decrease of the transparency of the potential barrier

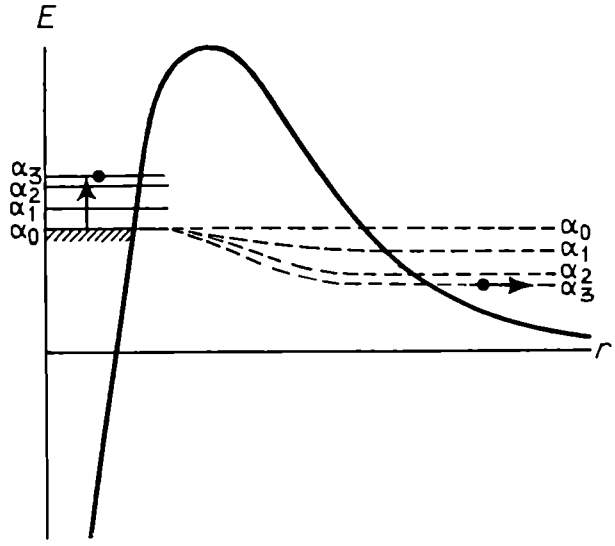


FIG. 36.

with decreasing energy of the emitted α-particles. In these cases we may predict that the spectra of α-rays will show several discrete groups corresponding to the different quantum levels of the product nucleus, as represented schematically in Fig. 36. The excited nucleus will emit this excess energy in the form of γ-radiation, which will follow immediately (due to the very short period of life of nuclear excited states) the emission of the α-particle. It is clear that the γ-rays originating in this way belong to the nucleus of the product element although they are experimentally observed together with the α-radiation of the original element. It is also clear that the energies of these γ-rays must fit in the level scheme of the product nucleus—which may be obtained directly by turning upside-down a figure showing the energies of the various groups of α-particles with respect to the group of smallest energy. We may notice here that, in general, the intensities of the slower α-components should decrease rapidly

† L. G. Elliott and M. Deutsch, *Phys. Rev.* **64** (1943), 321.

due to the increasing difficulty experienced by slower particles in penetrating the potential barrier.

Such weak groups of α-particles with smaller energies than the main group were first observed by Rosenblum† when investigating the deviation of the α-rays from ThC in strong magnetic fields. It appeared that there were actually five very close components referred to by Rosenblum as the components of 'fine structure' of the α-rays. The energy-differences and the relative intensities as measured by Rosenblum are given in the second and third columns of Table XXI.

TABLE XXI

'Fine structure' of the α-rays from ThC

α-group	*Energy-difference* $(E_{\alpha_0}-E_{\alpha_i})\times 10^{-6}$ *e.v.*	*Relative intensities obs.* $I_{\alpha_i}/I_{\alpha_0}$		*Relative intensities theor.* $I_{\alpha_i}/I_{\alpha_0}$
α_0	..	1·0	19·1%	1·0
α_1	0·040	3·3	77·0%	0·7
α_2	0·330	0·1	2·2%	0·03
α_3	0·477	0·01	0·2%	0·005
α_4	0·496	0·07	1·5%	0·004
α_5	0·626	..		..

The level scheme for the ThC″ nucleus obtained from these energies is shown in Fig. 37, which also includes the γ-ray lines from the measurements of Ellis. It will be seen that these fit well into the scheme. Since the relative intensities of the different α-components evidently fix relative values for the partial decay constants corresponding to the different groups of particles, we see at once that the probability of disintegration does not vary so regularly (or, as we shall see later, so rapidly) with energy as might be expected from simple considerations; in particular the intensity of the group α_1 is greater than that of the group α_0, although the energy of the α-particles is somewhat smaller.

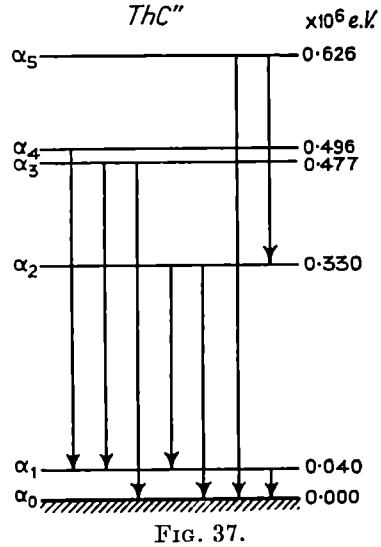


FIG. 37.

After the discovery of 'fine structure' with thorium C the same phenomenon was found with a number of other elements. Rutherford

† Rosenblum, *C.R.* **190** (1930), 1124; *J. de Phys.* **1** (1930), 438. For later references see Rosenblum in *Reports of the Solvay Congress*, 1933.

and his collaborators,† using first the differential counter for range measurements, and later also the magnetic method, proved that with radium C and actinium C there are at least two components of comparable intensity; the corresponding level schemes and the γ-rays are shown in Fig. 38, (*a*) and (*b*). Further investigations of Rosenblum, by the magnetic method, showed the presence of more complicated and well-marked 'fine structure' with many members of the actinium family; he found three components with actinon, three components with actinium X, and eleven components with radio-

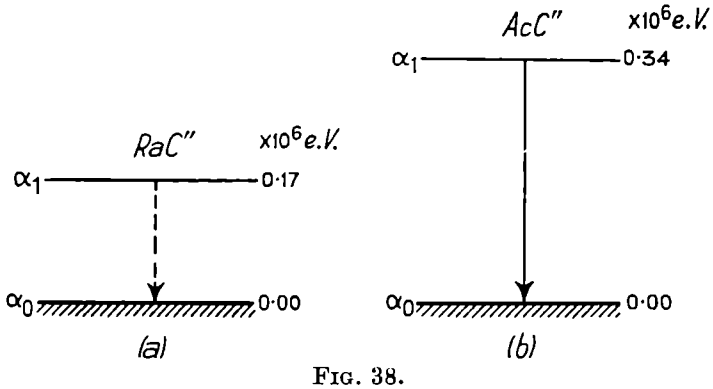


FIG. 38.

actinium. Very probably, also, such 'fine structure' exists in the case of protoactinium, which, although not investigated by the magnetic method, is known to possess rather strong γ-radiation which has to be explained by excitation through α-decay.

On the other hand, with all other elements so far investigated 'fine structure' is either not observed at all (ThX, Tn, ThA, Rn, RaA, RaF, AcA) or occurs in the form of an extremely faint component, corresponding to a γ-ray of small intensity (RaTh and Ra).

We shall investigate now in more detail the question of the relative intensities of different 'fine-structure' components in relation to the transparencies of the potential barrier for α-particles of the appropriate velocities. Using the formula (55) of the previous chapter and accepting $j = 0$ (and $r_0 = \text{const.}$) for all components of the structure, we can calculate the relative intensities which are to be expected. The results of such a calculation for the groups of the ThC α-spectrum are given in the last column of Table XXI. We see that in this case, and it is equally true in other cases of strong 'fine structure', the intensities of the components decrease much more slowly than would

† E. Rutherford, F. A. B. Ward, and C. E. Wynn Williams, *Proc. Roy. Soc.* A. **129** (1930), 211; **139** (1933), 617.

be expected theoretically—and notice even the inversion of order in the special case of the α_0 and α_1 groups of ThC. On the other hand, the faintness of the components with Ra and RaTh is in good agreement with the theoretical expectation. In the same way we might say that the failure to detect any slow components with other elements must be due to their extreme weakness† following the predictions of theory.

These considerations lead us to the conclusion that we must consider the presence of strong 'fine-structure' components as an

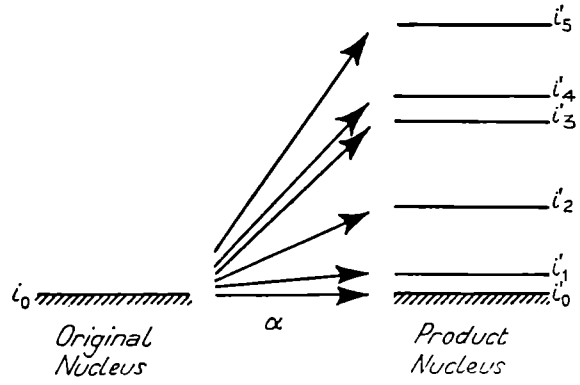


FIG. 39.

anomalous effect and try to find some explanation of it in terms of probabilities of different kinds of disintegration. The explanation of this effect proposed by Gamow‡ is based on the dependence—already discussed—of the probability of disintegration on the angular momentum of the emitted particle. Let us consider the process of α-disintegration of a nucleus possessing a spin i_0 when the product nucleus possesses, in its normal and excited states, spins i'_0, i'_1, i'_2, etc. (Fig. 39). If the spins of the normal states of the original and product nuclei are the same ($i_0 = i'_0$) the probability of normal disintegration (i.e. without excitation) will be given by expression (71) of Chapter VI with $j = 0$, as in this case the α-particle can be emitted without angular momentum. Also, the intensities of disintegrations with excitation of different levels of the product nucleus will be considerably less than this on account of the smaller energy of the emitted α-particle and the effect of the additional 'potential barrier of

† We say this, rather than ascribe the absence of 'fine structure' to an energetic impossibility; in fact one can feel perfectly sure that in radioactive nuclei there are always levels with excitation energies less than than energy of α-particles amounting to at least 4×10^6 e.v.

‡ G. Gamow, *Nature*, **131** (1933), 618.

centrifugal force' for those states with spin different from i_0. This represents just the above-mentioned case of very rapidly decreasing intensities of fine-structure groups characteristic of those elements for which a very faint structure or no structure at all has been observed. Let us imagine now that the normal state of the product nucleus possesses spin different from that of the original one. In this case the probability of normal α-decay will be considerably reduced as the emitted α-particle must take with it the spin-difference and escape from the nucleus with angular momentum $j = |i_0 - i_0'|$. On the other hand, it might now happen that some of the excited states of the product nucleus possess spin closer or equal to i_0, the spin of the original nucleus. For transitions to these states the decrease of disintegration probability due to the smaller energy of the emitted α-particle will be partially balanced on account of the absence of the additional 'centrifugal barrier' and the resulting intensity may become comparable with, or in certain cases even larger than, that of normal decay. These considerations, relating the existence of strong 'fine structure' with large spin-differences between the normal states of original and resulting nuclei, throw some light on the question why this effect is observed only for all three C products and for most members of the actinium family. For, as we have seen above (Chap. I), nuclear spins different from zero are to be expected only for elements with odd atomic number (C products) or odd mass numbers (actinium family). A glance at Fig. 30 shows us that it is just for the elements possessing strong 'fine-structure' components that the anomalously small values of the effective radius are observed, and this, as has been already mentioned, should be considered as indicating the emission of an α-particle with angular momentum different from zero. All these facts seems to prove, rather unambiguously, that the proposed explanation of the 'fine-structure' effect has the correct basis.

It is much more difficult, however, to give a quantitative treatment of that effect, because the formula (71), Chapter VI, used for calculating the effect of spin on the probability of disintegration does not take into account all the factors known to be concerned. In fact, in deriving this formula, we accepted a simplified model of the potential barrier with a *vertical fall at* $r = r_0$, and correspondingly cut off the potential barrier of centrifugal force at the same distance. In reality the fall of potential near the nuclear boundary cannot be so abrupt,

whilst the addition of centrifugal potential will somewhat change the whole distribution and slightly decrease† the value to be accepted for the nuclear radius; this effect cannot be taken into account until we know the exact form of the potential-energy curve near the nucleus. The second effect neglected in the formula can be expressed as the change in the number of collisions of an α-particle inside the nucleus with the potential barrier: an α-particle possessing non-zero angular

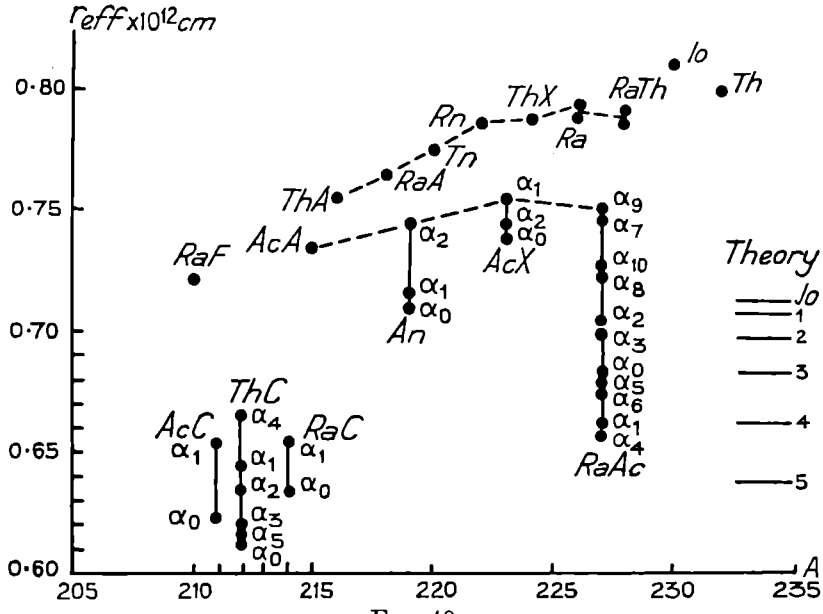


FIG. 40.

momentum will suffer fewer collisions with the 'wall' than a particle of the same energy, but without such momentum. At present we can only say that both these effects will act in the same direction as the effect already taken in account, so that the actual decrease in intensity due to spin-differences should be larger than that given by our formula (71).

Anyhow we shall make the attempt to deduce nuclear spin-differences from the relative intensities of 'fine-structure' components and we shall proceed in the following way. For each component we calculate the partial decay-constant, using the experimental values of total decay-constant and relative intensities of individual α-ray components. Substituting these, and the experimental values for the energies of the different components, in the formula (71), we

† Because it is necessary to come closer to the nucleus in order that the stronger repulsion (Coulomb + centrifugal forces) may be compensated by nuclear attractive forces.

obtain the effective radii corresponding to the components of 'fine structure'. These radii, as calculated by Gamow and Rosenblum,† are shown in Fig. 40‡ plotted against the atomic weight of decaying nucleus. We see that the points representing the effective radii for the different α-components of the same element are spread over a considerable region and that they show a tendency to accumulate in pairs, triplets, or other groups. If we agree, as stated above, that the reduction of effective radius is to be ascribed to the angular momentum of the emitted α-particle we must conclude that, for a given element, the disintegrations characterized by the same effective radius take place with the same change of angular momentum and those described by smaller radii with correspondingly large momenta. For example, the groups α_0, α_5, and α_6 in RaAc must have equal angular momenta and this momentum must be larger than that of the groups α_2 and α_3. In Fig. 40 are included, also, the points representing the effective radii for different j's as calculated from the formula (71). We see that, although there is a close similarity between theoretical and experimental schemes, the experimental points are situated somewhat farther apart from each other than are the calculated points. This can be accounted for by the roughness of the approximation used in obtaining formula (71)—as previously mentioned. For this reason it is difficult with certainty to assign values of j to the different groups; for example, in the case of ThC we could accept $j(\alpha_4) = 0$, $j(\alpha_1$ and $\alpha_2) = 1$,§ and $j(\alpha_0, \alpha_3$, and $\alpha_5) = 2$, although the values 0, 2, and 3 or 1, 3, and 4 are also possible. If, however, the above-mentioned correction can be taken into account it is not at all impossible that this method will permit us to give quite definite j-values to all components. In conclusion we must mention that our considerations lead only to the spin-differences ($j = i - i^1$) between the nuclei, and, in order to assign values to the nuclear spin itself, a knowledge of at least one spin in each family is necessary. In the Th and U families it is very likely that the elements of the main α-decay sequence possess the spin $i = 0$, and for the Ac family we can use the spin of Pa, which, according to Schüler and Gollnow, is equal to $\frac{3}{2}$.

† G. Gamow and S. Rosenblum, *C.R.* **197** (1933), 1620.

‡ The values in Fig. 40 are slightly different from those given above (Fig. 30), as in the new calculations somewhat improved experimental data have been used.

§ The difference between the points for α_1 and α_2 in ThC is within the limits of experimental error.

A serious difficulty with the spin-theory of fine-structure intensities arose as a result of the work of Chang,† who, by using a new technique, was able to study the very weak components of the α-fine-structure of polonium. The results of his measurements are shown in Table XXII which gives the energy values for different α-groups as

TABLE XXII

Energy groups of Po *α-particles*

Group	*Group energy M.e.v.*	*Excitation M.e.v.*	*Relative intensity*	$r_{\text{eff}} \times 10^{12}$ *cm.*	j
α_0	5·303	..	10^6	0·71	13
α_1	5·113	0·190	(96)	0·58	15
α_2	5·065	0·238	(126)	0·59	15
α_3	4·901	0·402	84	0·63	14
α_4	4·838	0·465	49	0·64	14
α_5	4·749	0·554	64	0·67	13
α_6	4·640	0·663	70	0·70	13
α_7	4·449	0·854	79	0·77	11
α_8	4·303	1·000	43	0·81	11
α_9	4·111	1·192	48	0·89	10
α_{10}	4·016	1·287	19	1·91	9
α_{11}	3·890	1·413	18	0·97	7
α_{12}	3·685	1·618	21	1·08	0

well as their relative intensities. In the last two columns we give the values of the effective nuclear radii calculated by the formula (71) of Chapter VI, and the values of angular momentum which must be ascribed to the various α-groups in order to account for the observed changes in intensity on the theory described above. It goes without saying that these extremely large values of j do not look real and that, although the effect of nuclear spin may account for minor variations of the observed intensities, the main increase of the effective nuclear radii with increasing energy of excitation in the product nucleus must be given an entirely different explanation in this case. The most straightforward way to account for these results would be to assume that the radius of the nucleus actually does increase with increasing nuclear excitation. It appears, however, that our present knowledge of the laws of nuclear forces excludes the possibility of such comparatively large changes in radius (it would, in fact, require the nuclear density to become 3 times smaller for a state excited by 1·6 M.e.v. than for the normal state!). Another possibility lies in the assumption of nuclear polarization produced by the Coulomb field of

† W. Y. Chang, *Phys. Rev.* **69** (1946), 60.

the outgoing particle. In this case the emitted α-particle can cross a good part of the potential barrier still going with a high velocity and lose part of its original energy only when it is a considerable distance from the original nuclear surface. An assumption of this kind of interplay could readily explain the absence of a rapid exponential decrease in the number of low-energy α-groups.†

† Calculations along these lines are being carried out by R. Peierls and M. A. Preston (private communication) and they appear to lead to a satisfactory interpretation.

VIII

NUCLEAR COLLISIONS

1. Properties of neutron beams

We now come to the problems arising in nuclear collisions which take place when a beam of particles of one kind pass through material composed of particles of another kind. Historically, these problems were first investigated in the case of the scattering of α-particles and, in fact, the observed deviations from the Coulomb scattering supplied us the first evidence of the dimensions of atomic nuclei. With the development of experimental technique for producing artificially accelerated beams these studies were extended to charged particles other than α-particles. Since, however, the scattering of charged particles is determined mostly by the external Coulomb field so that specifically nuclear effects appear as deviations from the inverse square scattering, much more direct information is obtained by the use of neutron beams.

Since neutrons exist permanently in nature only as constituents of atomic nuclei they have to be produced therefrom by bombarding nuclei with other particles. The classical example of such a *nuclear reaction* is the one by which neutrons were discovered, viz. the bombardment of beryllium with fast α-particles:

$$Be^9+He^4 \rightarrow C^{12}+n. \tag{1}$$

In the early experiments the alphas were obtained from the naturally radioactive elements, e.g. Po. Since energy must be conserved, the sum of the binding energies of Be^9 and He^4 plus the kinetic energy of the He^4 must be equal to the energy of the resulting C^{12} and neutron. The latter consists of (1) the kinetic energy of the neutron, (2) the kinetic energy of the C^{12} atom, and (3) the binding energy in the C^{12} atom (relative to hydrogen atoms and neutrons). If the C^{12} nucleus is formed in its lowest energy its binding energy is known (cf. Table II), and we can calculate the total kinetic energy of the particles produced in reaction (1) to be

$$(KE)_{nC} = E_\alpha + BE(Be^9+He^4-C^{12}). \tag{2}$$

Of this amount, $\frac{4}{13}E_\alpha$ appears in the motion of the centre of gravity of the system and one can readily obtain the energy of neutron, as a function of the angle with which it is emitted, by a straightforward

application of the laws of conservation of momentum. It is evident that, in the general case, the energy of a neutron produced in a given nuclear reaction will depend upon (1) the atomic weights involved, (2) the difference in binding energies (with attention to the possibility that the product nucleus need not be formed in its lowest state), (3) the angle with which the neutron comes off, and (4), the energy of the bombarding particle. With regard to the last point, if the bombarding particles are charged particles they will also lose energy to the atomic electrons in the target and, if in addition the target is 'thick', the energy of bombardment will vary from its maximum value to zero. Generally speaking, although by no means necessarily so, neutrons produced through nuclear reactions have kinetic energies of the order of millions of electron volts (M.e.v.) since this is the usual order of differences in binding energy among nuclei. The neutrons produced simultaneously with the fission of the heaviest nuclei are also in this energy range. All such neutrons are referred to as *fast neutrons*, and their energies are usually measured in M.e.v.

According to the fundamental laws of the quantum mechanics a free neutron with kinetic energy E is represented by a plane wave-function periodic with the wave-length

$$\lambda = \frac{2\pi\hbar}{\sqrt{(2ME)}} = \frac{0{\cdot}286}{\sqrt{E_{\text{ev}}}} \times 10^{-8} \text{ cm.}, \tag{3}$$

where E_{ev} means the value of the energy in electron volts. Thus, even at very low energy, neutron waves are not much longer than the distances between atoms in a crystal, say, and are certainly short compared with slit-widths and apertures used in optical experiments. It is justifiable, therefore, to consider a narrow stream of neutrons, that has been defined by a slit system, as a plane wave which, at a given instant, varies with the distance z along the beam as

$$e^{2\pi iz/\lambda}.$$

In fact, the wave-length region concerned is very similar to that customarily encountered in ordinary X-rays.

The parameter that enters into calculations of scattering, etc., is almost always $\lambda\!\!\!^{-} = \lambda/2\pi$, and, for fast neutrons

$$\lambda\!\!\!^{-} = \frac{4{\cdot}55}{\sqrt{E_{\text{mev}}}} \times 10^{-13} \text{ cm.} \tag{4}$$

For fast neutrons, therefore, $\lambda\!\!\!^{-}$ is of the order of nuclear radii.

It is possible to find nuclear reactions in which the difference in binding energies is small, or negative, so that the emitted neutron has a very low kinetic energy. For example, if the bombarding particle is a γ-ray of energy slightly greater than the binding energy of the last neutron in the target nucleus, the resulting neutron will be slow. Since the energy of a γ-ray is not affected appreciably by the atomic electrons, the neutrons produced in such a *photo-neutron source* will be not only slow but *mono-energetic* (sometimes called monochromatic) if the gammas have a sharply defined frequency themselves. For the purpose of obtaining neutrons of low energy, however, it is generally more efficient to take fast neutrons from an ordinary nuclear reaction, or fission, and slow them down by collisions with light nuclei. Obviously the most efficient atom for slowing down neutrons is hydrogen, which has practically the same mass as the neutron, so that, if the energy of the neutron is large compared with the binding of hydrogen atoms in molecules, the neutron loses half of its energy in each collision, on the average. If the energy of the neutron is not large enough, so that the hydrogen can be considered free, the mass with which it collides is effectively the mass of the molecule. The loss of energy in this case is then a particular example of general collision between a neutron with a nucleus (or molecule if the neutron is slow) of mass A. Let the angle by which the neutron is deflected in the centre of gravity system be θ and let the incident energy be E_1 and the scattered energy be E_2, then, in general

$$E_2 = E_1\left\{1-\frac{2A}{(A+1)^2}(1-\cos\theta)\right\}, \tag{5}$$

which for $A = 1$ becomes $E_2 = \frac{1}{2}E_1(1+\cos\theta)$. Thus a common method of producing slow neutrons is to surround a source of fast neutrons by paraffin in order to reduce the energy through collisions with the hydrogen nuclei. In the chain-reacting pile the same function is served by collisions with the carbon nuclei of the moderator. The natural limit of energy to which neutrons may be slowed by such methods is determined by the thermal motion of the atoms in the slowing-down medium, i.e. the velocity of the neutrons eventually assumes a Maxwellian distribution characteristic of the temperature of the slowing-down system. Neutrons that have come to equilibrium with a medium at temperature $T°$K are called *thermal neutrons* if T is in the neighbourhood of room temperatures, *liquid*

air neutrons if T is the boiling-point of liquid air, and so on. The value of $\lambda\!\!\!^{-}$ is given by

$$\lambda\!\!\!^{-} = \frac{4{\cdot}92}{\sqrt{T}}\,10^{-8}\ \text{cm.}, \tag{6}$$

so that thermal neutrons have wave-lengths comparable to the diameters of atoms, or the distances between atoms in condensed phases.

Except for the (very useful) property of neutrons that they may be slowed down readily, neutron beams are similar in many respects to electromagnetic beams (X-rays) of corresponding wave-length. The phenomena of diffraction, scattering, refraction, reflection (total), polarization, dispersion, and absorption are well established for neutron beams by one method or another. In Chapter I we have referred to the scattering of fast neutrons as a method of estimating nuclear radii. According to eq. (4), neutrons of 25 M.e.v. have $\lambda\!\!\!^{-} \simeq 10^{-13}$ cm., whereas the radius, R, of a nucleus with $A = 200$ is about 7×10^{-13} cm. The scattering of these neutron waves should thus be similar to the diffraction of optical waves by an obstacle of diameter $14\,\lambda\!\!\!^{-}$, since the scattering of waves in such cases is largely a matter of the effect of the principle of uncertainty. Applying the diffraction equation of Kirchhoff, Bethe and Placzek† obtain the familiar formulae for the differential cross-section for a circular disk of radius R

$$\sigma(\theta) \simeq R^2\left[\frac{J_1(R\theta/\lambda\!\!\!^{-})}{\theta}\right]^2 \quad (\theta \ll 1),$$
$$\int\sigma(\theta)d\cos(\theta) = \pi R^2. \tag{7}$$

Thus, counting up all rays that are bent by any amount whatsoever leads to an integrated cross-section of πR^2. From the form of $\sigma(\theta)$ as well as from the physical content of the problem, it is evident that most of the scattered rays are bent only slightly from their incident direction. In addition, all those incident rays that fall *inside* the area πR^2 of the disk are either absorbed or reflected. Hence the total cross-section of an absorbing (or reflecting) sphere of radius R is $2\pi R^2$. Actually, because of the finite range of forces between nucleons, the radius R is better estimated as the radius of the target nucleus plus the range of forces. The more precise manner in which nuclear forces influence the scattering of neutrons is not represented

† H. A. Bethe and G. Placzek, *Phys. Rev.* **57** (1940), 1075.

in eq. (7), of course, so that the formula is only approximate. Attempts to measure the distribution of scattered neutrons as a function of θ indicate, however, that the relation is roughly correct.

The diffraction, or scattering, of *slow neutrons* by atoms is of great interest because of the possibility of obtaining interference effects between several scattering centres. In Chapter II we have discussed already, in a semi-quantitative way, the interference between waves scattered by the two protons in the hydrogen molecule. There we related the scattered amplitudes to the 'phase-shift', δ, produced by a square potential well. For neutron energies approaching zero, $(\delta \to 0)$, the scattered wave has the approximate form

$$-\frac{\delta\lambda\!\!\!^{-}}{r}e^{ir/\lambda\!\!\!^{-}} = \frac{a_0}{r}e^{ir/\lambda\!\!\!^{-}}, \tag{8}$$

and the total cross-section is

$$\sigma = 4\pi(\delta\lambda\!\!\!^{-})^2 = 4\pi a_0^2.$$

Eq. (8) is equivalent to the definition of the scattering length a_0. The application to para- and ortho-hydrogen, made in Chapter II, is not complete for three reasons: (1) the thermal motion of the protons was not taken into account in calculating $\lambda\!\!\!^{-}$ and δ; (2) the separation of the protons is not entirely negligible compared with $\lambda\!\!\!^{-}$, although it is nearly so for liquid air neutrons; and (3) the value of δ was assumed to be equal to that computed for collision between the neutron and a single proton. The first point may be corrected by considering the statistical distribution of H_2-velocities, relative to the neutron beam, and integrating the scattered intensity over the Maxwell distribution. The second point merely requires a little more care in the description of the interfering scattered waves.

The third point above, the relation between the phase-shifts caused by free protons and those by protons that are bound to a heavier system, may be elucidated further as follows. In singlet collisions of a neutron with a free proton, for example, δ_0 is computed from the refraction of a spherical wave that converges upon, and subsequently diverges from, the *centre of gravity* of neutron and proton. The whole phase-shift in this wave is produced in that part of the collision during which neutron and proton are within the range of forces of each other, i.e. within half a range of the centre of gravity, and amounts to $2\delta_0$. If, however, the proton had been assumed to be fixed at the origin of coordinates, as is the case when it is chemically bound to a heavy

molecule, the refraction of the neutron wave would begin when the converging phase of the neutron wave is at the full range of forces from the origin and continue until the diverging phase reaches that radius. The resulting phase-shift of the neutron wave will then be twice as great as for collision with a free proton, i.e. $4\delta_0$, and the cross-section (when δ_0 is very small) will be 4 times as large. This factor of four was first pointed out by Fermi.† Since the whole question is one of converting the n–p phase-shift to the centre of gravity motion, it is readily seen that if the proton is bound to a system of total weight W (neutron masses) the phase-shift as calculated for a free proton must be multiplied by $2W/(W+1)$ and the cross-section for slow neutrons will be multiplied by this squared. Hence, for the scattering of neutrons by hydrogen molecules, $W = 2$, and the cross-section presented by each proton for slow neutrons is individually 16/9 times that for a free proton.

The interference phenomenon, exemplified in the case of scattering of slow neutrons by hydrogen molecules, can be extended to other molecules as well as to solids and liquids for which the neutron wave-length λ is of the order of the interatomic distances.‡ Many of these phenomena are so very similar to those occurring for X-rays of comparable wave-length that we shall refer to them only sketchily. In particular, the Bragg law for reflection of waves by crystal planes, separated by d:

$$\sin\theta = \frac{N\lambda}{2d}, \tag{9}$$

where N is an integer, has been found to apply to neutron waves with $\lambda \leqslant 2d$, just as to X-rays. The selective reflection of neutrons of definite wave-length has indeed proved to be a useful research tool for producing highly monochromatic neutron beams, especially when used with such copious sources of slow neutrons as the thermal column of a chain-reacting pile. An interesting consequence of the Bragg law is that neutrons of wave-length greater than 6·7 A are not deflected by any of the planes in graphite (cf. Fermi and Marshall, loc. cit.). Hence by filtering thermal neutrons ($T \simeq 300°$ K) through 20 cm. of graphite most of the wave-lengths shorter than 7 A have been scattered out of the beam and only the longer wave-lengths,

† E. Fermi, *Ric. Sci.* **7** (1936), 13.

‡ A summary of early experiments along these lines is given by J. H. Van Vleck, *Philadelphia Centennial*, 1939, and a great deal of the modern work with pile neutrons is presented by E. Fermi and L. Marshall, *Phys. Rev.* **71** (1947), 666.

i.e. cold neutrons ($T \sim 20°$ K), remain. The theory of crystalline refraction of neutron beams, including possibilities of absorption, has been reported in a paper by Goldberger and Seitz.†

When the neutron wave-length is too long for Bragg reflection, or if we are dealing with non-crystalline solids, we may simply compute a refractive index for the medium. Suppose, for simplicity, that all nuclei in the medium are equivalent, each producing a phase-shift of 2δ in S-waves of the neutron, and let $\lambda\!\!\!^{-}$ be the wave-length over 2π for the neutron beam *in vacuo* and $\lambda\!\!\!^{-}'$ the value inside the medium. The number of S-wave collisions per centimetre of path, and per square cm. of cross-section of the beam will be $\pi n_1 \lambda\!\!\!^{-2}$ where n_1 is the number of nuclei per cm.^{3} in the medium. Each collision advances the phase of the neutron wave by the (small) amount 2δ. By the principle of superposition, the advancement of phase per unit length of path is then $2\pi n_1 \lambda\!\!\!^{-2}\delta$ and by definition, this is $1/\lambda\!\!\!^{-}' - 1/\lambda\!\!\!^{-}$. Setting n, the *index of refraction*, equal to $\lambda\!\!\!^{-}/\lambda\!\!\!^{-}'$ and using the collision length, $a_0 = -\delta\lambda\!\!\!^{-}$, we get

$$n = 1 - 2\pi a_0 n_1 \lambda\!\!\!^{-2}. \tag{10}$$

Whether n is larger than or smaller than unity thus depends upon the sign of δ, or of a_0. This sign has been determined experimentally for a number of nuclei by comparing the scattering from composite crystals of various types of nuclei, and the sign of a_0 is found to be positive for most cases (H, Li, and Mn have been found to have negative a). Pure materials of this type will then produce total reflection of neutron beams that impinge at a glancing angle from the air-side of a polished surface, just as in the case of X-rays (where $n < 1$, also). The total reflection of neutron beams has indeed been demonstrated for a half-dozen elements (Fermi and Marshall, loc. cit.) using neutrons of wave-length 1·87 A. The limiting angles of reflection are about 10 minutes of arc.

Monochromatic neutrons have been produced also by the 'time of flight' method. This method, originated by Alvarez, derives its neutrons through a nuclear reaction from a high-voltage source of protons, or deuterons, such as a cyclotron; the cyclotron is not operated continuously, however, but in sharp bursts at regulated intervals. So many microseconds after a burst of charge particles is made and allowed to fall upon an appropriate target the neutron counting equipment is activated, also for a very short interval of

† M. L. Goldberger and F. Seitz, ibid. p. 294.

time. In this way neutrons of rather sharply defined velocity are recorded in the counters which are at a known distance from the target. Both the 'time of flight' and Bragg reflection methods have been applied to studying the *absorption spectrum* of various materials. Many of these spectra are characterized by rather sharp regions of energy in which there is high absorption, and the shape of the absorption curve in such regions is very similar to that in the resonant absorption of light. A typical example is shown in Fig. 41. The energy at the centre of such a peak is called the *resonant energy* and the *half-width* of the resonance is specified in the usual way. Over fifty such resonant levels have been found experimentally† for neutron energies mostly in the range of zero to 50 electron volts. They are found with rather uniform frequency in the range of atomic weights from 90 to 200, and the average spacing between neighbouring levels turns out to be about 10 e.v. Such irregularities and groupings in these levels as do occur are probably related to the periodic features of nuclear structure for which the current theory has no explanation.

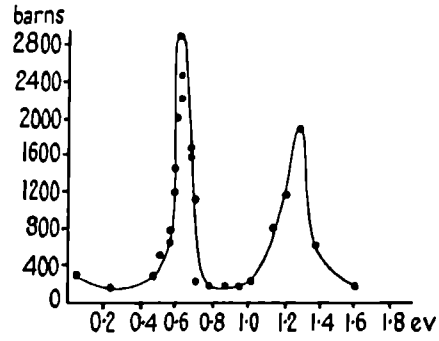


FIG. 41. Total cross-section for slow neutrons in iridium (after R. B. Sawyer, E. O. Wollan, S. Bernstein, and K. C. Peterson, *Phys. Rev.* **72** (1947), 109).

The existence of strong absorption lines in the spectra for slow neutrons is one of the most important characteristics of the interaction of neutrons with nuclei in this range (i.e. $A > 90$). In interpreting the observed distribution of levels, however, it must be remembered that the (negative) binding energy of the last neutron decreases steadily from about 10 M.e.v. for $A = 90$ to about 7 M.e.v. for $A = 200$ and to lower values for higher A. We shall return to the question of the distance between levels in Chapter IX.

Another salient feature of the interaction of neutrons with nuclei

† B. D. McDaniel, *Phys. Rev.* **70** (1946), 83 (In); W. W. Havens, Jr., and J. Rainwater, ibid. p. 154 (Sb, Hg); L. J. Rainwater, W. W. Havens, Jr., C. S. Wu, J. R. Dunning, ibid. **71** (1947), 65 (Cd, Ag, Ir, Sb, Mn); W. W. Havens, Jr., C. S. Wu, L. J. Rainwater, and C. J. Meaker, ibid. p. 165 (Au, In, Ta, W, Pt, Zr); C. S. Wu, L. J. Rainwater, W. W. Havens, Jr., ibid. p. 174 (Os, Co, Tl, Cb, Ge); R. E. Lapp, J. R. Van Horn, A. J. Dempster, ibid. p. 745 (Gd, Sm); W. J. Sturm, ibid. p. 757 (Rh, Au, Ir, Gd, Sm, Eu, Dy).

is illustrated best by those absorption spectra in which no sharp lines occur. For example, the absorption cross-sections for Li and B, for neutrons in the range of energy up to 100 e.v., are represented by the formulae:

$$\sigma_{\mathrm{Li}} = \left[1{\cdot}7 + \frac{11{\cdot}5}{\sqrt{E_{\mathrm{ev}}}}\right] 10^{-24} \text{ cm.}^2$$
$$\sigma_{\mathrm{B}} = \frac{118}{\sqrt{E_{\mathrm{ev}}}} \times 10^{-24} \text{ cm.}^2 \qquad (11)$$

This characteristic dependence of non-resonant absorption at low energy on $E^{-\frac{1}{2}}$ is known as the $1/v$-*law*; we shall take up its theoretical foundation in Chapter IX.

The analogy between neutron beams and beams of electromagnetic radiation extends to include polarization. Both beams are capable of two independent polarizations which, in the case of the neutron, are associated with the two possible orientations of the spin-vector along a chosen axis of reference. Physically, it is possible to define a preferential direction in space by applying a magnetic field. Then, since the neutron carries a magnetic moment, the two polarizations of neutron beam will be refracted, or otherwise behave, differently when passing through such a field. An ingenious method of putting such an effect into evidence has been proposed and used by F. Bloch.† This method is to pass the neutron beam through a magnetized sheet of iron. Interference between scattering by the iron nuclei and scattering by the atomic magnetic moments tends to remove (scatter) one spin-polarization from the original beam more effectively than the opposite polarization. The beam then emerges from the iron partially polarized, and this can be demonstrated by 'analysing' the transmitted beam with a second sheet of magnetized iron.

To illustrate the principles of Bloch's method in more detail, consider the scattering of a slow neutron beam by a single paramagnetic atom or ion. The nucleus of the atom will scatter a certain, spherically symmetrical intensity which determines the elastic cross-section, σ_n, for *nuclear* scattering. In addition to these scattered waves, the electron cloud about the nucleus will scatter the neutron waves through the interaction of their resultant magnetic moment and the magnetic moment of the neutron. If the resultant electronic

† F. Bloch, ibid. **50** (1936), 259; **51** (1937), 994.

magnetic moment, $\bar{\boldsymbol{\mu}}_e$, could be visualized as a point dipole coincident with the nucleus, the added potential would be

$$U_0 = \frac{(\bar{\boldsymbol{\mu}}_e, \boldsymbol{\mu}_n)r^2 - 3(\bar{\boldsymbol{\mu}}_e, \mathbf{r})(\boldsymbol{\mu}_n, \mathbf{r})}{r^5}.$$

Actually, the neutron interacts at each point in the electron cloud with the magnetic polarization, due to electrons, $\mathbf{m}_e(\mathbf{r}_e)$, at that point, so that more exactly:

$$U(\mathbf{r}_n) = \int \frac{(\mathbf{m}_e, \boldsymbol{\mu}_n)r^2 - 3(\mathbf{m}_e, \mathbf{r})(\boldsymbol{\mu}_n, \mathbf{r})}{r^5} d\tau_e \qquad (12)$$

$$\mathbf{r} = \mathbf{r}_n - \mathbf{r}_e.$$

Since we are dealing with slow neutrons, and very small phase-shifts, we may use the Born approximation which says that the perturbing potential acts like a source of Huygens wavelets of amplitude

$$\frac{2M}{\hbar^2} U(\mathbf{r}_n) e^{i(\mathbf{k}_0 - \mathbf{k}_n)\mathbf{r}_n}$$

where $\mathbf{k}_0$ is the wave-number vector of the incident beam, and $\mathbf{k}_n$ of the scattered beam. At large distances from the atom, the scattered wave has the form:

$$\psi' = -\frac{e^{ikr}}{4\pi r} \frac{2M}{\hbar^2} \int e^{i(\mathbf{k}_0 - \mathbf{k}_n)\mathbf{r}_n} U(\mathbf{r}_n)\, d\tau_n. \qquad (13)$$

We are primarily interested in those waves originating from $U(\mathbf{r}_n)$ that interfere with the waves scattered from the nucleus. It is assumed that the spin of the neutron is not changed by interaction with the nucleus, and since $U(\mathbf{r})$ is a weak potential we may restrict the calculation to those components of $U(\mathbf{r})$ in which $\boldsymbol{\mu}_n$ is diagonal. Taking the spin-effects into account, one then substitutes (12) into (13) and integrates to obtain the amplitude scattered by the magnetic field of the atom. The integral is not elementary, however, because of the singularity at $\mathbf{r}_n - \mathbf{r}_e = 0$.

Bloch (loc. cit.) has shown that, if one encloses the position of the neutron by a small volume bounded by the surface S, the angular distribution of neutrons scattered both by the nuclei and by the magnetic polarization of the electron cloud is given by the differential cross-section

$$\sigma(\theta) = \left| \sqrt{\frac{\sigma_n}{4\pi}} \pm \frac{\gamma_n \gamma_e}{2} \frac{e^2}{mc^2} (\cos^2\theta_q - C) \right|^2, \qquad (14)$$

where θ_q is the angle included by the vector difference $\mathbf{k}_0 - \mathbf{k}_n$ and

the vector $\int \mathbf{m}_e(\mathbf{r})e^{i(\mathbf{k}_0-\mathbf{k}_n)\mathbf{r}}\,d\tau$, γ_n is the magnetic moment of the neutron measured in nuclear magnetons, and γ_e is the magnetic moment of the electron measured in Bohr magnetons. The number C is the result of the integration over the surface of the excluded volume, and may be expressed in coordinates relative to the centre of that volume as

$$C = \frac{1}{4\pi}\int \frac{z\,dS_z}{r^3}. \tag{15}$$

The value of C thus depends upon the 'shape' of the neutron. If S is a needle-like volume such as one should use to enclose two monopoles forming a dipole in the z-direction, C obviously vanishes; if S is disk-shaped, presumably the case if the neutron's magnetic moment is due to a small Amperian current, $C = 1$; if S is a sphere, $C = \frac{1}{3}$, and so on.

The plus or minus sign in eq. (14) refers to one polarization, or the opposite, of the neutron's spin relative to the axis defined by the electronic magnetic moment. It is evident that the differential cross-section for a given angle of scattering will be larger or smaller according to whether the waves scattered magnetically interfere constructively or destructively with the waves scattered by the nucleus. Since, if waves of one spin-orientation interfere constructively, waves of the opposite orientation interfere destructively, the scattered intensities will be different and the scattered and transmitted beams will be partially polarized. Experimentally, the degree of polarization appears to be about 10 per cent. This method of producing polarized beams of neutrons was used in the first determination of the magnetic moment of the neutron.†

2. Elastic scattering

In Chapter II we have introduced the theory of scattering of one nucleon by another, with special attention given to the effect of short-range nuclear forces on the S-wave component of a plane wave. The restriction to S-waves was made possible by the fact that λ, for the cases considered, was long compared with the range of forces. Since, in general, the collision of particles in nuclear bombardment involves scattering by short-range forces, it is evident that we have merely to extend this theory to apply to scattering by complex nuclei. It is usually convenient to think of the scattering nucleus as an effectively

† L. W. Alvarez and F. Bloch, *Phys. Rev.* **57** (1940), 111.

infinitely heavy object representing a refractive region for nucleonic waves, which region extends to a definite radius R. In the mathematics, one automatically expresses the proper approach by going over to reduced masses. Let us imagine a plane through the centre of the scattering nucleus and perpendicular to the direction of bombardment, and let us draw circles on this plane about the centre and of radii $\lambda\!\!\!^{-}$, $2\lambda\!\!\!^{-}$, $3\lambda\!\!\!^{-}$,..., etc. If the radius of the refractive region, R, lies wholly within the first of these circles it is sufficient, in good approximation, to consider only the head-on collisions, or S-waves. If R extends into the second zone, however, collisions with orbital angular momentum unity are affected appreciably and the P-wave refraction has to be taken into account. It will be noted that the area presented to a plane wave by the zone of orbital angular momentum l is

$$\pi\lambda\!\!\!^{-2}[(l+1)^2-l^2] = (2l+1)\pi\lambda\!\!\!^{-2}.$$

The method of Chapter II is readily generalized to include refraction in the states of higher orbital angular momenta. The problem considered here has axial symmetry about the direction of bombardment which we shall denote by z or by $r\cos\theta$.† It is then most convenient to expand the bombarding plane wave in eigenfunctions of angular momenta by the well-known formula:

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} (2l+1)i^l \sqrt{\left(\frac{\pi}{2kr}\right)} J_{l+\frac{1}{2}}(kr) P_l(\cos\theta), \tag{16}$$

where $J_{l+\frac{1}{2}}(kr)$ is the Bessel function of half-integral order that is bounded at $r = 0$, and $P_l(\cos\theta)$ is the Legendre polynomial. As suggested above, only those states of angular momentum with zero component about the direction of bombardment can occur in this expression.

Ultimately, we are interested in the effect of a short-range region of refraction, around $r = 0$, on the plane wave (16), but we first concern ourselves with the net effect at large distances from this region. We may then use the asymptotic expressions for the Bessel functions:

$$J_{l+\frac{1}{2}}(kr) \to \sqrt{\left(\frac{2}{\pi kr}\right)} \sin(kr - \tfrac{1}{2}\pi l), \tag{17}$$

† The situation becomes less symmetrical in the case of polarized spins, in the beam, or in the target or both, as, for example, in the scattering of neutrons by magnetic iron discussed in § 1.

and eq. (16) becomes:

$$e^{ikr\cos\theta} \xrightarrow[\text{large } r]{} \sum_{l=0}^{\infty} (2l+1)\frac{P_l(\cos\theta)}{2ikr}[e^{ikr}-(-1)^l e^{-ikr}]. \qquad (18\,a)$$

The trigonometric functions are expanded to put into evidence the incoming part† of the incident wave, at large r, e^{-ikr}, and the outgoing part, e^{ikr}. The short-range region of refraction will induce a certain total shift in phase, $2\delta_l$, which is represented by inserting the factor $e^{2i\delta_l}$ in the outgoing part for each l. The asymptotic form of the refracted wave is then:

$$\psi_{\text{refr}} \to \sum_{l=0}^{\infty} (2l+1)\frac{P_l(\cos\theta)}{2ikr}[e^{i(kr+2\delta_l)}-(-1)^l e^{-ikr}]. \qquad (18\,b)$$

Subtracting (18 a) from (18 b) we get the scattered amplitude,

$$S = \sum_{l=0}^{\infty} (2l+1)P_l(\cos\theta)e^{i\delta_l}\frac{\sin\delta_l}{kr}. \qquad (19)$$

However, r^2 times the scattered intensity is just the differential cross-section for scattering into unit solid angle in the direction θ, and this is $\sigma(\theta)$

$$\sigma(\theta) = \frac{1}{k^2}|e^{i\delta_0}\sin\delta_0 + 3e^{i\delta_1}\sin\delta_1\cos\theta\ldots + (2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta)\ldots|^2. \qquad (20)$$

The fact that the sum of terms belonging to different l occurs inside the absolute square expresses the possibility of interference between these waves, which must be possible since all scattered waves originate from a single plane wave. Generally speaking, scattered waves of the incident particle will interfere if the *initial* and the *final* states are the same. Since we have been disregarding spin, a final state may be completely specified as a plane wave travelling in the direction θ and will contain, for example, an $S-P$ interference term given by:

$$6\sin\delta_0\sin\delta_1\cos(\delta_0-\delta_1)\cos\theta. \qquad (20\,a)$$

The rule for interference permits one to treat the collision of particles with spin, as we did in Chapter II, by setting up independent expressions for the differential cross-sections, one for each possible

† This is the incoming part in accordance with the convention that the time-dependent factor of the wave-function is $e^{-iEt/\hbar}$, whence a given phase progresses inward with time.

spin-combination, and then adding the $\sigma(\theta)$ with the statistical weight appropriate to each. If the colliding particles have spins s and i, there are $(2s+1)(2i+1)$ possible polarizations of the system, and these may be grouped into eigenstates of total spin, j,

$$s+i \geqslant j \geqslant |s-i|.$$

In the general case, the phase-shifts will be different for different j (double, or multiple, refraction!) and the combined scattering cross-section must be written:

$$\sigma(\theta) = \sum_j \frac{(2j+1)}{(2s+1)(2i+1)} \sigma_j(\theta). \tag{21}$$

The total elastic scattering cross-section is then

$$\sigma_{el} = 2\pi \int \sigma(\theta) d\cos\theta. \tag{22}$$

A complication arises if the forces that produce the refraction may also turn over the spins. In this case the final states are not always the same and non-interfering terms appear in the scattered intensity. The treatment of this question is a part of the general theory of transmutation that is considered in the next chapter, but a simple example will be taken up at the end of this section.

For the sake of completeness, we write down at this point also the analogous expressions for $\sigma(\theta)$ in case there are long-range forces, i.e. Coulomb forces. As discussed in § 1, Chapter II, we find that the particle waves are refracted everywhere and the analysis is more complicated than for short-range forces alone. One can follow through the method introduced in Chapter II for higher angular momenta, however, with the general result:†

$$\sigma(\theta) = \frac{1}{k^2}\Big| -\frac{\dot{\eta}}{1-\cos\theta} e^{i\eta \ln\{2/(1-\cos\theta)\}} + e^{i\delta_0}\sin\delta_0 + \\ +3\cos\theta e^{i\delta_1+i\phi_1}\sin\delta_1 + \ldots (2l+1) e^{i\delta_l+i\phi_l}\sin\delta_l P_l(\cos\theta)\Big|^2, \tag{23}$$

where $$\eta = \frac{Z_1 Z_2 e^2}{\hbar v} \qquad \phi_l = \arg\frac{(1+i\eta)(2+i\eta)\ldots(l+i\eta)}{(1-i\eta)(2-i\eta)\ldots(l-i\eta)}$$

for particles of charge Z_1 and Z_2 and with relative velocity v. For high velocities and small charges, $|\eta| \ll 1$, and ϕ_l may be approximated by

$$\phi_l \cong 2\eta\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots\frac{1}{l}\right), \quad (\eta \ll 1).$$

† W. Gordon, *Zs. f. Phys.* **48** (1928), 180.

We turn now to the rather more specifically nuclear part of the theory of scattering in which we attempt to evaluate the δ_l's in terms of the short-range forces between the incident particle and the scattering nucleus. To take the simplest case, let us suppose we are dealing with a slow neutron incident upon a heavy nucleus, so that $\lambda\!\!\!^{-} \gg R$, where R *is the radius of the nucleus plus the range of force.* In Chapter II, § 2, we had precisely this case for scattering of neutrons by protons. But we shall simplify the present example further by neglecting the spin of the neutron as well as of the scattering nucleus, if any. If we should represent the nucleus by a 'square well' of radius R and depth $\hbar^2k_0^2/2M$, the calculation of δ_0 would be formally the same as in the case of hydrogen and we should obtain

$$\delta_0 = -kR + \arctan\frac{kR}{\sqrt{(k^2+k_0^2)}R\cot\{\sqrt{(k^2+k_0^2)}R\}}, \tag{24}$$

and, since δ_l of higher l are negligibly small, the total cross-section is:

$$\sigma_{el} = \frac{4\pi}{k^2}\sin^2\delta_0 \tag{25}$$

in accordance with eq. (20).

The feature of these formulae (24) and (25) that is most readily checked against observation is the occurrence of resonant energies, i.e. values of $\hbar^2k^2/2M$ for which $\delta_0 = (n+\frac{1}{2})\pi$, where n is an integer and, therefore, for which the scattering cross-section reaches its maximum possible value for an S-wave, $4\pi^2\lambda\!\!\!^{-}$. From eq. (24) we may readily estimate the distance (difference in energy) between successive resonance levels to be $D_n \cong \frac{\pi^2\hbar^2}{MR^2}n$; if we substitute $n = 1$, $R = 6\times10^{-13}$ cm., as representative values for, say, Si^{28}, we find a value of D_1 to be 22 M.e.v.! This is to be compared with the observed level distance in Si^{28} of about 0·030 M.e.v. (at an excitation of 10 M.e.v., p. 236). This obvious inadequacy of the 'potential well model' springs from the fact that such a model assumes that the impinging neutron moves in the average field of the other nucleons while inside the radius R, and otherwise quite independently. Actually, the strong, short-range forces between nucleons makes it possible for the incident neutron to share quickly its energy with other nucleons in the nucleus. The energy of bombardment, plus the potential energy (binding energy of the next neutron), is then distributed throughout the composite system formed by the neutron and

the target nucleus. This composite system is the *compound nucleus*† in the currently accepted terminology, and the term applies as well if particles other than neutrons are used in the bombardment. Owing to this sharing of the excitation energy in rather statistical fashion, the neutron spends a much longer time in the field of the nuclear forces than if it moved independently, i.e. it has to wait until all its original energy has been restored to it in order to become elastically scattered. By the uncertainty principle, the longer the duration of the compound state the smaller the width of the resonant level. An essentially equivalent physical picture is that the strong coupling between nucleons splits the individual particle levels predicted by the potential well model into very many quantum states, the energy separations of which is rather to be determined from statistical theory considered in the following chapter.

We shall see later that the average distance between resonant levels for neutrons in rather heavy nuclei is in the range of 10–100 e.v. This means that δ_0 should go through π radians whenever the energy of bombardment is advanced by this amount rather than by some 10 M.e.v. which eq. (24) would require. In order to take account of the fact that resonant levels occur much more frequently than allowed by eq. (24), we shall replace the argument of the cot, $\sqrt{(k^2+k_0^2)}R$, in that equation by a much more rapidly varying function of the excitation energy, U, viz. $\phi(U)$.‡ The phase-shift for scattering of neutrons in S-waves is then

$$\delta_0 = -kR+\arctan\frac{k}{K\cot\phi} \qquad \{K=\sqrt{(k^2+k_0^2)}\}. \tag{26}$$

The new function, $\phi = \phi(U)$, is then a kind of 'phase of neutron waves' at the radius R and has to be determined from the experimental results on δ_0 as a function of the energy of excitation. It is evident that for low energies of bombardment, for which kR is very small, a resonant energy, E_r, will correspond to $\phi_r = (n_r+\frac{1}{2})\pi$. The most fundamental type of information concerning ϕ as a function of energy is then the location of the resonant energies, E_r. From these one may then calculate the resonant excitations $U_r = E_r+B_n$, where B_n is the binding energy for the neutron. Next in importance to finding the positions of the levels is to determine the width of each

† N. Bohr, *Nature* **137** (1936), 344.

‡ This approach to a description of resonant scattering is essentially that presented by H. Feshbach, D. C. Peaslee, and V. F. Weisskopf, *Phys. Rev.* **71** (1947), 145.

resonant level. The relation between the width of a level and the function $\phi(E)$ is evidently that the width Γ_r^n is proportional to $(dE/d\phi)_r$. We shall now turn to the problem of expressing δ_0 and the cross-section for scattering in terms of the positions and widths of the resonant levels.

Substituting (26) into eq. (25) and rearranging factors we obtain for the total scattering cross-section:

$$\sigma_{el} = \frac{4\pi}{k^2}\left| e^{ikR}\sin kR + \frac{1}{i-(K\cot\phi/k)}\right|^2. \tag{27}$$

If we are concerned with the neighbourhood of a resonant energy E_r (for which $\cot\phi$ vanishes) and the width of this level is known, it is convenient to approximate $K\cot\phi$ as a linear function:

$$\begin{aligned} K\cot\phi &= (E-E_r)\left[\frac{d(K\cot\phi)}{dE}\right]_r \\ &= -(E-E_r)k\left(\frac{d\delta}{dE}\right)_r = -(E-E_r)\frac{2k}{\Gamma_r^n}, \end{aligned} \tag{28}$$

where we define 'the neutron width', for the level r

$$\Gamma_r^n = 2\Big/\left(\frac{d\delta}{dE}\right)_r$$

and, in these terms,

$$\sigma_{el} = \frac{4\pi}{k^2}\left|e^{ikR}\sin kR + \frac{\frac{1}{2}\Gamma_r^n}{E-E_r+\frac{1}{2}i\Gamma_r^n}\right|^2. \tag{29}$$

The resonant feature of the scattering is contained in the second term under the absolute square in eq. (29). The first term is a 'potential scattering' term of the type that would be obtained by assuming the nucleus to be an impenetrable sphere. At low energies, $kR \ll 1$, and this term may be disregarded, giving,

$$\sigma_{el} \cong \frac{\pi}{k^2}\,\frac{(\Gamma_r^n)^2}{(E-E_r)^2+\frac{1}{4}(\Gamma_r^n)^2}. \tag{30}$$

At resonance, $E = E_r$, and σ_{el} takes its maximum value $4\pi\lambda\!\!\!^{-2}$. It will be noticed that in the more complete expression, eq. (29), the maximum cross-section does not occur exactly at E_r because of the interference with the potential term. Nevertheless, it is convenient to retain the definition of a resonant energy as that energy for which $\cot\phi$ vanishes. Equation (30) is a special case of the 'dispersion formula' first derived by Breit and Wigner† for the description of

† G. Breit and E. P. Wigner, ibid. **49** (1936), 519.

nuclear processes in the region of a resonant energy. In their derivation of this particular case it would be assumed that the compound nucleus can get rid of its energy of excitation *only* by ejecting a neutron. The probability of neutron emission is proportional to Γ_r^n and this, in the original form of the theory, was calculated from perturbation theory in the usual way as, $\Gamma_r^n = 2\pi|H_r^i|^2$, where H_r^i is the matrix element of the perturbation that couples the resonant state r with the initial state i (neutron incident upon the target nucleus) which is *normalized to unit energy*. Since, however, the nuclear forces are so strong, it is not clear in what sense the H_r^i may be considered as matrix elements of a small perturbation. Consequently many attempts have been made to formulate the resonant phenomena without recourse to perturbation theory, in addition to that followed above.† Certain features of these theories will be returned to later.

So far as the development that we have been following goes, eq. (29), or (30), holds only near a single resonance and in the event of many resonant energies one must specify how the function $K \cot \phi$ varies in the regions between them. Since, in all practical cases, the excitation energy, U, is very large compared with the level distances, we might expect the general behaviour of $K \cot \phi$ to be quite similar for a number of resonant energies in the neighbourhood of a given excitation. In particular, we may note that $\pi(dE/d\phi)$ should be of the order of magnitude of the level distance, for S-levels, and may be represented by an average value over a region that embraces many levels. Using such an average value, one sees also that the neutron width

$$\Gamma_r^n = 2k\left[\frac{dE}{d(K\cot\phi)}\right]_r = 2\frac{k}{K}\frac{dE}{d\phi}$$

will vary with the energy of bombardment essentially as k, or $E^{\frac{1}{2}}$. Hence, as $E \to 0$, the scattering cross-section (41) approaches a finite value.

The scattering formula for the more general cases of neutron waves of higher angular momentum in their orbital motion, or waves of charged particles in the Coulomb field of the nucleus, are readily deduced along the lines of the foregoing calculation. We shall continue to denote the phase-function for the internal wave by ϕ, it being under-

† P. L. Kapur and R. Peierls, *Proc. Roy. Soc.* **166** (1938), 277; A. J. F. Siegert, *Phys. Rev.* **56** (1939), 750; E. P. Wigner, ibid. **70** (1946), 15, 606; E. P. Wigner and L. Eisenbud, ibid. **72** (1947), 29.

stood that it may now depend upon the angular momentum and charge of the bombarding particle, as well as the energy of excitation, $\phi = \phi(U, l, \eta)$. Through this function we fit the internal wave smoothly on to an external wave of the general form

$$\frac{1}{r}\{F_{l,\eta}(r)\cos\delta_l + G_{l,\eta}(r)\sin\delta_l\}P_l(\cos\theta),$$

where $F_{l,\eta}(r)$ is the appropriate bounded radial wave-function (times r) for the particular values of l and η concerned, and $G_{l,\eta}(r)$ is the corresponding unbounded wave-function. Then, assuming that the internal waves may still be approximated as simple sine waves, and writing F' for $\left[\frac{dF_{l,\eta}(r)}{dr}\right]_R$ and similarly for G', we get:

$$K\cot\phi = \frac{F'\cos\delta + G'\sin\delta}{F\cos\delta + G\sin\delta}, \tag{31}$$

which we may readily solve for δ. The scattered amplitude is proportional to

$$e^{i\delta}\sin\delta = \frac{1}{\cot\delta - i} = i\frac{F' - FK\cot\phi}{F' - FK\cot\phi + i(G' - GK\cot\phi)}. \tag{32}$$

Even for this more general ϕ we shall retain the notion of a 'fundamental width', Γ_r^0, for each resonant level r such that near resonance $K\cot\phi$ may be approximated linearly by

$$K\cot\phi \cong -(E - E_r)\frac{2k}{\Gamma_r^0}. \tag{33}$$

It is evident from eq. (32) that the scattered intensity will have its maximum value whenever

$$G' - GK\cot\phi = 0,$$

$$E - E_r = -\frac{\Gamma_r^0}{2k}\frac{G'}{G} \cong 0, \tag{34}$$

since, in any practical case, $RG'/G \ll 1$ and $\Gamma_r^0/2kR$ is of the order of the distance between neighbouring levels.

The observed width of the resonance at E_r will not be Γ_r^0, which is more closely related to the level spacing, but will rather be

$$\Gamma_r^{l,\eta} = E_{\frac{1}{2}r} - E_{-\frac{1}{2}r},$$

where $E_{\frac{1}{2}r}$ means that value of the bombarding energy for which the scattering intensity has just dropped to one-half its value at E_r, as energy is increased, and $E_{-\frac{1}{2}r}$ is the corresponding value on the lower

side of E_r. In these simple scattering formulae it is evident from eq. (32) that $E_{\frac{1}{2}r}$ and $E_{-\frac{1}{2}r}$ correspond to $\cot\delta = 1$ and to -1, respectively. Substituting these values of δ into eq. (31), and the resulting values of $\cot\phi$ into eq. (33), we get

$$\frac{2kR}{\Gamma_r^0}[E_{\frac{1}{2}r}-E_{-\frac{1}{2}r}] \cong \frac{R(F'+G')}{F+G} - \frac{R(F'-G')}{F-G}$$
$$\cong \frac{2R(F'G-G'F)}{G^2-F^2} \cong \frac{2kR}{G^2},$$

since $G^2 \gg F^2$ under the conditions of our approximations (a more careful analysis shows the denominator to be G^2+F^2). From this calculation, therefore, we get the simple relation

$$\Gamma_r^{l,\eta} = \frac{\Gamma_r^0}{G_{l,\eta}^2}. \tag{35}$$

For neutrons in S-waves, $G = \cos kR \cong 1$. For higher angular momenta, however, and small kR

$$G_l = \sqrt{\left(\frac{\pi kR}{2}\right)} J_{-l-\frac{1}{2}}(kR) = \sqrt{\left(\frac{\pi kR}{2}\right)} \frac{(\frac{1}{2}kR)^{-l-\frac{1}{2}}}{(-l-\frac{1}{2})!}$$

and

$$\Gamma_r^l = \Gamma_r^0 \frac{(kR)^{2l}}{[1.3.5...(2l-1)]^2}, \quad kR \ll 1. \tag{36}$$

Thus, the width of a level of higher angular momentum is much narrower, if $kR \ll 1$, than that of an S-level for the same spacing between levels. Since $\Gamma^0 \sim k \sim v$, the level width for neutrons of angular momentum l and small k varies as $k^{2l+1} \sim v^{2l+1}$ (v is the velocity of the neutron outside the nucleus). This sharpening of the levels may be said to be due to the fact that the neutron must penetrate a potential barrier of centrifugal force in order to reach the nucleus and, therefore, the probability is large only very close to exact resonance. For high energy, and thus large k, the centrifugal barrier is ineffective and $\Gamma^{l,\eta} \to \Gamma^0$. The correct expression for Γ^l in intermediate ranges of kR may be derived from the more elaborate theories (cf. Wigner and Eisenbud, loc. cit.), and we list here the results for the l-values 0, 1, and 2:

$$\begin{aligned} \Gamma^l &= \Gamma^0 \quad (l=0) \\ &= \Gamma^0 \frac{(kR)^2}{1+(kR)^2} \quad (l=1) \\ &= \Gamma^0 \frac{(kR)^4[1+(kR)^2]}{[3+2(kR)^2]^2+(kR)^6} \quad (l=2). \end{aligned} \tag{37}$$

In a Coulomb field, e.g. for protons instead of neutrons, $G_{l,\eta}$ depends upon the value of $\eta = Ze^2/\hbar v$, as well as upon l, and Γ^l becomes multiplied by an additional factor which is essentially the penetration factor for a Coulomb barrier. For $kR \ll 1$, we have the approximate expression

$$\Gamma^{l,\eta} = \frac{(kR)^{2l}}{[1.3.5...(2l-1)]^2}\left(1+\frac{\eta^2}{l^2}\right)...(1+\eta^2)\frac{2\pi\eta}{e^{2\pi\eta}-1}\Gamma^0 \quad (\eta \gg kR). \tag{38}$$

In the case of $l = 0$, i.e. S-wave collisions of protons on a resonating nucleus, and for reasonably large η, say of the order unity:

$$\Gamma^{0\eta} = 2\pi\eta e^{-2\pi\eta}\Gamma^0. \tag{39}$$

Hence, owing to the necessity for penetration of the Coulomb barrier, resonance widths for protons, or α-particles, are much narrower than for neutrons of the same value of k. As kR increases, its most important effect appears first in the Coulomb penetration factor, and it becomes appropriate to apply the more complete expressions for barrier penetration presented in Chapter VI. According to the calculations in that chapter, one would apply a further (energy-independent) factor to eq. (38) and (39) of

$$e^{+4\sqrt{(2\eta kR)}} \tag{40}$$

for protons; of course, η must be made twice as large in all these formula when applied to α-particles. It will be noted, also, that if $\eta \gg l$ then the factor in front of $e^{-2\pi\eta+4\sqrt{(2\eta kR)}}$, in the general expression for $\Gamma^{l,\eta}$, will not depend sensitively upon v, since $(kR\eta)^{2l}\eta\Gamma^0 \sim v^0$. This means that by the time the proton wave has penetrated to a radius at which the centrifugal barrier is noticeable, the Coulomb barrier has reduced its amplitude so much that the effect of the centrifugal barrier is entirely unimportant.

Experimental material on resonant scattering is rather scarce, but there is one highly interesting example, namely, the scattering of neutrons by helium. The work of Staub and Tatel† shows that the backward scattering of neutrons by helium is a sensitive function of energy in the neighbourhood of 1 M.e.v. From eqs. (20) and (20 a) we expect preferential backward scattering to arise first as S–P interference. Referring to Chapter IV, we remember also that He^5, which would be the compound nucleus in this case, is not stable, but in its lowest state the third neutron should be most likely in a P-state. We

† H. Staub and H. Tatel, *Phys. Rev.* **58** (1940), 820.

should deduce, therefore, that what is being observed is a resonance of the neutron with this P-state. In the simplest form of scattering formula, then, we should include a resonant term for the P-wave but none for the S-wave (since we do not expect any). For the S-wave, we shall include a potential scattering term, but the possibility of such a term for the P-wave will be disregarded. The formula for the differential cross-section should then look like:

$$\sigma(\theta) = \lambda\!\!\!^{-2} \left| e^{i\delta_0} \sin\delta_0 + 3\cos\theta \frac{\frac{1}{2}\Gamma_1'}{E - E_1 + \frac{1}{2}i\Gamma_1'} \right|^2,$$

where E_1 is the resonant energy of the P-state and Γ_1' its half-width. The experimental results indicate, however, that there are *two* resonant energies, rather close together. The natural interpretation of this observation is that there is spin-orbit coupling in the resonant states such that the $P_{\frac{1}{2}}$ and $P_{\frac{3}{2}}$ states lie at somewhat different energies. Now the waves scattered from these two levels will interfere *if the spin-orientation of the neutron does not change*, so that, for these waves, we simply replace

$$3\cos\theta \frac{\frac{1}{2}\Gamma_1'}{E - E_1 + \frac{1}{2}i\Gamma_1'} \quad \text{by} \quad 2\cos\theta \frac{\frac{1}{2}\Gamma_{\frac{3}{2}}'}{E - E_{\frac{3}{2}} + \frac{1}{2}i\Gamma_{\frac{3}{2}}'} + \cos\theta \frac{\frac{1}{2}\Gamma_{\frac{1}{2}}'}{E - E_{\frac{1}{2}} + \frac{1}{2}i\Gamma_{\frac{1}{2}}'}, \tag{41}$$

where we allow for the fact that two-thirds of the scattered amplitude will spring from the fourfold degenerate $P_{\frac{3}{2}}$-state and one-third from the doubly degenerate $P_{\frac{1}{2}}$-state. But the mere existence of spin-orbit coupling means that in some of the collisions the spin of the neutron will turn over. The scattered amplitudes in such cases will not be coherent with those above and their contribution should be discussed more properly as a nuclear reaction.

We shall discuss the spin-reversing process here in the following elementary way. Consider a neutron with spin 'up' colliding with a He^4 nucleus in a P-wave. The partial wave of this neutron must be of the form $3\alpha\cos\theta$, where α stands for spin 'up' (β for spin 'down') and the $\cos\theta$ comes from eq. (16). This form is not a pure component of either a $P_{\frac{3}{2}}$- or $P_{\frac{1}{2}}$-wave, but, as can be seen easily, is rather a linear superposition:

$$\alpha\cos\theta \sim [\sqrt{2}\, P_{\frac{3}{2}} + P_{\frac{1}{2}}]_{m=\frac{1}{2}}.$$

This means that $\frac{2}{3}$ of the collisions may be considered as subject to

the forces characteristic of a $P_{\frac{3}{2}}$-state and $\frac{1}{3}$ to those of a $P_{\frac{1}{2}}$-state. These states, for $m = \frac{1}{2}$, look like:

$$P_{\frac{3}{2}} = \frac{1}{\sqrt{2}}\{2\alpha\cos\theta - \beta e^{i\phi}\sin\theta\} \qquad (m = \tfrac{1}{2})$$
$$P_{\frac{1}{2}} = \alpha\cos\theta + \beta e^{i\phi}\sin\theta. \tag{42}$$

Hence, if the neutron enters the $P_{\frac{3}{2}}$-state there is one chance in three that it will emerge as $-\beta e^{i\phi}\sin\theta$, i.e. with spin and sign reversed and with its orbital motion changed (reoriented). If it enters the $P_{\frac{1}{2}}$-state, there is a two-thirds probability of its coming out in the wave $\beta e^{i\phi}\sin\theta$. These two possible paths will interfere with each other, but not with the α-waves. To summarize: the incoming wave $3\alpha\cos\theta$ overlaps the eigenstate $P_{\frac{3}{2}}$ (which is normalized to 4π) by a factor $\sqrt{2}$ and may be scattered from this state with spin up as $\sqrt{2}\,\alpha\cos\theta$ (cf. eq. (42)) and it overlaps $P_{\frac{1}{2}}$ by a factor unity and may be scattered as $\alpha\cos\theta$; this gives just the substitution noted in (41); in addition, however, the neutron may emerge from the $P_{\frac{3}{2}}$-state in the wave $-\sqrt{2}\frac{1}{\sqrt{2}}\beta e^{i\phi}\sin\theta = -\beta e^{i\phi}\sin\theta$, and from the $P_{\frac{1}{2}}$-state in the wave $\beta e^{i\phi}\sin\theta$. Collecting these numbers, we find for the differential cross-section, including the non-interfering elements:

$$\sigma(\theta) = \lambda\!\!\!^{-2}\left| e^{i\delta_0}\sin\delta_0 + \cos\theta\left\{\frac{\Gamma_{\frac{3}{2}}}{E - E_{\frac{3}{2}} + \frac{1}{2}i\Gamma_{\frac{3}{2}}} + \frac{\frac{1}{2}\Gamma_{\frac{1}{2}}}{E - E_{\frac{1}{2}} + \frac{1}{2}i\Gamma_{\frac{1}{2}}}\right\}\right|^2 +$$
$$+ \sin^2\theta\left|\frac{\frac{1}{2}\Gamma_{\frac{3}{2}}}{E - E_{\frac{3}{2}} + \frac{1}{2}i\Gamma_{\frac{3}{2}}} - \frac{\frac{1}{2}\Gamma_{\frac{1}{2}}}{E - E_{\frac{1}{2}} + \frac{1}{2}i\Gamma_{\frac{1}{2}}}\right|^2. \tag{43}$$

This theory was first presented by F. Bloch.† The analysis of the data indicates $E_{\frac{3}{2}} = 0{\cdot}76$, $E_{\frac{1}{2}} = 1{\cdot}08$ M.e.v. with $\Gamma_{\frac{1}{2}} = \Gamma_{\frac{3}{2}} = 0{\cdot}32$. In other words, although unstable, the $P_{\frac{3}{2}}$-state of He^5 lies lower and with a 'binding energy' of 0·76 M.e.v., i.e. *not* bound.‡ It will be recollected also that the Li^7-$P_{\frac{3}{2}}$ lies lower than the $P_{\frac{1}{2}}$ by 0·44 M.e.v.

† F. Bloch, *Phys. Rev.* **58** (1940), 829.
‡ Cf. J. H. Williams, W. G. Shepherd, R. O. Haxby, ibid. **51** (1937), 888.

IX

NUCLEAR TRANSFORMATIONS

1. Energy-levels in nuclei

So far we have become acquainted with two general categories of excitation levels in nuclei. The first category includes the levels of daughter nuclei discussed in Chapter VII; these may be excited by α-decay and β-decay and, in some cases, by direct bombardment with X-rays. These levels correspond to the familiar excited states of atoms with the characteristic difference that the quanta emitted in transitions between nuclear states are usually of the order of M.e.v. instead of the order of volts, as in atoms. The excitation in such levels is generally lower than the binding energy of a single nucleon and we shall refer to them as the *low-lying levels*. According to the experimental results cited in Chapter VII, one may expect such low-lying levels that enter into γ-ray transitions to appear at intervals of about 0·4 M.e.v., on the average, as one increases the energy of excitation, and in nuclei with $A > 100$.

For nuclei with $A < 100$ the average distance between low-lying levels gradually increases, roughly as $1/A^{\frac{1}{3}}$, but in a way that is rather strongly influenced by shell-structure, symmetry, etc., especially in the very light nuclei. Independent evidence of the low-lying levels in light nuclei can be obtained, however, by studying the energy-balances in nuclear reactions. As a first example, consider the reaction,† $Mn^{55}(d, p)Mn^{56}$,

$$Mn^{55}+H^2 \rightarrow Mn^{56}+H^1. \qquad (1)$$

According to Table II, the neutron is bound in Mn^{56} by 8·8 M.e.v. and in H^2 by 2·2 M.e.v. Except for small corrections due to the kinetic energy of the Mn, the emitted proton will carry away the original kinetic energy of the deuteron plus 6·6 M.e.v., provided the Mn^{56} nucleus is left in its lowest state. Actually, Martin observes protons of six different energies, corresponding to six different states of the residual nucleus, one of which is the ground state. The energies of excitation of the other five states are 1·07, 1·77, 2·48, 3·61, and 4·38 M.e.v., or an average spacing of 0·88 M.e.v. The (d, p) reaction

† A. B. Martin, *Phys. Rev.* **71** (1947), 127.

has been used on numerous other nuclei, for example $Al^{27}(d, p)Al^{28}$† and on the separated isotopes of neon.‡ The average spacing found in Al^{28} was 1·3 M.e.v. and in Ne^{21} and Ne^{23} about 0·9 M.e.v. The (d, n) reactions also lead to information on excited levels, although it is more difficult to measure the energy of an emitted neutron than of a proton. For example, using proton recoils in a cloud chamber, Bonner and Brubaker§ have studied the neutron energies from

$$Be^9+H^2 \rightarrow B^{10}+n \tag{2}$$

and find excited levels of B^{10} at 0·5, 2·0, and 3·3 M.e.v.

α-Particle bombardment has been as useful as deuteron bombardment in studying the low-lying nuclear levels. Here, again, we shall only give representative examples, such as

$$Be^9+He^4 \rightarrow C^{12}+n, \tag{3}$$

from which Bernardini and Bocciarelli|| determined levels in the C^{12} nucleus at 3·0, 4·4, and 6·4 M.e.v., and

$$F^{19}+He^4 \rightarrow Ne^{22}+H^1, \tag{4}$$

which indicates excited levels of Ne^{22} at 0·6, 1·4, 3·4, and 4·5 M.e.v.†† Needless to say, the energy-release, or Q-value, for the lowest state of the residual nucleus is useful for checking, or determining, the differences in binding energy of the nuclei involved.

Perhaps the simplest type of 'nuclear reaction' from which information is obtained about excitation levels is that in which neither mass nor charge of the target nucleus is changed but only its internal energy. The bombarding particle is then *inelastically scattered.* Quantitative work of this type has been done by determining the energy lost by protons when scattered from various materials. In Al^{27}, for example,‡‡ energy-levels appear at 0·87, 2·03, 2·70, and 3·5 M.e.v., i.e. an average spacing of 0·9 M.e.v.

The data acquired to date on the low-lying levels cannot be considered *a priori* as representative of the complete term systems of

† E. M. McMillan and E. O. Lawrence, ibid. **47** (1935), 343.
‡ F. K. Elder, H. T. Motz, P. W. Davison, ibid. **71** (1947), 917.
§ T. W. Bonner and W. M. Brubaker, ibid. **50** (1936), 308.
|| G. Bernardini and D. Bocciarelli, *Accad. Lincei Atti*, **24** (1936), 132.
†† J. Chadwick and J. E. R. Constable, *Proc. Roy. Soc.* **135** (1931), 48, and A. N. May and R. Vaidyanathan, ibid. **155** (1936), 519. The reader is referred to M. S. Livingston and H. A. Bethe, *Rev. Mod. Phys.* **9** (1937), 246, for a systematic collection of the early data on reactions and excitation energies.
‡‡ R. H. Dicke and J. Marshall, Jr., *Phys. Rev.* **63** (1943), 86.

the nuclei; it is probably more justifiable to consider the levels found thus far to be the 'easy' levels to excite, since, in the usual nuclear bombardment, the particles that get close enough to the target to make a transmutation have very low angular momentum (S- and P-waves in the orbital motion and spin-unity at most (deuteron bombardment)). Hence there is little chance of exciting low-lying states of high angular momentum, even though they exist. Offhand, one might expect the number of low-lying levels with high angular momenta to be quite large, especially in heavy nuclei, since the moment of inertia is $I = \frac{2}{5}M_n A^{\frac{5}{3}} r_{00}$ ($r_{00} = 1{\cdot}45 \times 10^{-13}$ cm., M_n is the nucleonic mass) and the kinetic energy of rotation is

$$\Delta E = \frac{\hbar^2 J(J+1)}{2I} = \frac{25J(J+1)}{A^{\frac{5}{3}}} \text{ M.e.v.}$$

for the rotational quantum number J. Thus, even in Al^{27} there should be levels spaced by a few hundred kilovolts, and in Pb the spacing drops to a few kilovolts. Such a profusion of low levels is in contradiction with many of the observed properties of nuclei, such as rather high metastable states in heavy nuclei, and the existence of strongly forbidden β-spectra, etc. This question has been taken up by Teller and Wheeler,† who point out that a nucleus should behave like a body of rather high spatial symmetry and that, consequently, many states (especially with small values of J) will be forbidden by the symmetrization of the wave-function, just as in the rotation of molecules. In fact, they find that a nucleus behaves approximately as though there were $A^{\frac{1}{3}}$ α-particles equidistantly disposed around its equator. Then, if the lowest state of the nucleus corresponds to $J = 0$, the next lowest state of rotation that is allowed by the Einstein–Bose statistics is $J = A^{\frac{1}{3}}$. Substituting in the formula for ΔE, we get an estimate of the first rotational level to be $25/A$ M.e.v. which is compatible with experiment. It would appear, therefore, that the 'easy' levels give a more complete term system than might be expected offhand but that there are several demonstrated cases of low-lying, metastable levels that are excited only indirectly by bombardment (cf. Chap. VII).

Although the term systems of nuclei have not been anywhere near completely explored, one can detect a tendency of the average distance between low-lying levels to decrease with increasing A

† E. Teller and J. A. Wheeler, *Phys. Rev.* **53** (1938), 778.

approximately as $A^{-\frac{1}{3}}$. This tendency, as well as absolute magnitudes of the spacings, substantiate Bohr and Kalckar's† interpretation of the lower levels of a nucleus as characteristic of the vibrations of a liquid droplet. The lowest lying levels of a liquid drop will be those in which the energy of excitation is in surface vibration at constant density. Suppose the surface radius, R_s, moves according to the law:

$$R_s = R_0\{1+\alpha_2 P_2(\cos\theta)\sin\omega t\}. \tag{5}$$

We have seen in Chapter VI that the surface potential energy due to such a distortion (from a sphere) will be

$$E_{\text{pot}} = \frac{8\pi}{5}\Omega R_0^2\alpha_2^2\sin^2\omega t,$$

and we shall disregard the change in potential energy of the Coulomb field which is relatively unimportant in lighter nuclei. Then assuming the radial velocity at r to be $v_r(r) = \alpha_2 r\omega P_2(\cos\theta)\cos\omega t$, the tangential velocity for irrotational flow will be $v_\theta = -\frac{3}{4}\alpha_2 r\omega\sin 2\theta\cos\omega t$ and the kinetic energy

$$E_{\text{kin}} = \frac{3}{20}\alpha_2^2 R_0^2\omega^2 A M_n\cos^2\omega t.$$

In order that the sum of potential and kinetic energy be independent of time, the coefficients of $\sin^2\omega t$ and $\cos^2\omega t$ have to be the same, and we get

$$\omega^2 = \frac{32\pi\Omega}{3AM}, \qquad \hbar\omega = \frac{28}{A^{\frac{1}{3}}}\text{ M.e.v.}$$

Hence the magnitude of the quanta of surface vibration vary with $A^{-\frac{1}{3}}$. Since there are five linearly independent surface vibrations of the type (15), and since these are probably coupled rather strongly with other modes, we should expect the average spacing between low-lying levels to be of the order of $\hbar\omega/5$ or $6/A^{\frac{1}{3}}$ M.e.v., in reasonable agreement with observation.

We turn now to the second category of excited quantum states in nuclei which includes the resonant energies of neutron absorption discussed in the preceding chapter. It will be recalled that the average distance between such levels appears to be about 10 e.v. for nuclei in the range $90 < A < 200$. This level spacing is much smaller than that for the low-lying levels, and the reason for this is that the nucleus concerned is the nucleus formed by *adding* a neutron (of small kinetic energy) to the original target nucleus. This temporary, composite

† N. Bohr and F. Kalckar, *Kgl. Danske Videnskab. Selskab.* **14** (1937), No. 10.

system is called the *compound nucleus*, and it is evident that the energy of excitation, produced by adding one slow neutron, is just the binding energy of the last neutron in the newly formed isotope and this is usually in the range of 10 to 6 M.e.v. As we shall see, the smaller spacing of these levels is due to the higher excitation, and we shall refer to these levels as *high-lying levels*. In every practical case the high-lying levels that have been determined experimentally are also levels of a compound nucleus rather than levels of a product nucleus.

It was first suggested by Gurney† that if the energy of the bombarding particles were close to one of the virtual energy-states of the composite system there would be a resonance phenomenon resulting in a large increase of the probability of a nuclear reaction. At that time the neutron was unknown and one had in mind virtual levels of nuclei that can emit (or accept) charged particles, such as the natural α-emitters; such levels are relatively long-lived, and hence well defined in energy, because a potential barrier has to be penetrated. The first indication of a large increase in the yield of emitted particles for certain discrete energies of the bombarding particles was found by Pose‡ in his experiments on the artificial transformation of aluminium under bombardment by α-particles. In these experiments Pose used thick layers of Al ($\sim$ 0·04 mm.), so that the incident α-particles were gradually losing energy and were finally stopped before escaping on the other side of the target. In such a case we should expect that the protons produced by collision of α-particles with Al nuclei at different depths would have different velocities, so that their observed energy-distribution would be in the form of a continuous spectrum extending from an upper limit determined by the original energy of the α-particle to much smaller values. The measurements indicated, however, that the ejected protons belonged to several more or less discrete groups, as shown in Fig. 42, where the number of ejected protons is plotted against their range. This observation can be explained only on the hypothesis that the observed groups of protons were produced at a number of particular depths in the Al target, evidently the depths at which the α-particles (continuously slowing down) have energies just corresponding to resonance penetration. A somewhat more detailed study of the resonance

† R. W. Gurney, *Nature*, **123** (1929), 565.
‡ H. Pose, *Zs. f. Phys.* **30** (1929), 780; **64** (1930), 1.

phenomena with aluminium was carried out by Chadwick and Constable,† who found a still larger number of discrete groups than was originally observed by Pose. It was shown that to each resonance-level correspond several proton groups—as if the α-particle entering

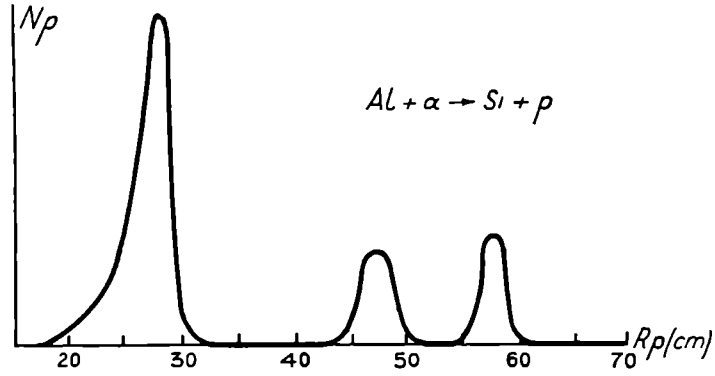


FIG. 42.

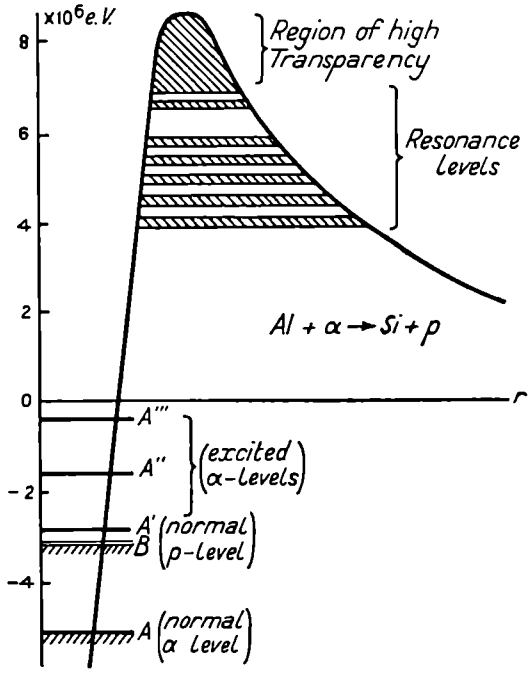


FIG. 43.

the nucleus with a particular resonance-energy might be captured on different inside levels and consequently give rise to an ejected proton of one of a number of different energies. The results of more recent investigations by Duncanson and Miller‡ are summarized in Fig. 43,

† J. Chadwick and J. Constable, *Proc. Roy. Soc.* **135** (1932), 48.
‡ W. Duncanson and H. Miller, ibid. **146** (1934), 396.

where the resonance-levels and the stationary levels for the α-particle are shown, together with the fundamental level of the proton before its ejection. We see that an incident α-particle may enter the nucleus through any one of six resonance-levels and that in each case four distinct modes of capture are possible—the particle may, first of all, occupy any one of the levels α_0, α_1, α_2, or α_3. Thus, with a beam of α-particles bombarding an aluminium target, four discrete groups of protons are produced for each of six resonance-energies of the α-particles. The shaded region near the top of the potential barrier in Fig. 43 is the region of high transparency through which an α-particle may enter the nucleus for a wide range of energies: the corresponding feature of the protons ejected from a thick aluminium target bombarded by α-particles is a more or less continuous energy-spectrum. This historical study thus illustrates the existence of low-lying levels, the 'excited α-levels' of Fig. 43, resonance penetration due to high-lying levels, and 'over the barrier', non-resonant, nuclear reaction, all in one. The 'over the barrier' alphas still give four distinct groups of protons corresponding to excitation of the residual nucleus at one of the 'α-levels'.

A somewhat simpler resonant reaction on Al is the radiative capture of protons

$$Al^{27}+H^1 \rightarrow Si^{28}+\gamma. \qquad (6)$$

Investigating this reaction with very thin aluminium targets and very well defined energies in the proton beam, Broström, Huus, and Tangen† have found thirty-six resonant levels of Si^{28} for proton energies between, 0·2 M.e.v. and 1·4 M.e.v. This is an average level distance of 30 k.e.v. at an average excitation energy of about 11·5 M.e.v. (binding energy of last proton plus 1 M.e.v.). The (p, γ) reaction has been studied in the same way on the other light monoisotopic elements Na,‡ F,§ and Be.‖ The apparent level distances are 87 k.e.v. for Mg^{24} at an excitation of about 12·5 M.e.v., 46 k.e.v. for Ne^{20} at about 13 M.e.v., and 400 k.e.v. for B^{10} at an excitation of about 8 M.e.v.

For one target nucleus at least, viz. C^{12}, resonances are observed††

† K. J. Broström, T. Huus, R. Tangen, *Phys. Rev.* **71** (1947), 661.

‡ R. L. Burling, ibid. **60** (1941), 340.

§ W. E. Bennett, T. W. Bonner, C. E. Mandeville, and B. E. Watt, ibid. **70** (1946), 882.

‖ W. J. Hushley, ibid. **67** (1945), 34.

†† C. L. Bailey, G. Freier, and J. H. Williams, ibid. **73** (1948), 274.

for (d, n) and (d, p) reactions. These results indicate an average spacing of the order of 150 k.e.v. in N^{14} at an excitation of about 11·5 M.e.v. Finally one has a suggestion of the level density in radioactive nuclei of the natural series at excitations of the order of 3 M.e.v. from their γ-spectra. Thus, Ellis and F. Oppenheimer have assigned levels to the RaC′ nucleus that would account for the γ-rays of that nucleus, Fig. 44. At 3 M.e.v. the average level spacing would appear to be about 30 k.e.v. in this nucleus for which $A = 214$.

The excitation energy in the high-lying levels is most probably shared among a large number of nucleons, at any given time, as a consequence of the strong short-range forces acting between them. The theory of the level density among these high states is then appropriately treated by the methods of statistical thermodynamics. The average number of levels per M.e.v., ρ, will then be proportional simply to the thermodynamic probability which is, in turn, e^S, where S is the entropy of the nucleus measured in units of k. The entropy is a function of the energy of excitation, U, and this relation serves to define a temperature of excitation, T, by

$$\frac{\partial S}{\partial U} = \frac{1}{T}. \tag{7}$$

If the excitation is sufficiently high so that all A-particles participate in sharing it, the entropy per particle will be some function of U/A, say $s = a(U/A)^\nu$, and the total entropy† will be of the form

$$S = aA^{1-\nu}U^\nu.$$

The value of ν depends upon the particular model adopted for the nucleus. If we assume that the A nucleons are independent particles in a Fermi gas near degeneracy,‡ the specific heat of the nucleus will be proportional to T, as in metals at low T, the entropy $S \sim T$, and the energy $U \sim T^2$; hence $\nu = \frac{1}{2}$ and

$$\rho_m \sim e^{a(AU)^{\frac{1}{2}}} \quad \text{(Fermi-gas model)}.$$

On the other hand, if the nucleus be considered more like an ordinary solid, for which the specific heat and entropy vary as T^3, then $U \sim T^4$ and $\nu = \frac{3}{4}$. This leads to

$$\rho(U) \sim e^{aA^{\frac{1}{4}}U^{\frac{3}{4}}} \quad \text{(solid-body model)}.$$

† This argument is due to E. Teller, private communication.

‡ H. A. Bethe, *Phys. Rev.* **50** (1936), 332; this theory plus making allowance for correlation (which results in replacing A by $\frac{1}{2}A$, effectively) has been worked out also by J. Bardeen, ibid. **51** (1937), 799.

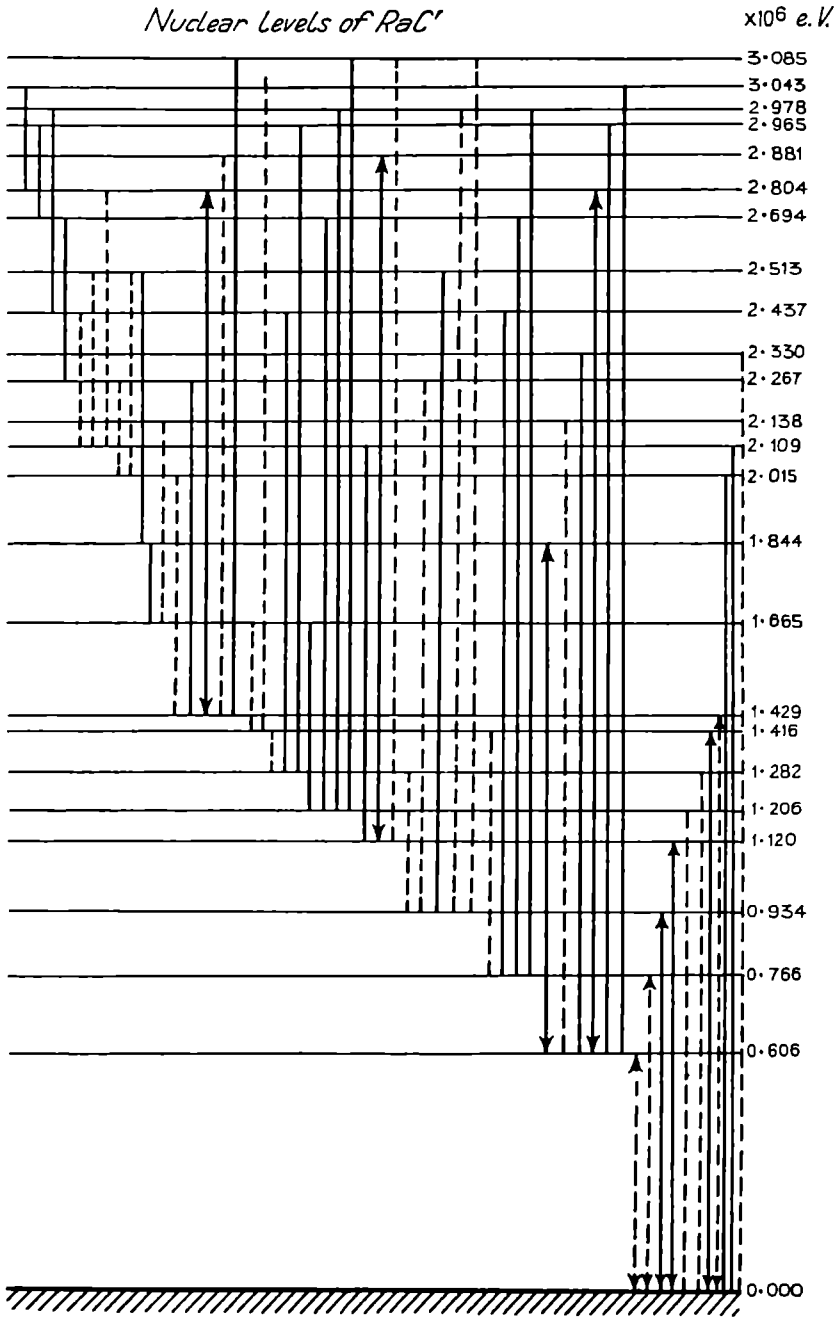

Nuclear Levels of RaC'
x10⁶ e.V.
3·085
3·043
2·978
2·965
2·881
2·804
2·694
2·513
2·437
2·330
2·267
2·138
2·109
2·015
1·844
1·665
1·429
1·416
1·282
1·206
1·120
0·934
0·766
0·606
0·000

FIG. 44.

Bethe† has developed the theory of statistical thermodynamics in its application to find level densities in general. The method is to construct the thermodynamic potential that pertains to definite values of the volume of the nucleus, V, temperature,‡ T, proton activity λ_Z, neutron activity, λ_N, and free energy of rotation about an arbitrary axis, $\zeta_M = T \ln \lambda_M$. This potential is then T times the logarithm of the grand partition function Ξ:

$$\Xi = \sum_{k,n',z',m'} \lambda_N^{n'} \lambda_Z^{z'} \lambda_M^{m'} e^{-E_k/T} = e^{\Phi/T}, \tag{8}$$

where the sum runs over all possible energy-levels of the nuclear systems, E_k, over all neutron numbers, n', proton numbers, z', and components of angular momentum, m'. If we write

$$\zeta_M = T \ln \lambda_M$$
$$\zeta_N = T \ln \lambda_N$$
$$\zeta_Z = T \ln \lambda_Z$$

we may define the level density $\rho(E', n', z', m')$ by expressing (8) as an integral:

$$e^{\Phi/T} = \iiiint \rho(E', n', z', m') e^{-(E' - n'\zeta_N - z'\zeta_Z - m'\zeta_M)/T} \, dE' dn' dz' dm'. \tag{8 a}$$

We now impose the conditions that the total energy have the average value U, total angular momentum M, neutron number N, and proton number Z. In a sufficiently large system, the integrand in eq. (8 a) will have a sharp maximum about $E' = U$, $n' = N$, $z' = Z$, and $m' = M$, and it is convenient to write (8 a) in the form

$$e^{\Phi/T} = \rho(U, N, Z, M) e^{-(U - N\zeta_N - Z\zeta_Z - M\zeta_M)/T} \lambda(U, N, Z, M), \tag{8 b}$$

where

$$\lambda(U, N, Z, M) = \iiiint \frac{\rho(E', n', z', m')}{\rho(U, N, Z, M)} \times$$
$$\times \exp \frac{(U - E') + (N - n')\zeta_N + (Z - z')\zeta_Z + (M - m')\zeta_M}{T} \times$$
$$\times dE' dn' dz' dm'. \tag{8 c}$$

From the general theory of statistical thermodynamics§ we know

† H. A. Bethe, *Rev. Mod. Phys.* **9** (1937), 81.

‡ In keeping with the usual practice in nuclear physics we shall measure temperature in M.e.v., i.e. T takes the place of kT in the familiar form of the equations.

§ Cf. R. H. Fowler and E. A. Guggenheim, *Statistical Thermodynamics* (Cambridge, 1939), especially chap. vi.

that the entropy S of the system is related to Φ and the average values, U, N, Z, and M through:

$$ST = \Phi + U - N\zeta_N - Z\zeta_Z - M\zeta_M. \qquad (8\,d)$$

Hence, in good approximation,

$$\frac{\rho(E', n', z', m')}{\rho(U, N, Z, M)} \cong e^{S(E',n',z',m')-S(U,N,Z,M)}. \qquad (8\,e)$$

Substituting (8 *e*) into (8 *c*), $\lambda\,(U, N, Z, M)$ may be evaluated in the usual way by expanding the exponent of e to second order in a Taylor series (first non-vanishing terms) and integrating.

This theory has been applied directly to several models for nuclei by Bethe† (loc. cit.). We shall consider here only the simplest type of nuclear model, viz. a degenerate Fermi gas of protons and neutrons. Nuclear excitations are then achieved by promoting relatively few ($\sim \sqrt{A}$) nucleons to higher orbits, and the entropies and derivatives will be calculated approximately from the known theory of the degenerate gas, e.g. the theory of metallic electrons. Moreover, we are not interested in the density of levels for a given component, M of the angular momentum, but rather the density for a definite total angular momentum, J. The latter can be found by subtracting $\rho(U, N, Z, M)$ from $\rho(U, N, Z, M+1)$. In this way Bethe finds for the average number of energy-levels per M.e.v., and for given J, and $N+Z = A$:

$$\rho_J(U) = \frac{2J+1}{1{\cdot}6S^4}\, e^S \text{ M.e.v.}^{-1}$$
$$S = S(U, A) = \pi\left(\frac{AU}{\zeta_0}\right)^{\frac{1}{2}} = \sqrt{(AU/2{\cdot}30)}, \qquad (9)$$

where U is measured in M.e.v., and ζ_0, the Fermi energy at $T = 0$, is 22·75 M.e.v. Considering that nuclei are not really very extensive statistical systems, the numerical factor in (9) is relatively meaningless. The general dependence of ρ and of S on A and U appears to be quite acceptable, however. The S–U relation may be used to define a nuclear temperature:

$$\frac{1}{T} = \frac{\partial S}{\partial U}, \qquad T = 2\frac{U}{S}. \qquad (10)$$

† In Bethe's work in *Rev. Mod. Phys.* **9** (1937), 91, the nuclei are assumed to have 'large' radii, i.e. $r_0 = 2{\cdot}0 \times 10^{-13}\, A^{\frac{1}{3}}$ cm., and this leads to too dense level systems in the free-particle model considered here. With these large nuclei it was found that the entropy of the surface vibrations was practically sufficient to account for the level density, but such is not the case for the radii adopted in this book, and now generally accepted, viz. $1{\cdot}45 \times 10^{-13}\, A^{\frac{1}{3}}$ cm. The theory presented here is essentially that of Bethe's original paper on the subject, *Phys. Rev.* **50** (1936), 332.

Comparison between $\rho_0(U) \equiv \rho_J(U)/2(2J+1)$ and the observed densities for $Si^{28}Al(p,\gamma)$ and $Eu^{152}(n,\gamma)$ are shown in Table XXIII. In that table is included also the 'nuclear temperature' corresponding to the excitations involved. It is evident that the values in the column labelled ρ_0 are well within the reach of the values under 'Observed' considering the uncertainties in the numerical factor in (9) and the fact that J remains unspecified. It is by no means clear at the present time that the agreement between eq. (9) and the

TABLE XXIII

Number of levels per M.e.v.

A	*Observed*	$U_{\text{M.e.v.}}$	S	$\rho_0(calc.)$	T
28	33	11·5	11·8	6	2·0
152	50,000	8	22·8	11,000	0·7

observed density of levels is not partly fortuitous. In our model no account has been taken of the contributions to the entropy due to the specific interaction between nucleons, effects near the nuclear surface, and so on.

In application of this theory to nuclear cross-sections we shall be interested, in many instances, in the average distance between levels of given J, M, and *parity*, $\bar{D}(U)$. The level density for given J and M will be given by (9) divided by the degeneracy factor $2J+1$ and then again by a factor 2 to take account of the fact that (9) includes both parities. The result is just the $\rho_0(U)$ defined above; and $\bar{D}(U)$ is the reciprocal of $\rho_0(U)$:

$$\begin{aligned} \bar{D}(U) &\cong 3S^4 e^{-S} \\ S &= \sqrt{(AU/2{\cdot}30)}. \end{aligned} \tag{11}$$

2. The theory of nuclear reactions

The concept of nuclear reactions has been used repeatedly in preceding sections and chapters, especially in connexion with the energy-balances involved, the determination of excited states, and the occurrence of resonant phenomena. We shall consider the general theory of such reactions further in this section. The first example of an artificial nuclear transformation was discovered by Rutherford† in 1919, who observed that protons are created by bombarding nitrogen

† E. Rutherford, *Phil. Mag.* Ser. 6, **37** (1919), 581.

with the fast α-particles from RaC′. Wilson chamber photographs of this (Pl. II, facing p. 256), taken by Blackett,† show the tracks of the bombarding particle, the ejected proton, and the recoil nucleus—but no track corresponding to the original α-particle moving away after collision. This nuclear reaction would then be written:

$$N^{14}+He^{4} \rightarrow O^{17}+H^{1}+Q, \tag{12}$$

where Q is the energy-balance.

The left-hand side of eq. (12) represents the reverse of α-disintegration, the right-hand side is the result of the disintegration of an excited, compound nucleus, F^{18*}, by proton emission. In accordance with the concept of the compound nucleus the reaction (12) is most conveniently divided into two *independent steps*, the first of which is:

$$N^{14}+He^{4} \rightarrow F^{18*}. \tag{13}$$

The excited nucleus, F^{18*}, may, in general, disintegrate in many ways, and it is assumed that the probabilities of the various modes of disintegration are independent of the particular method by which the F^{18*} was created. In other words, the F^{18*} is a radioactive nucleus in the sense used in Chapter VI. The number of possible modes depends upon the energy of excitation. If we suppose that the kinetic energy of the α-particle supplies 8 M.e.v., its binding energy (cf. Table II) adds another 4 M.e.v., giving a total excitation of 12 M.e.v. above the ground state of F^{18}. Such a highly excited state may disintegrate in the following simple ways:

$$\begin{aligned}
F^{18*} &\rightarrow N^{14}+He^{4} \text{ (8 M.e.v. kinetic energy) elastic scattering}\\
&\rightarrow N^{14*}+He^{4} \ (< 8 \text{ M.e.v. kinetic energy}) \text{ inelastic scattering}\\
&\rightarrow \text{ditto—as many ways as there are available energy-levels in } N^{14}\\
&\rightarrow O^{17}+H^{1} \text{ (6·5 M.e.v. kinetic energy)}\\
&\rightarrow O^{17*}+H^{1} \ (< 6{\cdot}5 \text{ M.e.v. kinetic energy}) \text{ as many ways as there are available energy-levels in } O^{17}\\
&\rightarrow F^{17}+n \text{ (3 M.e.v. kinetic energy)}\\
&\rightarrow F^{17*}+n \ (< 3 \text{ M.e.v. kinetic energy}) \text{ many ways, etc.}\\
&\rightarrow F^{18}+h\nu \text{ (12 M.e.v.)}\\
&\rightarrow F^{18*}+h\nu \ (< 12 \text{ M.e.v.}) \text{ many ways.}
\end{aligned} \tag{14}$$

† P. Blackett, *Proc. Roy. Soc.* A **107** 1925), 349; P. Blackett and D. Lees, ibid. **136** (1932), 325.

Owing to the fact that the residual nucleus may be left in one of a variety of states of excitation, the number of possible reactions increases very rapidly with the excitation of the compound nucleus. Each type, i, of disintegration (14) is characterized by a definite probability of emission, $\Gamma^i/\hbar$, so that the relative probability of the kth type of emission is equal to

$$P^k = \frac{\Gamma^k}{\sum_i \Gamma^i} = \frac{\Gamma^k}{\mathbf{\Gamma}} \tag{15}$$

with

$$\mathbf{\Gamma} = \sum_{all\ k} \Gamma^k$$

because of the assumption that the probabilities are independent. The sum of all 'widths', $\mathbf{\Gamma}$, is called the total width of the compound state in question, and the individual Γ^i are called partial widths.

The absolute values of the various Γ^k are to be calculated for neutrons, protons, and α-particles in just the same way as in the preceding section on resonant scattering. Each Γ^k is composed of a 'fundamental width', Γ^0, which is related to the distance between consecutive, quasi-stationary levels in the compound nucleus at the excitation involved (12 M.e.v. in the case of (14)), and of a penetration factor, $G_{l,\eta}^{-2}$. From the discussion in the preceding section we found a 'fundamental' particle width of the form

$$\Gamma_r^0 = 2kR\left[\frac{dE}{d(KR\cot\phi)}\right]_r = 2\frac{k}{K}\left(\frac{dE}{d\phi}\right)_r$$

for the resonant level r for which $\csc^2\phi_r = 1$. Since ϕ advances by π radians in going from one resonant level to the next, we may approximate the *mean distance between levels* by

$$\bar{D} = \pi\left(\frac{dE}{d\phi}\right)_{\bar{r}} \equiv \bar{D}(U),$$

where the bar over $\bar{r}$ indicates an average over levels in the vicinity of the excitation energy, U; we then express the mean fundamental width, Γ^0, in terms of $\bar{D}(U)$:

$$\Gamma^0 = \frac{2k}{\pi K}\bar{D}(U). \tag{16}$$

The simplest partial width entering eq. (15) is that for the emission of a neutron in an S-wave, for which the penetration factor is unity and (16) is the complete expression. In such a light compound nucleus one should appeal to experimental results for an estimate of

$\bar{D}(U)$ when possible; otherwise, the statistical formula has to be used, and with $A = 18$, $U = 12$ this leads to an average level distance (between levels of same J, M, and parity) of about 1 M.e.v. Similarly, the value of K, the wave-number (times 2π) inside the compound nucleus, may be subject to noticeable variations in light nuclei, but we shall estimate it in the usual way from the average potential energy per nucleon, viz. $\hbar^2K^2/2M \simeq 23$ M.e.v. Finally, the possible values of k must be estimated in the light of the binding energy of the neutron, plus considerations of selection rules where possible. In reaction (13) we shall assume that the α-particle enters the (even) N^{14} nucleus in an S-wave, so that the F^{18*} nucleus is an even state with spin-unity (same as N^{14}). The ground state of F^{17} has spin $\frac{1}{2}$ and is probably even. Consequently, it appears possible for a neutron to be emitted with the maximum energy for such a reaction (3 M.e.v. in this case) leaving the residual F^{17} in its ground state. Then, $\hbar^2k^2/2M = 3$ M.e.v., determines k. If there were an excited state, of the same spin and parity, less than 3 M.e.v. above the ground state of F^{17}, say at U_1, a second mode of neutron emission would be with $\hbar^2k'^2/2M = 3-U_1$ M.e.v. The Γ for this mode will be somewhat less than that for the former because of the smaller k. In fact, it seems unlikely that there is an excited level of F^{17} similar to the ground state and within 3 M.e.v. of the latter, considering that the estimated level distance at an excitation of 12 M.e.v. in F^{18} is around 1 M.e.v.

Individual neutron widths are of particular interest also to the interpretation of the resonant capture of neutrons in nuclei with $A > 90$. We shall return to this question when we consider the accompanying γ-radiation, but, using eq. (16) we estimate the S-neutron widths at once from the observed fact that the level distances seem to be about 20 e.v. Expressing the energy of the neutron in electron volts, E_{ev}, and estimating K as above, we find $\Gamma^n \simeq 3\times 10^{-3}\sqrt{E_{\text{ev}}}$ electron volts for slow neutrons on these nuclei.

If a compound nucleus is excited by an energy U above its ground state, and if the absolute value of the binding energy of one neutron is B ($< U$), the nucleus may emit a neutron of any kinetic energy between zero and $U-B$, provided there are available quantum states of the residual nucleus. In heavy nuclei, and for high excitations, the number of available quantum states is enormous and the transition probability into any one of them (including the ground state) is negligible compared with the sum over all such transitions. In such

cases it is convenient to replace the individual partial widths, of eq. (15) for example, by an *integrated width* for neutron emission. Then P^k gives the relative probability that a neutron of any energy was 'evaporated' from the nucleus, rather than the probability that a transition was made to one particular state of the residual nucleus. As an example, suppose a neutron of energy E falls on a nucleus and forms a compound nucleus in which it gains the binding energy B. The compound nucleus, which has an excitation $B+E$, may now emit a neutron of energy E' ($< E$) leaving the residual nucleus in a state of excitation $E-E'$. The probability of such an emission is proportional to the density of nuclear levels at this excitation and may be written in the form:

$$dP(E') = \sqrt{\left(\frac{E'}{E_0}\right)}\,\bar{D}(B+E)\rho(E-E')\,dE', \tag{17}$$

where, instead of the K in eq. (16), we use $E_0 = \frac{\pi^2}{8}\frac{\hbar^2K^2}{M}$. Under the conditions postulated, the statistical formula for $\rho(E-E')$ may be used, and since this increases so rapidly with its argument, the maximum of (17) comes at very small values of E'. It is then convenient to approximate $\rho(E-E')$ by an exponential that has the same value and same derivative as the correct statistical formula when $E' = 0$, viz.

$$\begin{aligned} \rho(E-E') &= \rho(E)e^{-\frac{1}{2}\{S(E)-4\}E'/E} \\ dP(E') &= \frac{\bar{D}(B+E)\rho(E)}{\sqrt{E_0}}\sqrt{E'}e^{-\frac{1}{2}\{S(E)-4\}E'/E}\,dE'. \end{aligned} \tag{18}$$

The expression (18) is a Maxwellian distribution of the 'evaporated' neutrons if the effective temperature is assumed to be related to the excitation of the *residual* nucleus through

$$T_{\text{evap}} = \frac{2E}{S(E)-4}\text{M.e.v.}, \tag{19}$$

i.e. a little higher than the thermodynamic temperature, $2E/S(E)$. According to the evaporation model, therefore, practically all of the neutron width is due to the emission of rather slow neutrons (provided always that E is large enough to justify the method just outlined) and the integrated width may be found, with negligible error, by integrating (18) from zero to infinity:

$$\Gamma^n_{\text{inel}} = \frac{\sqrt{\pi}\,\bar{D}(B+E)\rho(E)}{2\sqrt{E_0}}\{T_{\text{evap}}\}^{\frac{3}{2}}. \tag{20}$$

In principle, the integrated width would be a useful concept also for particles that have to go through a potential barrier, if the excitation of the compound nucleus is sufficiently high. In most practical cases, however, the penetration factor decreases so rapidly with decreasing energy of the emitted particle (especially charged particles) that the most probable transitions are those in which the proton or α-particle carry away the most energy rather than those that leave the residual nucleus with the most energy. The balance between penetration and high-level density is probably intermediate for the emission of neutrons with orbital angular momentum ($l\lambda > R$), but no practical cases of this type have been established; the principal example of this type of 'penetration' is afforded by the emission of γ-rays which we now consider.

In Chapter VII we found that the emission of a *single* γ-ray from RaC′ with an energy of 0·603 M.e.v. corresponded to the probability 3×10^{11} sec.$^{-1}$ This corresponds to the partial width

$$\Gamma^{0\cdot603} = \frac{3\times10^{11}\times1\cdot05\times10^{-27}}{1\cdot6\times10^{-12}} \cong 2\times10^{-4} \text{ e.v.}$$

In order to apply this measurement to estimating the partial widths for γ-rays in other nuclei and of other energies, we shall make two assumptions: (1) that $\Gamma^{h\nu}$, like the particle widths, is proportional to $\bar{D}(U)$, since a partial width should not be greater than the distance between neighbouring levels of the same J, M, and parity; and (2) that $\Gamma^{h\nu}$ is proportional to the fifth power of $h\nu$,† since γ-transitions appear to be predominantly of the quadrupole type, and a fifth power of the wave-length typifies the penetration of a D-wave. If we estimate the $\bar{D}(U)$ pertaining to the low-lying levels of RaC′ on the basis of the surface vibrations of the (charged) liquid droplet we get about 1·5 M.e.v. and this leads to our estimate of an individual γ-width

$$\Gamma^{h\nu} = \left(\frac{h\nu}{0\cdot603}\right)^5 \frac{\bar{D}(U)}{1\cdot5\,.\,10^6} 2\times10^{-4} \simeq 2\times10^{-9}(h\nu)^5\bar{D}(U), \qquad (21)$$

where $h\nu$ is measured in M.e.v. Essentially the same formula is obtained by Weisskopf (loc. cit.) on the basis of observations of (γ, n) cross-sections.

Ordinarily, a compound nucleus may emit any of a large number of γ-rays which leave the residual nucleus in different states of

† V. F. Weisskopf, *Phys. Rev.* **59** (1941), 318.

excitation, so that one is again interested in the integrated width rather than the individual γ-width. The only question remaining is whether the integrated γ-width can compete favourably with the probability of particle emission. Let $\rho_c(E)$ be the density of *combining* levels at the excitation E of the residual nucleus, i.e. the final state of the first γ-transition. Then the integrated γ-width for a compound nucleus of excitation U is estimated as

$$\Gamma^\gamma = 2\times 10^{-9}\bar{D}(U)\int_0^U (U-E)^5\rho_c(E)\,dE. \tag{22}$$

According to § 1, we may express $\rho_c(E)$ in the form

$$\rho_c(E) = CS(E)^{-4}\exp S(E),$$

where C is an average of $(2J+1)/3$ over the combining final states, and hence is a coefficient of the order unity. The integrand in (22) has a maximum at a certain energy, E_m, such that

$$n = \left[\frac{d\ln\rho_c(E)}{d\ln E}\right]_{E_m} = \tfrac{1}{2}S(E_m)-2 = \frac{5E_m}{U-E_m}. \tag{23}$$

Equation (23) determines both the energy of the most probable final state, E_m, and the exponent n which we may use to approximate the functional dependence of $\rho_c(E)$ in the following form:

$$\Gamma^\gamma = 2\times 10^{-9}\,3C\,\frac{\bar{D}(U)}{\bar{D}(E_m)}\int_0^U (U-E)^5\left(\frac{E}{E_m}\right)^n dE, \tag{24}$$

in which we use also the relation $\rho_c(E_m) = 3C/\bar{D}(E_m)$. Equation (24) may be integrated readily, and, with the help of Stirling's formula for the factorials involved, we finally obtain:

$$\Gamma^\gamma = 6\times 10^{-9}C\sqrt{(10\pi UE_m)}\,\frac{(U-E_m)^6}{6U-E_m}\,\frac{\bar{D}(U)}{\bar{D}(E_m)}. \tag{25}$$

As mentioned above, the integrated γ-width is of particular interest for the elements with $A > 90$ that absorb slow neutrons. If we take Eu^{151} as an example, we see from Table XXIII that $\bar{D}(U) \simeq 90$ e.v. (using the statistical theory) and from the solution of (23), setting $U = 8$, we get $E_m = 4{\cdot}56$. This value of E_m indicates that the $(h\nu)^5$ factor has a strong influence on the spectral distribution and, in fact, leads to a most probable γ-ray energy of 3·5 M.e.v. From E_m we find $\bar{D}(E_m) = 9000$ e.v., and substituting into (25) we finally obtain

$$\Gamma^\gamma = 0{\cdot}08C \text{ e.v.} \tag{26}$$

We note, first of all, that the result (26) is in satisfactory agreement with the observed widths for absorption of neutrons which are generally of the order of 0·1 e.v. Secondly, we should compare the width (26) with that for the emission of the neutron which formed the compound state: $\Gamma^n \sim 3\times 10^{-3}\sqrt{E_{\text{ev}}}$ e.v. For neutron energies $< 1{,}000$ e.v., $\Gamma^\gamma > \Gamma^n$, and the more probable process is the emission of a γ-ray thus leading to capture. For neutron energies $> 1{,}000$ e.v. the more probable process is the re-emission of the neutron. At very high neutron energies the residual nucleus will be left in an excited state in the usual event, as cited above. These remarks are founded on the further assumption that only neutron or γ-emission is important to the decay of the compound nucleus. Since the remaining possibilities involve charged particles, this assumption is mostly valid for heavy nuclei in which the Coulomb barrier is very high, especially when the reaction is due to slow neutrons. One very important exception is the nucleus U^{235} which, when a slow neutron is added, may show fission as well as γ-emission or re-emission of the neutron. The probability of fission will be represented also by an integrated width Γ^f since the two fragments may be left with a great variety of states of excitation. The fraction of disintegrations of the U^{236*} that are fissions will then be

$$P^f = \frac{\Gamma^f}{\Gamma^f+\Gamma^n+\Gamma^\gamma}. \tag{27}$$

A preliminary study of the variations of Γ^f, Γ^n, Γ^γ with the energy of excitation of the compound, U^{236}-nucleus has been carried out by Bohr and Wheeler,† and the results of their calculations are shown in Fig. 45. It can be seen from this figure that the fission width, Γ^f, is a much more rapidly increasing function of excitation energy than Γ^γ. The curves representing these two widths cross at a certain energy, and at higher values of the energy the competition of the γ-radiation becomes of minor importance. It must be remarked here that, whereas the Bohr–Wheeler curves give a fair qualitative picture of the situation, they cannot be relied upon for quantitative results.

The foregoing discussion has been chiefly concerned with the development of that part of the theory of nuclear reactions which is exemplified by the second phase of reaction (12), viz. (14). We shall consider now the first phase, i.e. (13), which we shall represent in

† N. Bohr and J. A. Wheeler, *Phys. Rev.* **56** (1939), 426.

terms of a *cross-section for the formation of the compound nucleus*† which we shall denote $\sigma^{\alpha *}$, the α-exponent indicating that in this particular case the compound nucleus is formed by an α-particle. Then the cross-section for a certain reaction (α, k) may be expressed

$$\sigma(\alpha, k) = \sigma^{\alpha *} \frac{\Gamma^k}{\Gamma}. \tag{28}$$

For simplicity, let us assume that the incident α-particle has such an

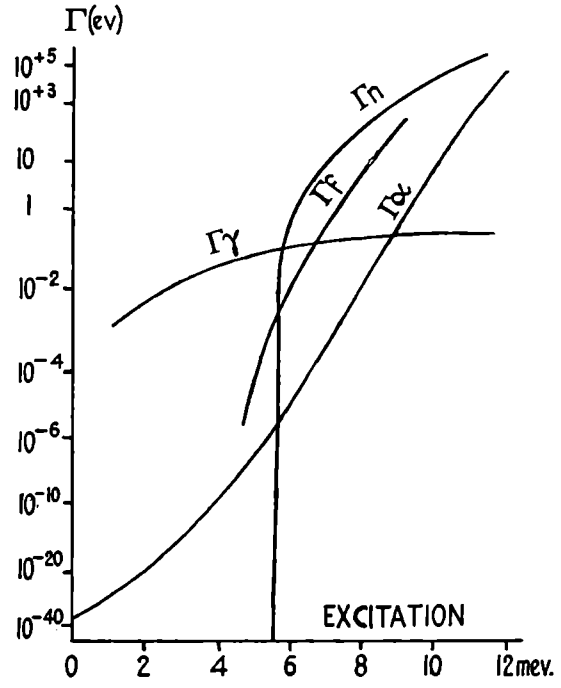


FIG. 45. The variation of Γ_γ, Γ_n, Γ_f, and Γ_α with the excitation energy of the compound nucleus after Bohr and Wheeler. (Neutron binding energy is assumed to be 5·6 M.e.v.)

energy that the collision takes place close to a single S-resonant level. We have seen in § 2, Chapter VIII, that if the compound nucleus could do nothing but scatter the α-particle elastically it would lead to the cross-section (disregarding the Coulomb and other potential terms which are of no interest to the argument)

$$\sigma_0(\alpha, \alpha) = \frac{\pi \lambda_\alpha^2 (\Gamma_r^\alpha)^2}{|E - E_r + \frac{1}{2} i \Gamma_r^\alpha|^2}. \tag{29}$$

We now have to generalize this formula to take account of the fact that the compound state, r, may disintegrate in many ways in

† Cf. V. F. Weisskopf and D. H. Ewing, ibid. **57** (1940), 472.

addition to re-emitting the α-particle. According to Chapter VI, this fact may be expressed by adding to the eigenvalue E_r just $-\frac{1}{2}i$ times the sum of the Γ^k for such additional modes of disintegration. This results, therefore, in replacing the Γ_r^α in the denominator of (29) by the total width $\mathbf{\Gamma}$! The general, single-level formula for elastic scattering of the α-particle is then

$$\sigma_{el}(\alpha,\alpha) = \frac{\pi\lambda_\alpha^2(\Gamma_r^\alpha)^2}{|E-E_r+\frac{1}{2}i\mathbf{\Gamma}|^2}. \tag{30}$$

Comparing (30) with the expression (28) we obtain the desired cross-section for the formation of a compound nucleus by an α-particle in an S-wave:

$$\sigma^{\alpha *} = \frac{\pi\lambda_\alpha^2\,\Gamma^\alpha\mathbf{\Gamma}_r}{|E-E_r+\frac{1}{2}i\mathbf{\Gamma}_r|^2}.$$

This formula may be generalized at once to an arbitrary type, j, of incident particle with orbital angular momentum $l\hbar$:

$$\sigma^{j*} = \frac{\pi(2l+1)\lambda_j^2\,\Gamma^j\mathbf{\Gamma}}{(E-E_r)^2+\frac{1}{4}\mathbf{\Gamma}^2}. \tag{31}$$

The immediately preceding considerations are limited to a single resonant state of the compound nucleus. In order to extend them to many such states we shall assume that each level contributes independently to the cross-section† and that the spread in energy of the incident particles is sufficient to cover a number of resonant levels. We then integrate over E to obtain the contribution of each level, r, to σ^{j*} and call the contribution, $\Delta_r\sigma^{j*}$:

$$\begin{aligned}\Delta_r\sigma^{j*} &= (2l+1)\pi\lambda_j^2\int_{-\infty}^{\infty}\frac{\Gamma_r^j\mathbf{\Gamma}_r}{(E-E_r)^2+\frac{1}{4}\mathbf{\Gamma}_r^2}dE \\ &= (2l+1)\pi\lambda_j^2\,2\pi\Gamma_r^j.\end{aligned} \tag{32}$$

But $(2l+1)\pi\lambda_j^2$ is the maximum cross-section for the particle j to form a compound nucleus. Hence $2\pi\Gamma_r^j$ is, so to speak, the width of the energy-band in which the particle j surely enters the level r. The probability that the incident particle will have an energy such as to fall within this band, or within the similar band for any of the neighbouring levels, defines an average transmission coefficient, $T(E)$,

† The intermediate situation in which just a few levels are important and in which one might be interested in the phase relations between particles emitted from each requires more detailed consideration. An example of this type is the scattering of neutrons by He considered in § 2, Chapter VIII.

for the formation of a compound nucleus as the result of a single collision. If $\overline{\Gamma}^j$ is the average particle width in this region of energy, and U the resulting excitation, the transmission coefficient may be written:

$$T(E) = \frac{2\pi\overline{\Gamma}^j}{\overline{D}(U)}. \tag{33}$$

On the other hand, the average transmission coefficient may be calculated from wave-mechanical considerations, as presented in Chapter VI:

$$\begin{aligned} T(E) &= 4\frac{v_e}{v_i} \times (\text{penetration factor}) \\ &= 4\frac{k}{K} G_{l,\eta}^{-2}. \end{aligned} \tag{34}$$

Combining eqs. (33) and (34) we get an expression for $\overline{\Gamma}^j$

$$\overline{\Gamma}^j_{l,\eta} = \frac{2}{\pi}\frac{k}{K}\overline{D}(U) G_{l,\eta}^{-2}, \tag{35}$$

which is just what one gets also by multiplying the average 'fundamental width' (16) by the penetration factor for the particle under consideration.

We are now prepared to write expressions for the cross-section for a general reaction (P, Q), at least in certain limiting cases. Let us assume first that a single resonant level is important to the reaction. If the incident particle and target nucleus have no spins the cross-section may be written:

$$\sigma(P, Q) = \frac{(2l+1)\pi\lambda\!\!\!^{-2}_P \Gamma^P\Gamma^Q}{(E-E_r)^2+\frac{1}{4}\mathbf{\Gamma}_r^2}, \tag{36}$$

where Γ^P is interpreted as the partial width for emission of the particle (P), with 'wave-length' $\lambda\!\!\!^{-}_P$ from the compound nucleus into a normalized eigenstate of orbital angular momentum $l\hbar$. For convenience in the comparison between states of general l and S-states, we shall normalize the angular dependence of these eigenstates to 4π. The angular dependence of a normalized state, with $m = 0$, is then $(2l+1)^{\frac{1}{2}}P_l(\cos\theta)$. The angular dependence in the incident plane wave is $(2l+1)P_l(\cos\theta)$, whence the intensity in the incident wave is $2l+1$ times normalized, thus accounting for the factor $(2l+1)$ in eq. (36) and similar formulae. The interpretation of Γ^Q, on the other hand, depends upon the physical nature of the cross-section being sought. We could ask, for example, for the cross-section for the process

leading to the particle (Q) in a particular state of energy, spin, and orbital angular momentum. In most cases the individual states of spin and (especially) orbital angular momentum cannot be disentangled, so that it is physically meaningless to ask for such particular information. We have seen already a general emission probability for inelastic scattering would be obtained by summing over all possible energies of the emitted particles (neutrons, in particular). In the same way, one would sum over all possible states of angular momentum to obtain an over-all cross-section. In some cases, however, one is interested in the angular distribution of the Q-particles, i.e. $\sigma(P, Q, \theta)$ becomes a differential cross-section. Then one includes in the individual $\Gamma^Q_{l',m}$ the transition probability from the normalized eigenstate (l', m) to a plane wave of direction θ, viz.

$$(2l'+1)\frac{l'-m!}{l'+m!}\{P^m_{l'}(\cos\theta)\}^2,$$

and sums over all l' and m. The result cannot be a more complicated function, i.e. a higher power of $\cos\theta$, than that contained in the square of the incident harmonic $P_l(\cos\theta)$. In the following the meaning of Γ^Q will be understood to be determined by the nature of the cross-section desired.

We shall retain the convention, however, that Γ^P pertains to the emission-width for a normalized state. This complicates the formulism a little if incident particle and target nucleus have spins. Let these spins be $s\hbar$ and $j\hbar$ respectively, and let the relative orbital motion be $l\hbar$. There are then $(2s+1)(2j+1)(2l+1)$ possible states of which $2J+1$ may form a particular compound nucleus of 'spin' $J\hbar$.† Thus, if a single compound nucleus of spin $J\hbar$ is involved, the formula becomes:

$$\sigma(P, Q) = \frac{2J+1}{(2s+1)(2j+1)}\frac{\pi\lambda^2_P\Gamma^P_l\,\Gamma^Q}{(E-E_r)^2+\frac{1}{4}\mathbf{\Gamma}^2_r}. \tag{37}$$

If the spins involved are high enough, the same compound level may be formed by incident waves of different l (but same parity). The waves of Q belonging to indistinguishable final states will then not interfere, since the incident partial waves will belong to different spin-polarizations. However, those reactions that spring from the same initial state and go to the same final state through different compound nuclei produce interfering waves of Q-particles, so that one

† For the proof see H. A. Bethe and G. Placzek, *Phys. Rev.* **51** (1937), 450, or G. Breit and B. T. Darling, ibid. **71** (1947), 402.

should add amplitudes of emitted waves and then take the absolute square to compute the cross-section. This implies taking the (absolute) square root of each Γ^Q which, in turn, admits an arbitrary phase-factor. The theory described herein gives no information about this new factor. In the example of elastic scatting (Chap. VIII) Γ^Q itself appears. All these rules are exemplified in the theoretical treatment of the angular distribution of α-particles from the $Li^7(p, \alpha)$-reaction,† but they will not be considered further here.

The question of phase relations between waves emitted through various compound states remains unsettled for the other limiting case which we shall discuss, viz. that in which the resonant levels are too close together to be resolved. In deriving eq. (32) it was assumed that the phases were uncorrelated, so that the levels could be considered statistically independent. We shall continue with that assumption.‡ Disregarding spins (or rather, summing over all J) the average cross-section for the formation of a compound nucleus, if the orbital motion is restricted to $l\hbar$, is, using eq. (32):

$$\sigma^{P*} = 2\pi^2(2l+1)\lambda\!\!\!^{-2}_P \frac{\Gamma^P_l}{\bar{D}(U)}. \tag{38}$$

The reaction cross-section $\sigma(P, Q)$ will be $\Gamma^Q/\mathbf{\Gamma}$ times (38), and, in the event that many values of l can contribute to the emission of Q, the effective cross-section may be written

$$\sigma(P, Q) = \sum_l 2\pi^2(2l+1)\lambda\!\!\!^{-2}_P \frac{\Gamma^P_l}{\bar{D}(U)} \frac{\Gamma^Q}{\mathbf{\Gamma}}. \tag{39}$$

Since each Γ contains $\bar{D}(U)$ as a factor, the expressions (38) and (39) are essentially independent of the level density and are functions of the particle velocities and penetration factors involved.

Returning to the cross-sections in the resonance region, say eq. (36), it is interesting to note that in exact resonance ($E = E_r$, *not* necessarily the maximum cross-section) and supposing that Γ^P and Γ^Q represent the only possibilities of emission,

$$\sigma_{\text{res}}(P, Q) = \frac{4\pi\lambda\!\!\!^{-2}_P \Gamma^P \Gamma^Q}{(\Gamma^P + \Gamma^Q)^2},$$

† C. L. Critchfield and E. Teller, ibid. **60** (1941), 10.

‡ A simple physical example of the importance of phase relations is given by the collision of billiard balls. The collision excites numerous elastic modes which if they were independent in phase would mostly warm up the balls. Actually the phase relations are such as to restore most of the energy to relative motion. The situation with the phases of the wave-functions in nuclei can be equally decisive, but there is, as yet, no good evidence on this point.

which shows that a reaction cross-section may never be larger than $(2l+1)\pi\lambda\!\!\!^{-2}_P$ (as compared with 4 times this for resonant scattering where the scattered wave is coherent with the incident wave). From eq. (36) also, one can deduce the behaviour of the cross-section for the absorption of slow neutrons, if there is no resonance very close to $E = 0$, by taking $l = 0$. As we have seen above, the significant emission probabilities for a heavy compound nucleus formed by an incident slow neutron are Γ^n and Γ^γ such that $\Gamma^\gamma \gg \Gamma^n$. Near zero neutron energy, $\Gamma^n \sim v_n$, whereas Γ^γ does not vary appreciably if the excitation is changed by a few volts and neither does the denominator in (36). In this case, therefore,

$$\sigma(n,\gamma) \sim \lambda\!\!\!^{-2}_n v_n \sim \frac{1}{v_n}. \tag{40}$$

Equation (40) thus expresses the $1/v$-law referred to in § 1, Chapter VIII. The same law applies in any case where Γ^Q is relatively insensitive to the neutron energy, when the latter is low, for instance in the $B^{10}(n,\alpha)Li^7$-reaction, from which the α-particle carries over 3 M.e.v., the $1/v$-law applies also.

The magnitude of the reaction cross-section $\sigma(P,Q)$ for low energy of bombardment, and for light elements (in which the resonant levels will be widely separated in general), is approximately (cf. eq. (36)):

$$\sigma(P,Q) \cong \pi\lambda\!\!\!^{-2}_P \frac{\Gamma^P\Gamma^Q}{\bar{E}_1^2}, \tag{41}$$

where E_1 is the resonant energy nearest $E = 0$. A reasonable estimate of E_1 would be $\frac{1}{2}\bar{D}(U)$, and combining this estimate with eq. (33) we get an expression for the cross-section in terms of the transmission coefficient $T_P(E)$ of the incident particle P:

$$\sigma(P,Q) \cong \pi\lambda\!\!\!^{-2}_P T_P(E) \frac{2\Gamma^Q}{\pi\bar{D}(U)}. \tag{42}$$

Substantially this form for the cross-section is used in the following chapter on thermonuclear reactions. In the applications that have been made of that work (to astrophysics) $\bar{D}(U)$ has been estimated from the individual particle model, along lines pursued in § 2, Chapter VIII, but allowing for the coupling between nucleons by dividing the characteristic energy-interval, $\hbar^2/MR^2$, by the number of nucleons,† A:

$$\bar{D}(U) \simeq \hbar^2/MAR^2. \tag{43}$$

† The formulae derived in this chapter and those that have been used in the astrophysical work still differ by a numerical factor of the order of $\frac{1}{4}Z$, which, however, is both unimportant and uncertain.

Certain general rules are often useful in application of the simple theory of the compound nucleus, especially for the heavier elements. One of these rules is that the total width of a level, $\mathbf{\Gamma}$, is often dominated by a single type of integrated width. If the excitation of the compound nucleus is below the threshold for neutron emission, this dominant contribution is the integrated γ-width, $\mathbf{\Gamma} \simeq \Gamma^{\gamma}$. Above the neutron threshold the dominant width becomes the integrated neutron width, $\mathbf{\Gamma} \simeq \Gamma^{n}$. The region in which γ-ray emission competes with (very slow) neutron emission is discussed above. The neutron width remains dominant so long as the nuclear excitation is not sufficient to emit a proton 'over the top' of the Coulomb barrier. At energies sufficient to emit such protons, the integrated neutron and proton widths become equally important and at energies of excitation over 20 M.e.v. α-particles may be emitted over the barrier, so that they must be taken into account also.

These general rules do not apply well in very light nuclei because the Coulomb barriers are not high enough to be decisive. It often happens that α-particle emission is very important in light particle reactions and, in fact, it is the (n, α) reaction on B^{10} that makes this isotope such a strong absorber of neutrons. As the charge on the target nuclei is increased, however, the (n, α) reaction becomes very weak because α-emission cannot compete with neutron-emission and it becomes detectable again (with the customary neutron sources) only for the very heavy elements from which the α-particles carry away an excess of energy.†

Another general rule is that the various expressions for the cross-section of $\sigma(P, Q)$ are essentially proportional to the factor $\lambda_P^2 \Gamma^P \Gamma^Q$, and this is well confirmed experimentally in cases where P or Q, or both, are charged particles. In such cases the Γ's contain penetration factors that are such strong functions of the energy as to completely dominate the variation of the cross-section with bombarding energy. In Fig. 46 we compare the logarithms of the yields as observed by Rumbaugh, Roberts, and Hafstad‡ for the reactions

$$Li^6 + H^1 \rightarrow He^3 + He^4 + 3{\cdot}7 \text{ M.e.v.} \qquad (44a)$$

$$Li^7 + H^1 \rightarrow 2He^4 + 17{\cdot}1 \text{ M.e.v.} \qquad (44b)$$

† Cf. R. Sherr, K. T. Bainbridge, and H. H. Anderson, *Phys. Rev.* **60** (1941), 473.

‡ L. H. Rumbaugh, R. B. Roberts, and L. R. Hafstad, ibid. **54** (1938), 657.

The reaction (44 *b*), and the related

$$Li^6+H^2 \rightarrow 2He^4+22{\cdot}1 \text{ M.e.v.} \tag{45}$$

are illustrated in Plate III, as observed by Dee and Walton.† In the reactions (44 *a*, *b*) the energy released by the transformation is so large that the variation of the cross-sections with proton energy is determined primarily by the penetration factor for the proton. And, in fact, the slopes of the curves in Fig. 46 are given substantially by:

$$\begin{aligned} \ln N &\sim \frac{2\pi Ze^2}{\hbar v} = \frac{10^3}{10{\cdot}4\sqrt{(KV)}} \\ \log_{10} N &\sim \frac{10^3}{25\sqrt{E_{kv}}}, \end{aligned} \tag{46}$$

i.e. the principal term in the exponential of the penetration factor.

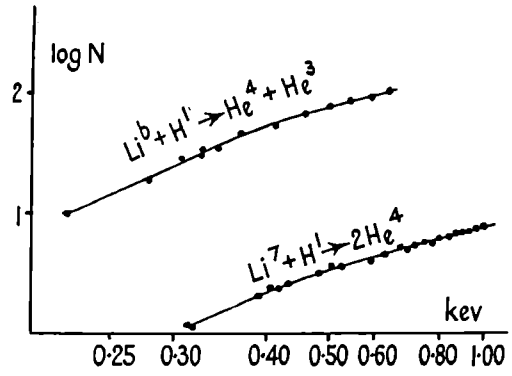
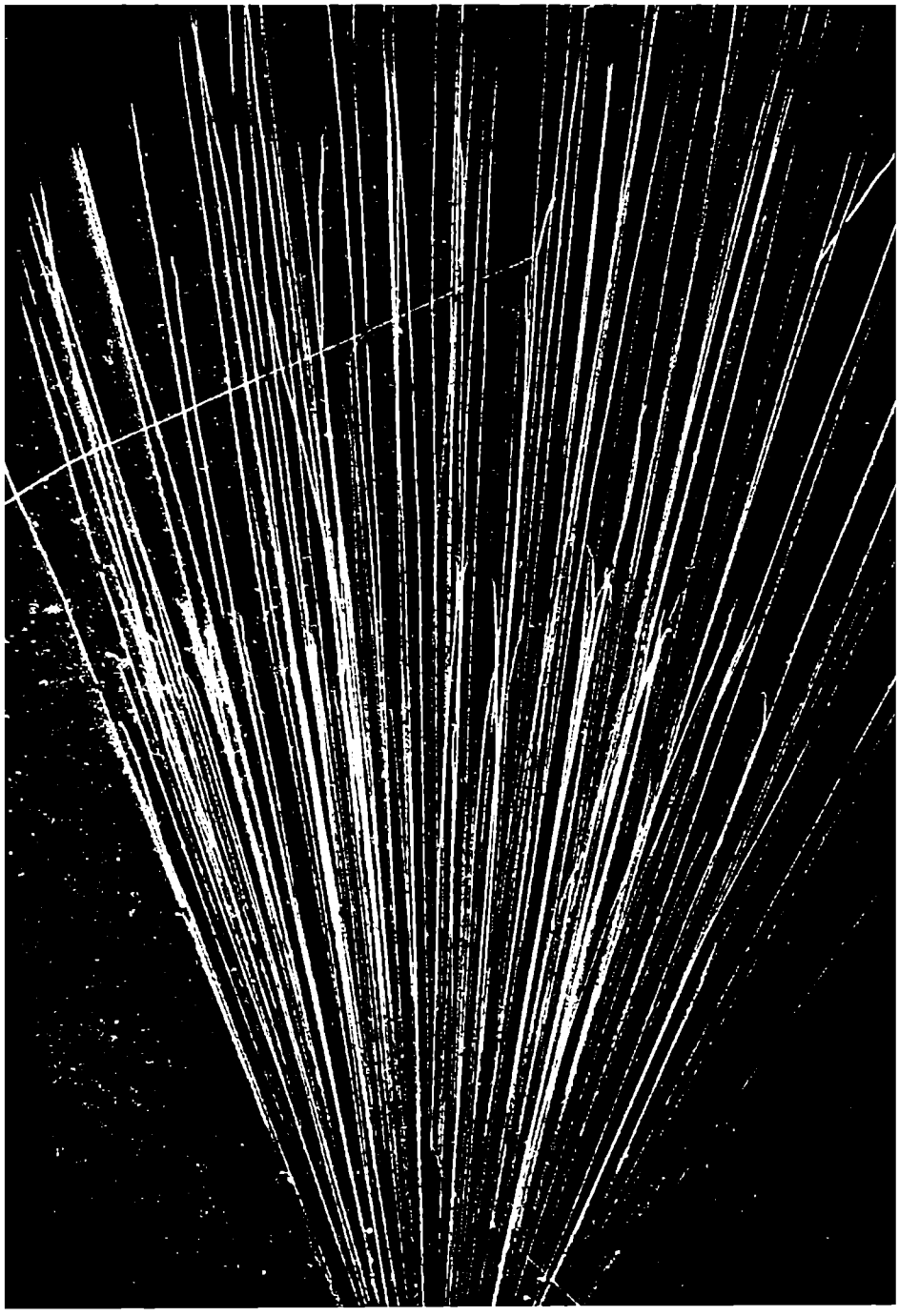


FIG. 46.

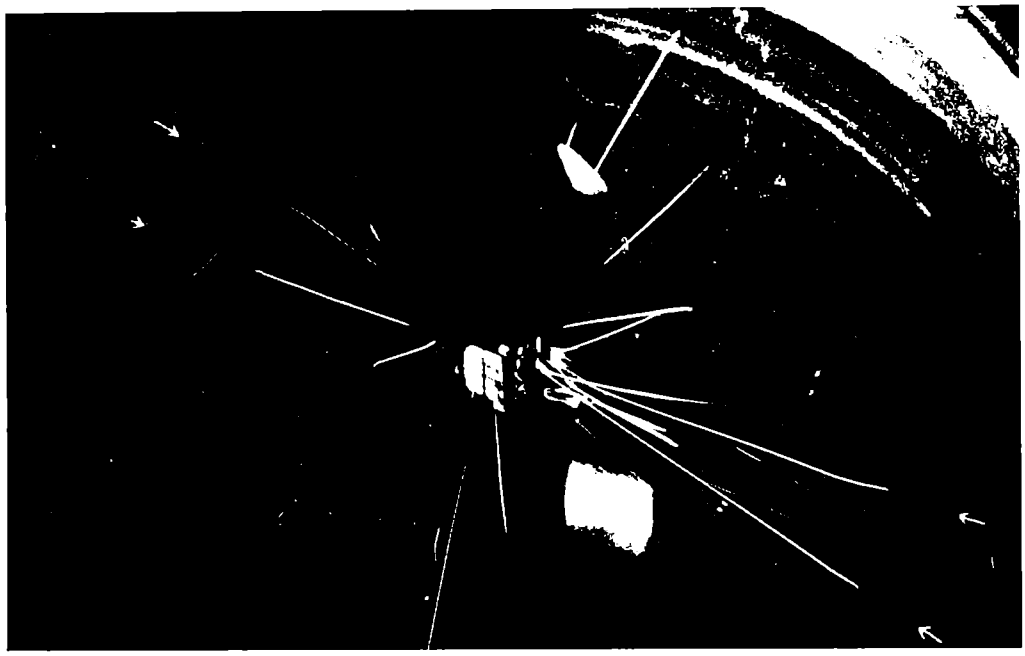
An interesting feature of Fig. 46 is that the Li^7 $(p,\alpha)\alpha$-reaction is about 30 times less probable than the $Li^6(p, He^3)\alpha$, despite the fact that the former releases much more energy. This observation appears to indicate, at least in part, the operation of selection rules in nuclear reactions.‡ According to the general results of Chapter IV, the Li^7-nucleus should have an odd, P wave-function. Hence, if the proton collides with it in an S-state, the compound nucleus for (45) will be an odd state also. It is impossible for two α-particles, moving about their centre of gravity, to form an odd state because they obey Bose–Einstein statistics. Thus, for S-wave collisions, $\Gamma^\alpha = 0$. The α-particles emitted in (45) probably arise from an even compound nucleus formed by collisions between the proton and the Li^7-nucleus

† P. Dee and E. Walton, *Proc. Roy. Soc.* **141** (1933), 733.
‡ M. Goldhaber, *Proc. Camb. Phil. Soc.* **30** (1934), 561.



Reaction ${}_7N^{14} + {}_2He^4 \rightarrow {}_8O^{17} + {}_1H^1$.

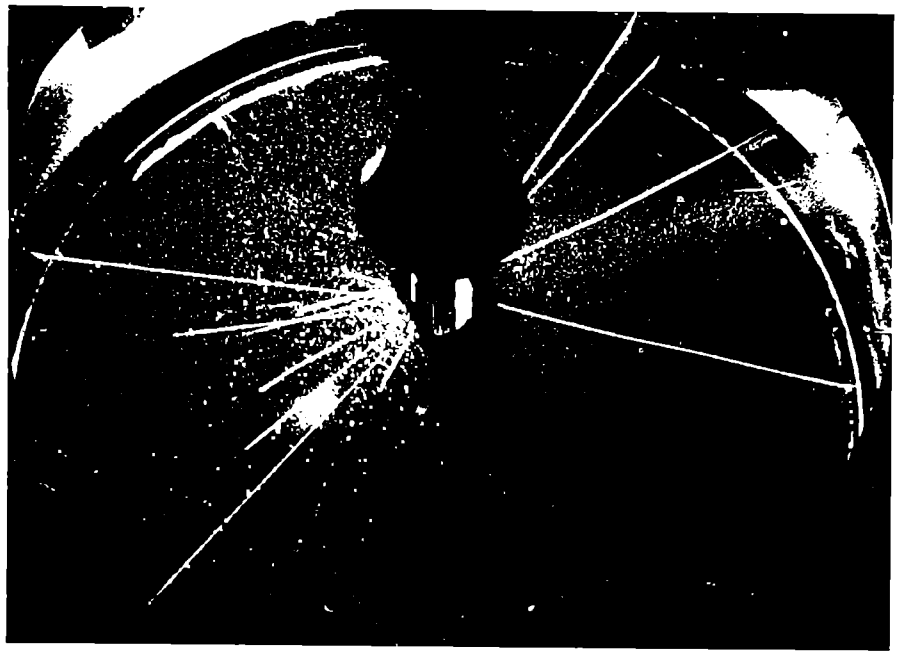
PLATE III



(*a*)

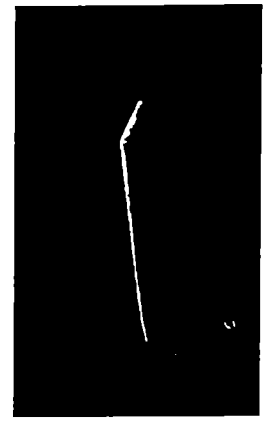
(*b*)

(*a*) Reaction $_3Li^7 + {}_1H^1 \rightarrow {}_2He^4$; (*b*) Reactions $_3Li^6 + {}_1H^2 \rightarrow {}_2He^4$ and $_3Li^6 + {}_1H^2 \rightarrow {}_3Li^7 + {}_1H^1$.



(*a*)

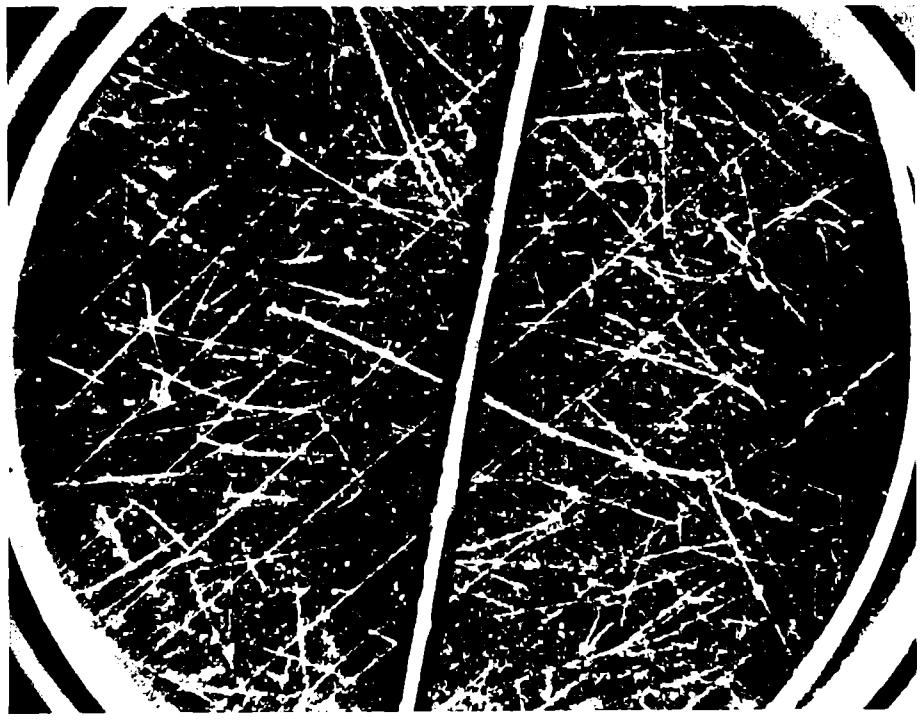
(*b*)



(*c*)

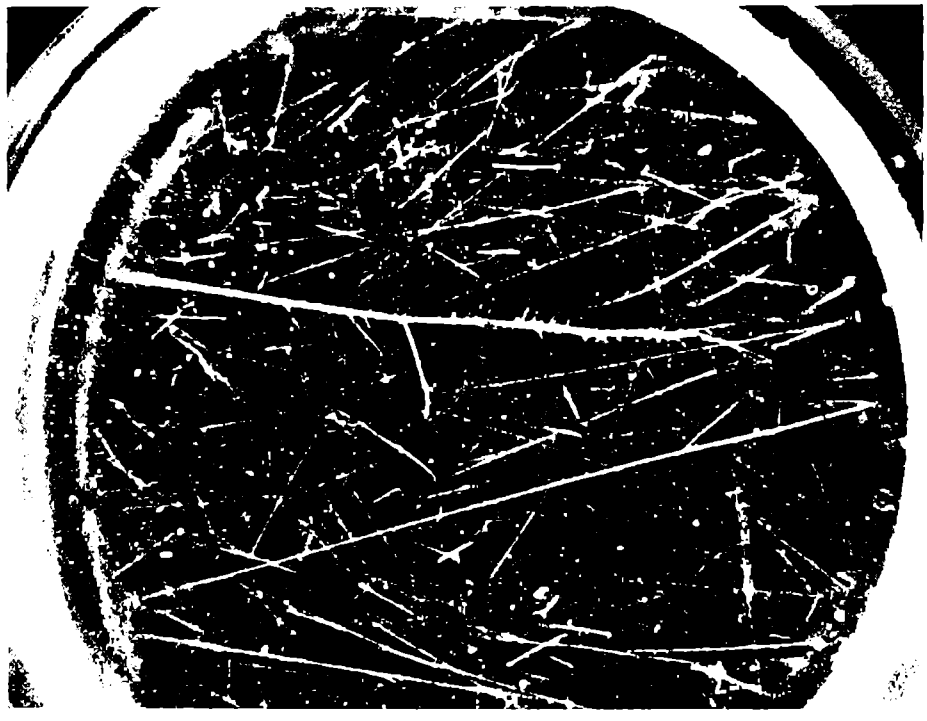
(*a*) Reaction ${}_1H^2 + {}_1H^2 \rightarrow {}_1H^1 + {}_1H^3$; (*b*) Reaction ${}_5B^{11} + {}_1H^1 \rightarrow 3\,{}_2He^4$;
(*c*) Reaction ${}_7N^{14} + {}_0n^1 \rightarrow {}_5B^{11} + {}_2He^4$.

PLATE V



(*a*)

The tracks of two fission fragments originating in the aluminium-supported uranium layer.



(*b*)

The fission track in low pressure hydrogen originating at the uranium-covered wall of the chamber. In addition to many fine proton branches, the picture shows one branch resulting from a collision with oxygen nucleus.

in a P-wave, and the proton width will be somewhat less ($\sim 0{\cdot}3$) than for S-wave collisions. There remains a factor 10, however, that is not accounted for in this way.

The formation of compound nuclei by deuteron bombardment occupies a special position among nuclear reactions because of the very high excitation achieved. This comes about because the excitation of the compound nucleus is equal to the sum of binding energies of a neutron and a proton plus the kinetic energy and less only the binding energy of the deuteron, 2·18 M.e.v. The high Q-value in (45) arises in this way. Deuteron reactions on Li^6 are shown in Plate III *b*. Owing to the high excitation, all deuteron-induced reactions produce particles rather than γ-rays. The (d, n) reactions are valuable sources of high-energy neutrons, especially on Li^7:

$$Li^7 + H^2 \rightarrow 2He^4 + n + 14{\cdot}5 \text{ M.e.v.} \qquad (47)$$

The energy-release in (47) is shared among the two α-particles and the neutron, so that the latter is by no means 'mono-energetic'. One of the most useful sources of neutrons of well-defined energy is the d-d reaction:

$$H^2 + H^2 \rightarrow He^3 + n + 3{\cdot}1 \text{ M.e.v.}, \qquad (48)$$

which, on account of the small nuclear charges involved, has an appreciable yield even at low deuteron energy. The reaction (48) competes, however, with the alternative

$$H^2 + H^2 \rightarrow H^3 + H^1 + 3{\cdot}75 \text{ M.e.v.} \qquad (49)$$

on a practically equal footing, i.e. half of the d-d reactions follow (48), half follow (49). These reactions were produced first by Oliphant, Harteck, and Rutherford† and a Wilson chamber photograph of (49), taken by Dee,‡ appears in Plate IV *a*. The longer tracks showing in the plate are those of ordinary protons and the shorter tracks are due to the isotope H^3.

Deuteron reactions among light nuclei probably follow the simple theory of the formation of a compound nucleus and subsequent, independent disintegration of the latter. Among heavy nuclei, however, there appears to be an additional important possibility that the deuteron is polarized by the target nucleus so strongly that only the neutron is actually caught, the proton flying on. This process has been calculated first by Oppenheimer and Phillips.§ The result is, of

† M. Oliphant, P. Harteck, and E. Rutherford, *Proc. Roy. Soc.* A **144** (1934), 692.
‡ P. Dee, *Nature*, **133** (1934), 564.
§ J. R. Oppenheimer and M. Phillips, *Phys. Rev.* **48** (1935), 500.

course, that a (d, p) reaction on heavier elements is somewhat more probable than one would compute from complete penetration by the deuteron followed by penetration by the proton. This effect has been verified experimentally by Lawrence, McMillan, and Thornton.†

Among heavier nuclei, the yields of nuclear reactions are largest, by far, when the bombarding particle is a neutron, since the neutron is the only particle that does not encounter the Coulomb repulsion (not counting the γ-quantum as a particle). A photograph of the first neutron reaction to be discovered:‡

$$N^{14}+n \rightarrow B^{11}+He^{4} \tag{50}$$

is shown in Plate IV *c*. Equation (50) is an example of an (n, α) reaction. Other, very common, types of reactions are (n, p) and (n, n) (inelastic scattering). The (n, γ) reactions for slow neutrons have been discussed above. If the residual nucleus in an (n, n) reaction is sufficiently excited it may emit a second neutron, giving rise to an $(n, 2n)$ reaction. For instance, for the detection of the fast neutrons used in determining the radii of nuclei with 25 M.e.v. neutrons, Sherr (Chap. I) used the reaction

$$C^{12}+n \rightarrow C^{11}+2n-17{\cdot}5 \text{ M.e.v.} \tag{51}$$

The intensity of the bombarding neutrons was then determined from the intensity of the β-radiation due to the C^{11}-nuclei.

Of all the reactions induced by neutrons the most spectacular is that of the fission of the heaviest nuclei, (n, f) reactions. In Plate V *a* is shown a Wilson chamber picture of a thin uranium foil (supported by aluminium) from which two fission fragments are emerging from a nucleus that has been hit by a neutron. In Plate V *b* the details of the track of a fission fragment, as revealed in low-pressure hydrogen, are shown. In addition to many fine branches due to recoil protons there is one branch resulting from collision with an oxygen nucleus. These photographs were taken by T. Lauritsen.

Occasionally one finds nuclear reactions that lead simultaneously to the emission of several particles. An historic example is given by

$$B^{11}+H^{1} \rightarrow 3\ He^{4}+8{\cdot}7 \text{ M.e.v.}, \tag{52}$$

which is shown in Plate IV *b* as taken by Dee and Gilbert.§ Currently, however, many such reactions are being produced by the

† E. O. Lawrence, E. McMillan, and R. Thornton, *Phys. Rev.* **48** (1935), 493.
‡ N. Feather, *Proc. Roy. Soc.* A **136** (1932), 709.
§ P. Dee and C. Gilbert, ibid. A. **154** (1936), 279.

ultra-high-energy particles of the Berkeley cyclotron. Since the excitation of target nuclei hit by neutrons from that machine exceeds 100 M.e.v., the mechanism of nuclear reactions is quite different (compare Appendix V).

3. Nuclear photo-effect

The excitation of nuclei through γ-ray bombardment has been discussed already in Chapter VII in connexion with the resonant absorption of X-rays. The general theory of nuclear reactions applies to such processes if we take for Γ^P the width for a single γ-ray and for Γ^Q the sum over all possible γ-emissions. The product of these widths is so small, however, that an observable cross-section is obtained only in the immediate neighbourhood of a resonance, and it is this fact that was used in applying the method of X-ray excitation to the determination of low-lying levels.

If the energy of the incident γ-ray is greater than the (absolute value) of the binding energy of the last neutron in the target nucleus, on the other hand, the dominant width is that for neutron emission; the most probable reaction then becomes (γ, n), sometimes called the nuclear photo-effect. There are several established cases of (γ, p) reactions which, in the range of γ-ray energies used, are generally much less probable than the (γ, n) reactions but which represent a nuclear photo-effect equally well. A systematic study of the cross-section, $\sigma(\gamma, n)$, has been made by Bothe and Gentner† using 12 M.e.v. γ-rays from $B(p, \gamma)$ and 17 M.e.v. rays from $\mathrm{Li}(p, \gamma)$. Weisskopf‡ has analysed these data and arrived at the conclusion that $\sigma(\gamma, n)$ is relatively independent of the size of the target nucleus and is proportional to the cube of the γ-ray energy $(h\nu)^3$. The excitations of the compound nuclei are quite high, in these cases, so that the cross-section averaged over many levels, eq. (39), may be applied. It is then readily seen that if $\sigma(\gamma, n)$ varies as $(h\nu)^3$ the individual width $\Gamma^{h\nu}$ is proportional to $(h\nu)^5$ and to $\bar{D}(U)$. This is, in fact, the basis on which the form of the individual γ-width was chosen, eq. (21), and, as remarked in § 2, the numerical factor as derived from the results of Bothe and Gentner is practically the same as that derived from the known emission probability of 0·603 M.e.v. γ-ray from RaC′. It must be emphasized, however, that this description of the interaction of

† W. Bothe and W. Gentner, *Zs. f. Phys.* **112** (1939), 45.
‡ V. F. Weisskopf, *Phys. Rev.* **59** (1941), 318.

γ-rays with nuclei is only semi-empirical, and it is to be expected that further study of the action of γ-rays in producing nuclear reactions will lead to a more complete understanding of this interaction.

The particular example of photo-effect that is of greatest fundamental interest is the disintegration of the deuteron into a neutron and a proton

$$H^2+h\nu \rightarrow n+H^1, \tag{53}$$

first reported by Chadwick and Goldhaber,† who determined also the energy-balance in the reaction and, thereby, the binding energy of the deuteron. From the mass-spectrographic measurement of the mass of deuterium one then obtains the mass of the neutron (cf. Table II). Moreover, the reaction (53) is of great interest because it should be possible to apply the usual theory of interaction between electromagnetic radiation and charged particles to predict the angular distribution, and total cross-section, in the reaction. In principle, such a calculation is complicated by the fact that the wave-function of the deuteron is not simple but is a mixture of S- and D-waves, due to the existence of tensor forces. Typical examples for the exchange nature of nuclear forces, i.e. the charged, symmetrical and neutral theories of Chapter III, have been assumed by Rarita and Schwinger‡ and the corresponding angular distributions of photo-protons (or neutrons) calculated. Such studies, combined with careful determinations of the angular distributions at various energies of γ-rays, promise to be of great usefulness to the final interpretation of the nature of the forces between nucleons.

The complete theory, including the small admixture of D-wave in the ground state of the deuteron, will not be presented here, but we shall sketch briefly the more elementary theory which assumes the deuteron to be a pure S-state as originally worked out by Bethe and Peierls.§ The part of the nuclear photo-effect that is due to the electric vector will be expressed in the same form as the cross-section for the ordinary photo-effect in the outer atom, in terms of the frequency of the quantum, ν, and the dipole matrix element, z_{0E}:

$$\sigma_E = \frac{8\pi^3 e^2}{c} |z_{0E}|^2 \tag{54}$$

$$z_{0E} = \int \psi_0 \tfrac{1}{2} z \, \psi_E \, d\tau. \tag{55}$$

† J. Chadwick and G. Goldhaber, *Nature*, **134** (1934), 237; *Proc. Roy. Soc.* **151** (1935), 479. ‡ W. Rarita and J. Schwinger, *Phys. Rev.* **59** (1941), 436.

§ H. A. Bethe and R. Peierls, *Proc. Roy. Soc.* A. **148** (1935), 146.

Here $\frac{1}{2}z$ is the projection on to the direction of polarization of the displacement of the proton from the centre of mass of the neutron-proton system, ψ_0 is the wave-function in the relative coordinates of neutron and proton forming the ground state of the deuteron, and ψ_E is the wave-function representing the system after disintegration has been effected. We shall assume not only that ψ_0 is spherically symmetric but also, since the main contribution to the matrix element (55) arises at rather large r, that ψ_0 can be represented in the simplified (normalized) form having the functional dependence of the 'tail' of the square-well deuteron function (cf. Chapter II):

$$\psi_0 = \sqrt{\left(\frac{\alpha}{2\pi}\right)}\frac{1}{r}e^{-\alpha r} \tag{56}$$

$$\alpha = \frac{\sqrt{\epsilon M}}{\hbar} \tag{57}$$

and ϵ is the binding energy of the deuteron. Since the transition is induced by the dipole moment, $\frac{1}{2}z$, the final neutron-proton state must be a P-wave that varies as $z/r = \cos\theta$. The short-range forces will have little effect on such a wave, so that ψ_E may be taken to represent free particles having the correct kinetic energy, $\hbar^2k^2/M$, and normalized to unit energy interval (as required by the form of eq. (54) in the usual way):

$$\psi_E = \frac{1}{2\pi\hbar\sqrt{Mk}}\left\{\frac{\sin kr}{kr} - \cos kr\right\}\frac{\sqrt{3}\cos\theta}{r}, \tag{58}$$

$$k = \frac{\sqrt{\{M(h\nu-\epsilon)\}}}{\hbar}. \tag{59}$$

Substituting the wave-functions (58) and (59) into (55) and (54) one obtains the total cross-section due to the electric vector

$$\sigma_E = \frac{16\pi^2}{3\hbar^2c}\frac{Me^2\nu\alpha k^3}{(\alpha^2+k^2)^4}$$

$$\simeq 1{\cdot}25\times10^{-26}\frac{(\gamma-1)^{\frac{3}{2}}}{\gamma^3}\text{ cm.}^2 \tag{60}$$

with

$$\gamma = \frac{h\nu}{\epsilon}.$$

To obtain the angular distribution, one does not integrate over angles. If we let ϕ be the angle included between the direction of irradiation and the direction of the emitted protons it is readily seen

that the differential cross-section (summed over the polarizations in the incident beam) is

$$\sigma(\phi) = \sigma_E \tfrac{3}{2}\sin^2\phi,$$

i.e. there is a maximum in the directions at right angles to the γ-ray beam in agreement with what was found in the original investigations of Chadwick and Goldhaber (loc. cit.).

Under ordinary conditions the transitions due to electric dipole moments are far more probable than those due to higher moments and to the magnetic dipole moment. In the special case of the deuteron, however, it must be remembered that there is a virtual, singlet S-level lying just above zero energy and therefore it is possible to obtain resonant absorption of γ-rays due to the magnetic dipole transition. The energy of coupling with the radiation is then given by the product of the magnetic field strength, H, with the magnetic moments of neutron and proton, μ_N and μ_P respectively. The calculation then proceeds along lines similar to those followed above and the details may be found in the review by Bethe and Bacher,† for example. The result may be expressed as the ratio of the total cross-section for magnetic dipole transitions, σ_M, to the total cross-section for electric dipole transitions and in terms of $h\nu$, the binding energy of the triplet deuteron ϵ, and of the singlet deuteron, $\epsilon' = 0{\cdot}11$ M.e.v.:

$$\frac{\sigma_M}{\sigma_E} = \frac{(\mu_P-\mu_N)^2}{4}\,\frac{(\sqrt{\epsilon}+\sqrt{\epsilon'})^2(h\nu)^2}{(h\nu-\epsilon+\epsilon')Mc^2(h\nu-\epsilon)}, \tag{61}$$

where μ_N and μ_P are measured in nuclear magnetons. For the ThC′ γ-ray, $h\nu = 2{\cdot}62$ M.e.v., the ratio (101) is just about $\frac{1}{2}$, so that even at this energy the magnetic effect is appreciable. The combined effect is in reasonable agreement with the observed cross-section of $6{\cdot}7\times10^{-28}$ cm.^{2}

It is evident from the fact that the magnetic dipole transitions lead to the ^1S-state that the resulting angular distribution of protons (or neutrons) is spherical. The complete differential cross-section for this simple theory is then

$$\sigma(\phi) = \tfrac{3}{2}\sigma_E \sin^2\phi + \sigma_M. \tag{62}$$

The more refined theories, including tensor forces, give a somewhat different angular distribution, of course, and it is through careful determination and comparison that one might be able to decide

† H. A. Bethe and R. F. Bacher, *Rev. Mod. Phys.* **8** (1936), 82.

among various possibilities for the nuclear forces. It is evident also that as $h\nu$ is decreased towards the threshold value, ϵ, the magnetic effect far outweighs the electric effect, i.e. the emission of very slow neutrons and protons is predominantly a magnetic effect. Conversely the capture of slow neutrons by protons is predominantly an effect of the magnetic dipole transitions and, in fact, this mechanism was introduced first by Fermi† to account for the extraordinarily large capture cross-section for neutrons in hydrogen.

Since the capture of neutrons by protons is simply the reverse of the photo-effect on the deuteron the total cross-section is readily computed from $\sigma(\gamma, n) = \sigma_E + \sigma_M$. Let $\sigma(A, B)$ be the cross-section for transformation of the particle system A into the system B, and let g_A and g_B be the statistical weights, and $\lambda\!\!\!^{-}_A$ and $\lambda\!\!\!^{-}_B$ be the wave-lengths for systems A and B, respectively. Then, from the general form of the reaction cross-sections derived in the preceding sections we see at once:

$$\frac{\sigma(B, A)}{\sigma(A, B)} = \frac{g_A}{g_B} \frac{\lambda\!\!\!^{-2}_B}{\lambda\!\!\!^{-2}_A}. \tag{63}$$

In our example, system A is a photon plus the stable deuteron; the photon is capable of two polarizations and the deuteron of three, so that $g_A = 2 \times 3 = 6$. System B comprises one neutron and one proton, each of which has two possible polarizations, so that $g_B = 2 \times 2 = 4$. The cross-section for capture of neutrons by protons is then:

$$\sigma(n, \gamma) = \tfrac{6}{4} \left(\frac{2\pi\nu}{kc} \right)^2 (\sigma_E + \sigma_M). \tag{64}$$

For thermal neutrons, eq. (64) leads to a cross-section of about $0{\cdot}2 \times 10^{-24}$ cm.^{2} in reasonable agreement with observation (Fermi, loc. cit.).

† E. Fermi, *Phys. Rev.* **48** (1935), 570.

X

THERMONUCLEAR REACTIONS AND ASTROPHYSICAL APPLICATIONS

1. Reaction-rate formula

In the previous chapter we have been considering nuclear transformations caused by the impact of a beam of fast particles against the nuclei of the atoms forming the target material. We have seen that the efficiency of bombardment with charged particles is generally very low due to the fact that the incident particles lose their initial kinetic energy through the electric interaction with the electronic envelopes of the bombarded atoms.

It must be remembered, however, that violent nuclear collisions can result also from thermal motion if the material is heated to a sufficiently high temperature. Thus, for example, the reaction $Li^7 + H^1 \rightarrow 2He^4$ can still be observed when the energy of the incident protons is as low as 8 k.e.v.,† which is to be compared with the value 1·7 k.e.v. corresponding to the kinetic energy of thermal motion at a temperature of 2×10^7 °C. occurring in the interior of the sun. We must expect, however, that such a *thermonuclear* reaction between lithium and hydrogen would take place at an observable rate at temperatures considerably lower than that. In fact, whereas the densities of bombarding ion-beams, obtainable in electric accelerators, are only of the order of magnitude of 10^{-14} g.cm.^{3} (for a current density of 1 milliampere per cm.^{2}), the effective density of the 'isotropic' bombarding beams in thermonuclear processes is comparable to the density of the material considered. In comparing thermonuclear processes with the results of beam-bombardment experiments it must be kept in mind, also, that in the former case the bulk of the transformations is not due to particles that possess the mean energy of thermal motion, but to much faster particles in the Maxwell distribution for which increased penetration effectiveness over-compensates the decrease in relative number. In Fig. 47 we give a schematic presentation of the relative number of particles dN/dE and of their effective reaction cross-sections σ as a function of energy E. It shows that the number of disintegrations, which is given essentially by the product $\sigma(E)(dN/dE)$, reaches a sharp

† E. H. S. Burhop, *Proc. Cambr. Phil. Soc.* **32** (1936), 643.

maximum for a certain energy E_{opt} well above the mean energy corresponding to the given temperature. In order to estimate the total reaction-rate one should integrate $\sigma(E)(dN/dE)\ dE$ over the entire range of energies forming 0 to ∞. Calculations of this type were carried out for the first time by Atkinson and Houtermans,† and were later improved by Gamow and Teller.‡

In deriving the formula for the rate of thermonuclear reactions we can assume that the atoms participating in this process are com-

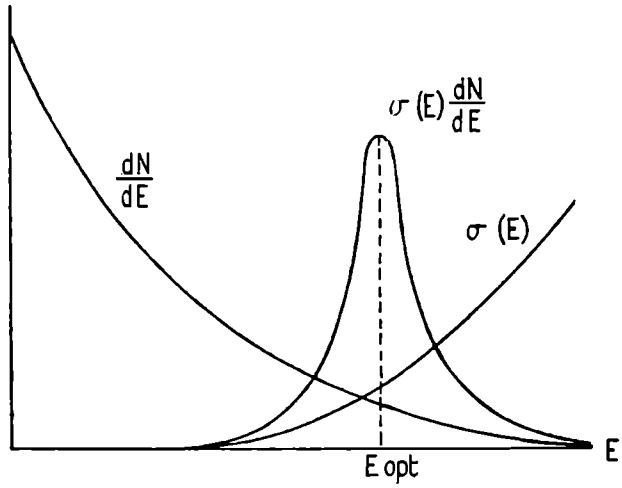


FIG. 47.

pletely stripped of their electronic envelopes, since the collision energies necessary for effective nuclear penetration are higher than the total ionization potentials of the corresponding atoms. Thus, even at very high densities, we may treat the material as an ideal gas formed by bare nuclei and free electrons.

Consider a gas mixture containing two reacting elements with atomic numbers Z_1 and Z_2, and atomic weights A_1 and A_2. Let c_1 and c_2 be the relative concentrations (by weight) of the two

† R. Atkinson and F. Houtermans, *Zs. f. Phys.* **54** (1928), 656. These calculations contained two errors which were due to the lack of knowledge concerning the theory of nuclear reactions at this early stage of nuclear physics. The probability of radiative capture of protons was calculated under the assumption of dipole (and not quadrupole) radiation, thus leading to numerical values about 1,000 times too large. On the other hand, the collision cross-section for thermal particles was taken to be πr_0^2 (instead of $\pi\Lambda^2$). Since for thermal protons at the inner-solar temperature of 2×10^7 °C. the effective de Broglie wave-length Λ is about 30 times larger than the nuclear radius r_0, this error increased the numerical values by a factor of 1,000, thus compensating the first error. Altogether the numerical values for the rate of energy-production in the sun came out approximately correct.

‡ G. Gamow and E. Teller, *Phys. Rev.* **53** (1938), 608.

elements in question, and ρ and T the density and temperature of the gas. According to Maxwell's theory, the number dN (per cm.^{3} per sec.) of reactive collisions between the nuclei of the two kinds, with a collision energy between E and $E+dE$, is given by:

$$dN = \frac{4c_1 c_2 \rho\sigma}{\sqrt{(2\pi)M^{\frac{1}{2}}A_1 A_2 \sqrt{A}(kT)^{\frac{3}{2}}}} e^{-E/kT} E\, dE, \tag{1}$$

where $A = \frac{A_1 A_2}{A_1+A_2}$, M is the nucleonic mass, and σ the effective cross-section for the reaction. The effective cross-section σ is given as the product of the cross-section for penetration of the Coulomb barrier, eq. (42), Chapter IX, and the ratio of the reaction probability $\Gamma/\hbar$ to the proper nuclear frequency $\hbar/MR^2A$. Thus we have

$$\sigma = \frac{\Lambda^2}{4\pi}\exp\left[\frac{-2\pi e^2\sqrt{(MA)}Z_1 Z_2}{\hbar\sqrt{E}} + \frac{4e\sqrt{M}\sqrt{(2AZ_1 Z_2 R)}}{\hbar}\right]\cdot\frac{\Gamma MAR^2}{\hbar^2}, \tag{2}$$

where $\Lambda = 2\pi\hbar^2/\sqrt{(2MAE)}$ is the de Broglie wave-length, and R can be taken to be equal to the radius $1{\cdot}7+1{\cdot}22\times10^{-13}(A_1+A_2)^{\frac{1}{3}}$ cm. of the compound nucleus formed in the collision. Substituting (2) into (1), we find that dN/dE has a sharp maximum at

$$E_{\text{opt}} = \frac{(\pi e^2\sqrt{M}\sqrt{A}\, Z_1 Z_2 kT)^{\frac{2}{3}}}{(\sqrt{2}\hbar)^{\frac{2}{3}}}, \tag{3}$$

the width of the maximum being given by

$$\Delta E = \left(\frac{8}{3kT}\right)^{\frac{1}{2}}\frac{2\pi e^2\sqrt{M}\sqrt{A}\, Z_1 Z_2 (kT)^{\frac{1}{2}}}{(\sqrt{2}\hbar)^{\frac{1}{2}}}. \tag{4}$$

Owing to the sharpness of the maximum, we can approximate the integral $\int_0^\infty dN$ by the integral of the error-curve with the same height and width, thus obtaining:

$$N \cong \frac{\pi^{\frac{1}{6}}c_1 c_2}{3^{\frac{1}{2}}(kT)^{\frac{2}{3}}}\frac{e^{\frac{1}{3}}\hbar^{\frac{1}{3}}Z_1^{\frac{1}{3}}Z_2^{\frac{1}{3}}R^2\Gamma}{M^{\frac{7}{3}}A_1 A_2 A^{\frac{1}{3}}}\exp\left[\frac{4e(2MAZ_1 Z_2 R)^2}{\hbar} - 3\left(\frac{\pi^2 e^4 MAZ_1^2 Z_2^2}{2\hbar^2 kT}\right)^{\frac{1}{3}}\right]. \tag{5}$$

Introducing, for the sake of brevity, the notation

$$a = \frac{\hbar^2}{MAe^2 Z_1 Z_2} \tag{6}$$

and

$$\tau = 3\left(\frac{\pi^2 Me^4 AZ_1^2 Z_2^2}{2\hbar^2 kT}\right)^{\frac{1}{3}}, \tag{7}$$

we can write (5) in the form:

$$N = \frac{4}{3^{\frac{1}{2}}} \frac{c_1 c_2 \rho \Gamma}{M^2 A_1 A_2 \hbar} a R^2 e^{4(2R/a)^{\frac{1}{2}}} \tau^2 e^{-\tau}; \tag{8}$$

or, numerically,

$$N = 5{\cdot}3 \times 10^{25} \rho c_1 c_2 \Gamma \phi_{1,2} \tau^2 e^{-\tau}, \tag{9}$$

$$\tau = 42{\cdot}7 (Z_1 Z_2)^{\frac{1}{3}} (A/T)^{\frac{1}{3}}, \tag{10}$$

$$\phi_{1,2} = \frac{1}{A_1 A_2 (Z_1 Z_2 A)^3} \left(\frac{8R}{a}\right) e^{2(8R/a)^{\frac{1}{2}}}, \tag{11}$$

where ρ is measured in g./cm.^{3}, T in million degrees centigrade, and Γ in electron volts.

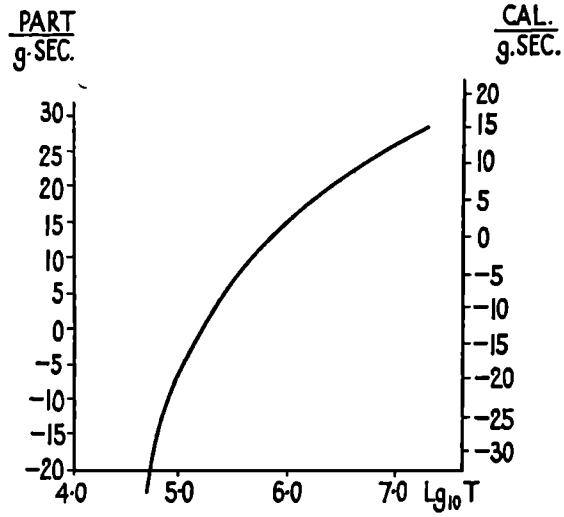


FIG. 48.

As the first example, we consider the application of the above formula to the reaction:

$$H^2 + H^2 \rightarrow H^3 + n \tag{12}$$

in hot deuterium gas, which is evidently the fastest of all thermonuclear reactions. We put here, $Z_1 = Z_2 = 1$, $A_1 = A_2 = 2$, and $c_1 = c_2 = \frac{1}{2}$. We also know that in this case, $\Gamma = 3 \times 10^5$ e.v. Assuming for ρ the value $\frac{1}{7}$g./cm.^{3}, which represents the density of liquid deuterium, we calculate for different temperatures the reaction-rates shown graphically in Fig. 48.† Remembering that each individual process liberates an energy of

$$3{\cdot}5 \text{ M.e.v.} = 5{\cdot}6 \times 10^{-6} \text{ erg} = 1{\cdot}3 \times 10^{-13} \text{ cal.},$$

we can find also the rates of thermonuclear energy production which

† The calculation of this curve was performed by R. A. Alpher.

are shown on the scale on the right-hand side of the figure. We see that, whereas at a temperature of 5×10^5 °C. the energy-production is still quite negligible (10^{-7} cal./g.sec.), it rises to 300, 3×10^7, and 3×10^{12} cal./g.sec. when the temperature rises to 10^6, 3×10^6, and 10^7 °C. respectively. It goes without saying that the problem of obtaining such extremely high temperatures on earth is of almost unsurpassable technical difficulty.

2. Physical conditions in stellar interiors

Whereas the temperatures necessary for thermonuclear reactions, even among the lightest elements, lie almost beyond any laboratory possibilities on the earth, they are quite common in the cosmos and, as a matter of fact, it is now quite certain that the principal sources of stellar energy lie in various thermonuclear reactions taking place in the hot interiors of the stars. Detailed information concerning the physical conditions in the interior of stars is provided by the theory of stellar structure, developed mainly by the work of Eddington.† The comparative simplicity of the theory of stellar structure is due to the fact that, under the conditions of very high temperature such as obtain in stellar interiors, the material of the star is almost completely ionized and thus can be considered, up to very high densities, as an ideal gas. All the physical properties of such an ideal gas mixture formed by bare nuclei and free electrons can be predicted with great accuracy and certainty by the present quantum theory of matter.

The fundamental *equation of hydrostatic equilibrium* in the interior of a star can evidently be written in the form:

$$\frac{dp}{dr} = -\frac{G\rho M_r}{r^2}, \tag{13}$$

where M_r (the mass inside of the radius r) is subject to the equation

$$\frac{dM_r}{dr} = 4\pi r^2\rho. \tag{14}$$

The total pressure p is composed of the gas-pressure p_{gas} and the radiative pressure p_{rad} and can be written in the form

$$p = \frac{R}{\bar{\mu}}\rho T + \tfrac{1}{3}aT^4, \tag{15}$$

† A. S. Eddington, *Internal Constitution of Stars*, Cambridge University Press, 1926; see also S. Chandrasekhar, *Introduction to the Study of Stellar Structure*, University of Chicago Press, 1939.

where R is the gas constant, a the Stephan–Boltzmann constant, and $\bar{\mu}$ the mean molecular weight of the stellar matter. For a completely ionized element of atomic number Z and atomic weight A we have the obvious relation $\bar{\mu} = A/(Z+1)$. This gives $\bar{\mu}_{H^+} = \frac{1}{2}$, $\bar{\mu}_{He}^{++} = 4/3$, and a value close to 2 for all heavier elements. It is customary to represent the heavier components of stellar constitution by the so-called *Russell mixture* which contains the most abundant elements found spectroscopically in stellar atmospheres. This 'mixture' consists of 50 per cent. O, 25 per cent. Na and Mg, 6 per cent. Si, 6 per cent. K Ca, and 13 per cent. Fe, and being completely ionized has the mean molecular weight $\bar{\mu}_R = 1{\cdot}85$. It may be noticed here that the possible changes in relative amounts of elements forming the Russell mixture have only a very small influence on the values of $\bar{\mu}_R$. Writing X and Y for the hydrogen- and helium-content of stellar matter, we evidently obtain

$$\bar{\mu} = \frac{1}{2X+\frac{3}{4}Y+0{\cdot}54(1-X-Y)}. \tag{16}$$

Another fundamental equation of the theory of stellar structure pertains to the energy-transport through the body of the star. As first indicated by Eddington, the energy originating in the interior of the star is carried towards the surface mainly by radiation, i.e. by diffusion of light quanta through the highly ionized stellar material. The *equation of radiative energy-transport* can be written easily if we remember that in this case the flux of momentum must be proportional to the gradient of the radiation pressure. We have:

$$\frac{J_r}{c} = \frac{1}{\rho \text{æ}} \frac{d}{dr}(\tfrac{1}{3}aT^4), \tag{17}$$

where J_r is the radial flux of energy at the distance r from the centre, and æ is the so-called *opacity coefficient* (per unit mass) of the stellar matter. If L_r is the total amount of radiation passing through a sphere of radius r around the centre of the star, and equal to the total energy production within this radius, we can rewrite (17) in the form:

$$\frac{dT}{dr} = -\frac{3}{4}\frac{\text{æ}}{ac}\rho T^{-3}\frac{L_r}{4\pi r^2}. \tag{18}$$

The theory of absorption of high-frequency radiation (at a temperature of $2\times10^7{}^\circ$ C. the maximum of the black-body spectrum lies at 1·2 A) was first developed by Kramers,† who has shown that

† H. A. Kramers, *Phil. Mag.* **46** (1923), 836.

the opacity-coefficient must be proportional to density and inversely proportional to the 3·5th power of the temperature. It was later indicated by Eddington that the original Kramer formula must be corrected slightly by the introduction of the so-called *guillotine-factor t* which performs a task similar to that of the famous machine of the French Revolution by cutting off radiation frequencies at a sharp absorption edge. Another correction factor g was introduced by Gaunt on the basis of the wave-mechanical treatment of the problem. On the basis of that theory we can write for the opacity coefficient of stellar matter the expression:

$$\varkappa = \varkappa_0 \rho T^{-3\cdot5}, \tag{19}$$

where
$$\varkappa_0 = 7\cdot23\times10^{24}(1+X)(1-X-Y)\sum_i \frac{w_i Z_i^2}{A_i}\frac{g}{t}, \tag{20}$$

in which, as before, X and Y represent the hydrogen- and the helium-content of the stellar matter, and w_i is the relative concentration of the element with atomic number Z_i and atomic weight A_i in the Russell mixture. For the Russell mixture which consists of the elements in the ratios given above,

$$\sum_i \frac{w_i Z_i^2}{A_i} = 5\cdot9.$$

Numerical values of the correction factor g/t have been calculated for different densities and temperatures of stellar matter by Strømgren† and, more elaborately, by Morse,‡ who investigated several different mixtures of heavier elements. As in the case of the molecular weight, the numerical value of $\varkappa_0$ is influenced very little by uncertainties in the concentrations w_1. For the temperatures and densities occurring in the sun the tabulated values of t/g can be represented with sufficient accuracy by the expression:

$$\log'(t/g)' = +0\cdot6+\log\rho^{0\cdot25}. \tag{21}$$

Finally, we have the *equation of energy production*:

$$\frac{dL_r}{dr} = 4\pi r^2\rho\epsilon\, dr, \tag{22}$$

where ϵ is the rate of energy production per unit mass of which the dependence upon T and ρ, as well as on chemical constitution of the

† B. Strømgren, *Zs. f. Aphys.* **4** (1932), 118.
‡ P. Morse, *Ap. J.* **92** (1940), 27.

stellar material, is determined by special assumptions about the way in which energy is produced. Since the rate of a thermonuclear reaction depends exponentially on temperature, and since the temperature increases rather rapidly towards the centre of the star, one can assume in first approximation that *all* the energy is produced at the centre. This leads to the so-called *point-source model* and we can put, in this case, $L_r = \text{const.} = \text{observed luminosity of the star.}$ Making this assumption, we are left with four fundamental equations of equilibrium, (13), (14), (15), and (18), which contain four unknown functions of r: ρ, p, T, and M_r (the quantity $\varkappa$ entering into (18) being determined as a function of T and ρ by (19), (20), and (21)). Setting up the boundary conditions:

$$\left.\begin{aligned} T = \rho = p = 0 \\ M_r = M \text{ (observed mass)} \end{aligned}\right\} \text{at } r = R \text{ (observed radius)}, \qquad (23)$$

we can integrate (numerically) our system of equations, with any arbitrary set of values for $\bar{\mu}$ and $\varkappa_0$ all the way into the interior of the star.

It may be noted here that, as first pointed out by Cowling,† the solution of this system of equations which determine the radiative equilibrium in the star becomes unstable with respect to convection at a certain distance from the stellar centre where the temperature gradient, given by (18), exceeds the adiabatic temperature-gradient determined by the calculated gradient of the density ρ. Within this radius radiative equilibrium is broken up and replaced by convective equilibrium, so that, instead of (18), we must use the adiabatic equation:

$$\frac{1}{T}\frac{dT}{dr} = (\gamma - 1)\frac{1}{\rho}\frac{d\rho}{dr}, \qquad (24)$$

where γ is the ratio of specific heats. This so-called *Cowling convective zone* extends up to 17 per cent. of the stellar radius and contains 15 per cent. of the stellar mass.

Continuing the integration of the equilibrium equations all the way to the centre of the star, we arrive at definite values of T_0, ρ_0, p_0, and M_0. Generally speaking, the value of M_0 resulting from such an integration comes out different from zero. Since, on the basis of the physical picture, we must have $M_0 = 0$ (no 'mass-point' and no 'vacuum' in the centre) we must adjust the solution, by adjusting the parameters $\bar{\mu}$ and $\varkappa_0$ used in the integration, so that the condition

† T. G. Cowling, *M.N.* **96** (1935), 42.

$M_0 = 0$ is satisfied. Since these two quantities depend essentially only on the hydrogen- and helium-content (equations (16) and (20)), the internal boundary condition, $M_0 = 0$, establishes a definite relation between X and Y. For each possible set $[X, Y]$ we get corresponding values of the central temperature, density, and pressure.

The set of fundamental equations of equilibrium was integrated numerically by Cowling (loc. cit.) under the assumption of constant $\mathfrak{x}_0$. Applying Cowling's solution to the sun, and using the *mean value* of

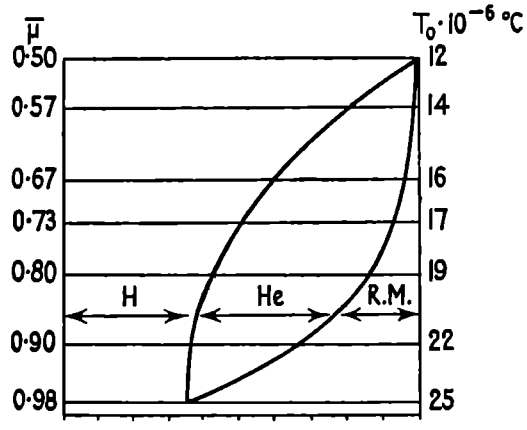


FIG. 49.

$\mathfrak{x}_0$ for the conditions inside the sun, we get with the assumption $Y = 0$: $X = 0{\cdot}35$; $T_0 = 2 \times 10^7$°C., and $\rho_0 = 80$ g./cm.^{3} More detailed integrations, which take into account the variations in $\mathfrak{x}_0$ (guillotine factor), were performed by Blanch, Lowan, Marshak, and Bethe† and also by Henrich‡ who obtain $T_0 = 2{\cdot}57 \times 10^{\circ}$ C., $\rho_0 = 110$ g./cm.^{3}, and $T_0 = 2{\cdot}5 \times 10^7$, $\rho_0 = 100$ g./cm.^{3}, respectively.

If we abandon the arbitrary assumption, $Y = 0$, the problem of the constitution of the star becomes indefinite, and, as indicated above, the boundary conditions can be satisfied by various sets of X and Y. This problem was investigated first by Strømgren,§ who has found that the possible solutions range from a constitution of almost pure hydrogen, with negligible amounts of helium and heavier elements, through a more or less even distribution among the three components to the solution with no helium, 35 per cent. hydrogen, and 65 per cent. heavier elements. In Fig. 49 we give the results

† G. Blanch, A. N. Lowan, R. E. Marshak, and H. A. Bethe, *Ap. J.* **94** (1941), 37.
‡ L. R. Henrich, ibid. **96** (1942), 106.
§ B. Strømgren, ibid. **84** (1938), 520.

of Strømgren's analysis modified so as to fit the more recent integrations of the equilibrium equations. The horizontal sections indicate the relative proportions of H, He, and R.M., consistent with the conditions of equilibrium, whereas the figures on the vertical axis indicate the corresponding values of mean molecular weight $\bar{\mu}$, and central temperature T_0.

Which of the possible solutions shown in Fig. 49 corresponds to reality is a question which cannot be answered on the basis of pure equilibrium theory; as we shall see later, the complete solution of the problem of solar constitution can be given only on the basis of a second relation between X and Y which is supplied by the nuclear theory of the energy production in the star.

3. Thermonuclear reactions in the sun

Knowing the temperature and the density in the central region of the sun, we can ask ourselves which particular thermonuclear reaction can go under these conditions at the appropriate rate to secure the necessary energy-supply. This question was first attacked by Atkinson and Houtermans (loc. cit.) who came to the conclusion that the only thermonuclear reactions that would proceed at reasonable rates in the interior of the sun are those between hydrogen and various light elements. They also have visualized this process as the successive capture of four protons by some light nucleus resulting in the emission of an α-particle ('wie kann man einen Helium-Kern im Potenzial-Topf kochen'), thus introducing the notion of *cyclic nuclear reaction*, the importance of which was proved only ten years later. Considering the possible results of proton-penetration into the nucleus of another element, we must take into account three different possibilities:

1. *The* (p, α)*-reaction* resulting in the capture of the incident proton and the emission of an α-particle.† For such processes the reaction probability $\Gamma/\hbar$ is comparable to the proper nuclear frequency $\hbar/MR^2$, so that the transformation takes place almost for every penetration. The exact values of Γ for different reactions can be estimated from the observed yields in bombardment experiments.

2. *The* (p, γ)*-reaction*, or the radiative capture of the incident proton, the probability of which is considerably smaller (by a factor

† The (p, n)-reactions are excluded in our case owing to their endothermic nature, and the fact that thermal protons have rather low kinetic energy.

10^4 or 10^5) than that in the previous case, owing to the small probability of the γ-ray emission.

3. *The (p, β)-reaction*, i.e. the capture of the incident proton through the spontaneous emission of an electron. The probability of this type of reaction is much smaller than in the case of radiative capture, due to the fact that (for equal energies) the mean life for a β-transformation exceeds that for γ-emission by a factor of about 10^{16}. It was suggested, however, by Bethe and Critchfield† that one particular process of that type, namely the reaction

$$H^1 + H^1 \rightarrow H^2 + e^+, \tag{25}$$

may be of importance for the problem of stellar energy-production, since in this case the low probability of β-transformation can be overcompensated by the very high probability for the mutual penetration of two thermal protons. Since, owing to its very small effective cross-section, this reaction cannot be expected to be observed in the laboratory, it is necessary to calculate the corresponding value of ϵ on the basis of the existing theory of β-transformations.

Formally, the reaction (25) is a β-transition between He^2 and H^2. There is no stable nucleus for He^2, so that one must consider the parent nucleus in this case to exist momentarily during the collision of two protons. The ground state of the daughter nucleus is the deuteron, an even state with spin one. Two protons colliding in the S-state form an even 'nucleus', but because of the exclusion principle the spin must be zero. Hence the transition (25) is allowed only if the spin of the nucleon may turn over, i.e. Gamow–Teller selection rules are obeyed. If this is the case the transition will be 'really allowed', by which we mean that the parent and daughter nucleus belong to the same supermultiplet (100). Then the transition probability is given by the formula, eq. (6), Chapter V. Substituting the indicated value of the Fermi constant, as found in that chapter, this formula may be written

$$\lambda = \frac{1}{\tau_0} |M|^2 I_0(W/mc^2) \tag{26}$$

$$\tau_0 = 6{,}200 \text{ sec.}$$

Although the binding energy of the deuteron is $4{\cdot}3mc^2$, the energy available to the positron emission (25) is only $1{\cdot}8mc^2 = W$, which is the mass-difference between two protons and a deuteron. The corresponding value of I_0 is $I_0(1{\cdot}8) = 0{\cdot}132$.

† H. A. Bethe and Chas. Critchfield, *Phys. Rev.* **54** (1938), 248, 862.

The distinguishing feature of this type of transition is contained in the matrix element M which is essentially an overlap-integral between the wave-function of the deuteron ψ_d on the one hand and the wave-function of colliding protons ψ_p on the other. If we normalized ψ_p to *unit incident current* we get directly the expression for the *cross-section* for the process which, with the numerical values above, becomes:

$$\sigma = 2{\cdot}2 \times 10^{-5} \left| \int \psi_p \psi_d \, d\tau \right|^2.$$

The wave-functions ψ_p are essentially the same as those presented in the discussion of proton-proton scattering, Chapter II, but, in the present instance, we are primarily concerned with these functions at very low energy and close to the origin. In this region, ψ_p may be represented in good approximation as the product of a function of radius, which is then integrated out after being multiplied by the known wave-function of the deuteron, and a function of relative velocity, which is retained as a factor in σ (essentially the penetration factor in a Coulomb field) and subsequently integrated over the thermal distribution of the protons. Let v be the relative velocity of the protons at large distance and b the radius of the deuteron

$$b = \hbar(M\epsilon_0)^{-\frac{1}{2}} = 4{\cdot}37 \times 10^{-13} \text{ cm}.$$

Then the expression for σ turns out to be

$$\sigma(v) = \Lambda^2 \frac{I_0(1{\cdot}8)}{\tau_0} \frac{e^2}{\hbar v} 16\pi^2 b^3 \, e^{-2\pi e^2/\hbar v}, \tag{27}$$

where Λ^2 is the integral over the function of radius. If the range of nuclear forces between the protons is taken to be e^2/mc^2 and the depth of the (square) well is 10·3 M.e.v., the value of Λ^2 has been found by Bethe and Critchfield (loc. cit.) to be 8·08.

To obtain the number of reactions per cm.^{3} per sec. in a gas of N protons per cm.^{3} we have to integrate $v\sigma(v)$ over the Maxwell–Boltzmann distribution function $\phi(\mathbf{v})d\mathbf{v}$. This integral will be multiplied by the number of proton pairs, $\frac{1}{2}N^2$. Other factors are: symmetrizing the two-proton wave-function and the fact that either proton may turn into a neutron† each introducing a factor 2, but the product is just compensated to unity by the fact that only one-fourth of the collisions between protons take place in the singlet state. Substituting for $\sigma(v)$ and $\phi(\mathbf{v})$ we get p' (per cm.^{3} per sec.)

$$p' = \frac{2\sqrt{\pi}}{\tau_0} I_0 N^2 b^3 \Lambda^2 \left(\frac{M}{kT}\right)^{\frac{3}{2}} \int \frac{2\pi e^2}{hv} v^2 \, dv \, e^{-(2\pi e^2/\hbar v)-(Mv^2/4kT)}.$$

† This factor was inadvertently omitted in the reference cited.

The integrand has a strong maximum very close to

$$v = (4\pi e^2 kT/\hbar M)^{\frac{1}{2}},$$

so that the function may be approximated by fitting it to a Gaussian 'error-curve' at this maximum and integrated to give

$$p' = \frac{32\pi}{3^{\frac{1}{2}}\tau_0} I_0(1{\cdot}8) N^2 b^3 \Lambda^2 \tau^2 e^{-\tau}$$

with $$\tau = 3(\pi^2 M e^4/4\hbar^2 kT)^{\frac{1}{3}}.$$

Or, if we measure T in millions of degrees, $T = 10^6 t$

$$\tau = 33{\cdot}8 t^{\frac{1}{3}}.$$

Substituting numerical values into p' we may now express the number of processes *per gramme* per second in terms of the density ρ, the concentration of hydrogen (by weight) c_H and τ:

$$p = 3{\cdot}4 \times 10^7 \rho c_H^2 \tau^2 e^{-\tau}.$$

Each process leads ultimately to the formation of an α-particle through the cycle:

$$\begin{aligned} H^2 + H^1 &\rightarrow He^3 + \gamma \\ He^3 + He^4 &\rightarrow Be^7 + \gamma \\ Be^7 &\rightarrow Li^7 - e^- \\ Li^7 + H^1 &\rightarrow 2He^4 \end{aligned} \qquad (28)$$

releasing $4{\cdot}3 \times 10^{-5}$ erg. The energy-production per gramme per sec. is then:

$$\epsilon = 1460 \rho c_H^2 \tau^2 e^{-\tau} \text{ ergs/g.sec.} \qquad (29)$$

Substituting in eq. (29) values appropriate to the centre of the sun: $\rho = 100$, $c_H = 0{\cdot}35$, $\tau = 11{\cdot}6$ ($T = 2{\cdot}5 \times 10^7$), we obtain

$$\epsilon = 24 \text{ ergs/g.sec.}$$

at the centre. Hence, *if the β-transition in eq.* (25) *is an allowed transition, it leads to an energy-production of the right order of magnitude for our sun.*

In Table XXIV we give theoretical results pertaining to various thermonuclear reactions under the conditions existing in the sun. These data are calculated on the basis of formula (10) on the assumptions: $T = 2 \times 10^7$° C. and $\rho X = 30$ g./cm.^{3}† In the third column we give the 'mean reaction-times' which are evidently independent of the concentration of the heavy element involved. The fifth column

† These data correspond to Cowling's old integration which assumes $Y = 0$. According to more detailed integrations (cf. Fig. 49) the same central conditions are obtained for $X \simeq Y \simeq 0{\cdot}4$.

TABLE XXIV

Mean reaction-times and rates of energy-production for $T = 2\times10^{7}$° C., $\rho = 80$ g./cm.^{3}, *concentrations* = 0·35

Reaction	Γ_{ev}	*Mean reaction-time*	Q(*M.m.u.*)	*Rate of energy-production erg/g.sec.*
$H^1 + H^1 \rightarrow H^2 + e^+$	..	$1{\cdot}5\times10^{10}$ yrs.	1·53	1·0†
$H^2 + H^1 \rightarrow He^3 + \gamma$	1	2 sec.	5·9	4×10^{19}
$He^3 + H^1 \rightarrow He^4 + \gamma$	10	0·2 sec.	21·3	9×10^{20}
$H^2 + H^2 \rightarrow He^3 + n$	3×10^{5}	30 sec.	3·5	1×10^{18}
$Li^6 + H^1 \rightarrow He^3 + He^4$	5×10^{5}	5 sec.	4·1	3×10^{18}
$Li^7 + H^1 \rightarrow 2He^4$	4×10^{4}	1 min.	18·6	1×10^{18}
$Be^9 + H^1 \rightarrow Li^6 + He^4$	10^{6}	15 min.	2·4	8×10^{15}
$B^{10} + H^1 \rightarrow C^{11} + \gamma$	10	1,000 yrs.	9·2	8×10^{8}
$B^{11} + H^1 \rightarrow 3He^4$	10^{6}	3 days	9·4	9×10^{13}
$C^{11} + H^1 \rightarrow N^{12} + \gamma$	0·02	10^{8} yrs.	0·4	3×10^{2}
$C^{12} + H^1 \rightarrow N^{13} + \gamma$	0·6	$2{\cdot}5\times10^{6}$ yrs.	2·0	6×10^{7}
$C^{13} + H^1 \rightarrow N^{14} + \gamma$	30	5×10^{4} yrs.	8·2	1×10^{6}
$N^{14} + H^1 \rightarrow O^{15} + \gamma$	60	4×10^{6} yrs.	7·8	1×10^{5}
$N^{15} + H^1 \rightarrow C^{12} + He^4$	10^{7}	20 yrs.	5·2	2×10^{10}
$O^{16} + H^1 \rightarrow F^{17} + \gamma$	0·02	10^{12} yrs.	0·5	4×10^{-2}
$F^{19} + H^1 \rightarrow O^{16} + He^4$	10^{5}	3×10^{7} yrs.	8·8	1×10^{4}
$Ne^{22} + H^1 \rightarrow Na^{23} + \gamma$	10	2×10^{13} yrs.	10·7	2×10^{-2}
$Mg^{26} + H^1 \rightarrow Al^{27} + \gamma$	10	10^{17} yrs.	8·0	3×10^{-6}
$Si^{30} + H^1 \rightarrow P^{31} + \gamma$	10	3×10^{20} yrs.	7·0	6×10^{-10}
$Cl^{37} + H^1 \rightarrow A^{38} + \gamma$	10	2×10^{25} yrs.	12·0	1×10^{-14}
$He^4 + He^4 \rightarrow Be^8 + \gamma$	5×10^{-9}	10^{15} yrs.	0·05 ?	1×10^{-3}
$Li^7 + He^4 \rightarrow B^{11} + \gamma$	1	4×10^{14} yrs.	9·1	2×10^{-3}
$Be^9 + He^4 \rightarrow C^{11} + \gamma$	1	2×10^{20} yrs.	8·0	5×10^{-9}
$C^{12} + He^4 \rightarrow O^{16} + \gamma$	1	10^{33} yrs.	7·8	1×10^{-21}

contains the rate of energy-liberation calculated under the assumption that the densities of both reactants are equal to 0·30 g./cm.^{3}

Comparing the figures of Table XXIV with astrophysical evidence we must remember that:

(1) since the mean energy-production of the sun is

$$4{\cdot}10^{33} \text{ erg/sec.} \div 2{\cdot}10^{33} \text{ g.} = 2 \text{ erg/g.sec.},$$

the rate of energy-liberation in the central regions, where the energy is actually produced, must be considerably higher than that figure;

(2) since, as evidenced by the continuity of evolution of life on our globe, the sun must have been shining at approximately its present rate for at least 10^9 years, the mean life of the process

† This value is smaller than that given on p. 276, since the values of the table have been calculated on the assumption of lower temperature and density in the centre of the sun.

responsible for the energy-production must be considerably longer than that.

Looking through the figures of Table XXIV we notice first of all that one possibility consists in the p–p reaction followed by the sequence of reactions (28). Since the mean periods of the subsequent reactions are much shorter than the solar life, we must suppose the sequence to be in equilibrium and thus producing energy at the rate calculated above. The estimated average energy production for the sun is then 1 erg/g.sec., i.e. half enough to account for the entire radiation. Unfortunately, at the present time, it is not certain that the β-transition involved is allowed. If the transition is of the allowed type, the p–p reaction is of primary importance in the sun, and increasingly so for less massive stars.

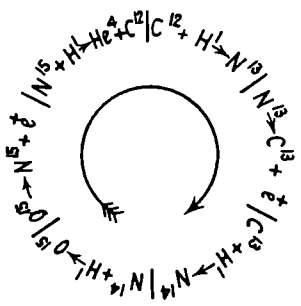


FIG. 50.

Looking farther through Table XXIV we may first get the impression that there is no other reaction suitable for the energy-production of the sun. In fact, the mean reaction-times for Li, Be, B, C, N, and F are much shorter than 10^9 years, and, on the other hand, the calculated rates of energy-production involving O, Ne, Mg, Si, etc., as well as those involving He, are much too low. This apparent difficulty, however, is straightened out, unexpectedly, by the peculiar nature of the reactions involving carbon and nitrogen. In fact, it was shown independently by Weizsäcker† and by Bethe‡ that the reactions involving these two elements represent different links in a single cyclic reaction-chain, in which the nuclei of C and N are regenerated after each complete cycle. This reaction-chain, now known as the *carbon cycle*, is represented schematically in Fig. 50, which shows that the net result of each cycle is the transformation of four hydrogen atoms into one atom of helium. Since carbon and nitrogen nuclei are not destroyed in this process, the comparatively short reaction-times of the individual links do not play any role in determining its duration, and the liberation of energy will continue uninterrupted as long as there is hydrogen left in the star (or rather, in its internal convective zone).§

† C. v. Weizsäcker, *Phys. Zs.* **39** (1938), 633.

‡ H. A. Bethe, *Phys. Rev.* **55** (1939), 434.

§ We may note at this point that such a cyclic reaction proceeding in the solar

Calculating the total energy-balance of the carbon cycle, we have:

$C^{12} + H^1 \rightarrow N^{13} + \gamma$	2·0
$N^{13} \rightarrow C^{13} + e^+ +$ (neutrino)	$0{\cdot}5 = (\frac{3}{8} \times 1{\cdot}35)$
$C^{13} + H^1 \rightarrow N^{14} + \gamma$	8·2
$N^{14} + H^1 \rightarrow O^{15} + \gamma$	7·8
$O^{15} \rightarrow N^{15} + e^+ +$ (neutrino)	$0{\cdot}7 = (\frac{3}{8} \times 1{\cdot}85)$
$N^{15} + H^1 \rightarrow C^{12} + He^4$	5·2
Total	24·4 M.m.u.
or	$3{\cdot}66 \times 10^{-5}$ erg

Here, as also in the case of the p–p reaction, we have subtracted the energy carried away by the neutrinos ($\frac{5}{8}$ of the total energy of β-transformation) which amounts to about 7 per cent. of the total energy-balance. Since the duration of a single cycle is the sum of durations of the individual links, the cycle period will be determined essentially by the sum of C^{12} and N^{14} lives:

$$2{\cdot}5 \times 10^6 + 4 \times 10^6 = 6{\cdot}5 \times 10^6 \text{ years} = 1{\cdot}95 \times 10^{14} \text{ sec.}$$

Thus the rate of energy-production (for $\rho X = 30$) becomes:

$$\epsilon = 4 \times 10^{-5} \frac{C_c}{12 \times 1{\cdot}65 \times 10^{-24}} \frac{1}{1{\cdot}95 \times 10^{14}} = 10^4 C_c \frac{\text{erg}}{\text{g.sec.}}, \quad (30)$$

where C_c is the concentration of carbon. Thus we see that even for carbon concentrations as low as 1 per cent., we still have

$$\epsilon = 100 \text{ erg/g.sec.},$$

which is high enough to account for the energy-production in the sun.

Although the calculated rates of energy-production by the H—H

interior must necessarily establish a definite dynamic equilibrium between the species of atomic nuclei participating in the process. In this state of equilibrium the relative amounts of different nuclei must become directly proportional to the corresponding mean periods of the reactions. Using the data of Table XXIV we find that in such an equilibrium C^{12} and N^{14} must be present in the ratio

$$C^{12}:N^{14} = 2{\cdot}5 \times 10^6 : 5 \times 10^4 = 5:8$$

whereas the relative abundances of their stable isotopes must be given by

$$C^{12}:C^{13} = 2{\cdot}5 \times 10^6 : 5 \times 10^4 = 50 \quad \text{and} \quad N^{14}:N^{15} = 4 \times 10^6 : 20 = 2 \times 10^5.$$

Comparing these figures with the relative abundances of these nuclei on the earth

$$(C^{12}:N^{14} = 3;\ C^{12}:C^{13} = 140\ N^{14}:N^{15} = 250)$$

we fail to find any agreement, especially in the case of nitrogen isotopes. This is, of course, not surprising in view of still larger disagreements in the case of such nuclei as H^2, Li, Be, and B which must be completely absent from the reaction region of the sun, but are present in noticeable amounts in the solar atmosphere and on the earth. This result suggests that the material forming our globe has never been a constituent part of the interior of the sun.

reaction and by the carbon cycle appear to favour the latter as the main source of solar energy, it is of interest to make an independent test based on the difference between the two processes with respect to their dependence on temperature. The temperature dependence of the rate of reaction, which is generally given by an exponential function, can be approximated within a given temperature interval by a power law:

$$\epsilon \sim \epsilon_0 T^n. \tag{31}$$

Comparing the derivative of this formula and of the exact formula (10) we obtain:

$$n = \frac{d\log(\tau^2 e^{-\tau})}{d\log T} = \tfrac{1}{3}(\tau - 2). \tag{32}$$

For the H—H reaction near 2×10^{7}° C. this gives $n = 3{\cdot}5$, whereas for the carbon cycle we obtain $n = 17$.

Substituting the expression (31) for ϵ into the fundamental equation of energy-production (22), neglecting the variability of the guillotine factor, and assuming that the radiation pressure is negligible compared with the gas pressure (for the sun, and stars of comparable mass, the former is only 0·3 per cent. of the latter),† we obtain a system of five *homogeneous* equations, (13), (14), (15), (18), and (22), containing nine independent variables T, ρ, p, L, R, M, $\bar{\mu}$, $\varkappa_0$, and ϵ_0. This set of equations is invariant to *homology* (*or similarity*) *transformations* in which the scale-factors of any five variables are determined as functions of those of the remaining four. Assuming $\bar{\mu}$, $\varkappa_0$, and ϵ_0 constant, and increasing the mass M by the factor a, we find that the remaining characteristics of the star must be changed by the following factors:‡

$$\begin{aligned} L &\;.\;.\; a^{(31+10n)/(5+2n)} \\ R &\;.\;.\; a^{11-2(31+10n)/(5+2n)} \\ T &\;.\;.\; a^{-10+2(31+10n)/(5+2n)} \\ \rho &\;.\;.\; a^{-32+6(31+10n)/(5+2n)} \\ P &\;.\;.\; a^{-42+8(31+10n)/(5+2n)}. \end{aligned} \tag{33}$$

Using the values of n derived above, we obtain the following

† The ratio of the radiation pressure to the total pressure, usually denoted by $(1-\beta)$, depends essentially only on the mass of the star and is given by the following table calculated by Eddington (loc. cit.).

Stellar mass (in sun masses)	1	5	10	20	40
$(1-\beta)$	0·003	0·5	0·15	0·30	0·45

‡ G. Gamow, *Zs. f. Astrophys.* **16** (1938), 113.

expressions for the dependence of stellar luminosity, radius, and central temperature on the stellar mass:

H—H *reaction*	C *cycle*	
$L \sim M^{5.5}$	$L \sim M^{5.2}$	
$R \sim M^{0}$ (const)	$R \sim M^{0.7}$	(34)
$T^{1} \sim M^{1.0}$	$T \sim M^{0.4}$	

In Fig. 51 *a*, *b* we give the plot of $\log L$ and $\log R$ against $\log M$ for 15 stars with masses between 0·4 and 2·5 sun masses.†

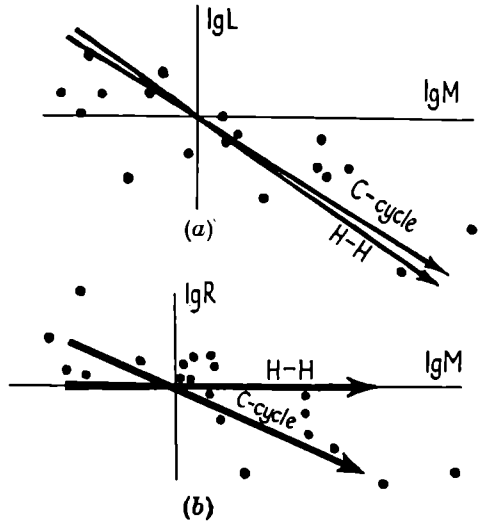


FIG. 51

The lines marked H—H and C–cycle represent the theoretical dependence for the H—H reaction and the carbon cycle, as given by (34), respectively. Whereas the [$\log L$, $\log M$] graph does not permit one to distinguish between the two possibilities, the [$\log R$, $\log M$] graph shows beyond any doubt that for stars of these masses the main part of the energy-production must be due to the carbon cycle. It should be noted, however, that for less massive stars the situation is reversed. In fact, since

$$\frac{\text{rate of (C—H)}}{\text{rate of (H—H)}} \sim \frac{T^{17}}{T^{3.5}} = T^{13.5},$$

and since for 2×10^{7}° C. the energy-production by H—H reactions amounts to a fraction of that for the carbon cycle, the two rates become equal at some lower temperature which depends upon the exact ratio of the two sources in the sun which, in turn, is not known.

† G. Gamow, *Ap. J.* **89** (1939), 130.

If this ratio is 1 per cent., the central temperature for which the rates are equal is $T_0 = 2\times10^7(0{\cdot}01)^{1/13{\cdot}5} = 1{\cdot}5\times10^{7\,\circ}$ C., and from (34) (for the carbon cycle) the corresponding stellar mass is about 0·4 suns. According to astronomical data, stars of this and still smaller mass form the bulk of the stellar population of our galaxy, so that we can say that, whereas the carbon cycle is used by comparatively few exceptionally brilliant stars, the rank and file of the stellar population quite possibly draws its life energy from the H—H process.

Having established that the energy production in the sun, and in stars of comparable mass, is due to the carbon cycle, we can use this knowledge for a more detailed analysis of the solar constitution. In fact, we have seen in the previous section that purely astrophysical evidence gives us one relation between the hydrogen content X and the helium content Y (Fig. 49). Knowing the nature of the energy-source, we can now get another relation between these two quantities that will permit us to estimate X and Y separately. Calculations of this type have been performed by Schwarzschild.† He assumed, on the basis of analyses of the solar atmosphere, that carbon represents 2 per cent. of the Russell mixture of heavier elements, so that $C_c = 0{\cdot}02(1-X-Y)$. Remembering that at $T = 2\times10^7$, $\rho X = 0{\cdot}30$, $\epsilon = 10^4 C_c$ erg/g.sec., one can write for neighbouring temperatures and densities:

$$\epsilon(\rho, T) = 200(1-X-Y)\frac{X\rho}{30}\left(\frac{T}{2\times10^7}\right)^{17} \tag{35}$$

and calculate the total energy-production L as the integral

$$L = \int_0^{r_{\text{conv}}} \epsilon(\rho, T)4\pi r^2\rho\, dr, \tag{36}$$

where r_{conv} is the radius of the convective core. The results of these calculations are shown in Fig. 52, where the curve A represents the $[X, Y]$ dependence obtained from equilibrium conditions, whereas the curve B gives the $[X, Y]$ relation which follows from the considerations of energy-production. The point of intersection corresponds to

$$X = 0{\cdot}47 \qquad Y = 0{\cdot}41 \qquad \text{R.M.} = 0{\cdot}12$$

$$\bar{\mu} = 0{\cdot}76 \qquad T_c = 1{\cdot}98\times10^{7\,\circ}\text{ C.} \qquad \rho = 112\text{ g./cm.}^3$$

The lighter curves in Fig. 52 represent the effect of possible (rather

† M. Schwarzschild, *Ap. J.* **104** (1946), 201.

liberally estimated) errors in X_c and $\mathfrak{X}_0$ of the Russell mixture. It seems certain that for the sun the value of X lies between 0·35 and 0·60, the value of Y between 0·25 and 0·50, whereas the total amount of heavier elements is between zero and 0·30. This result is in excellent agreement with the more recent results of spectroscopic analysis of solar† and stellar‡ atmospheres which indicate 54 per cent. hydrogen, 44 per cent. helium, and only about 1 per cent. of the heavier elements.

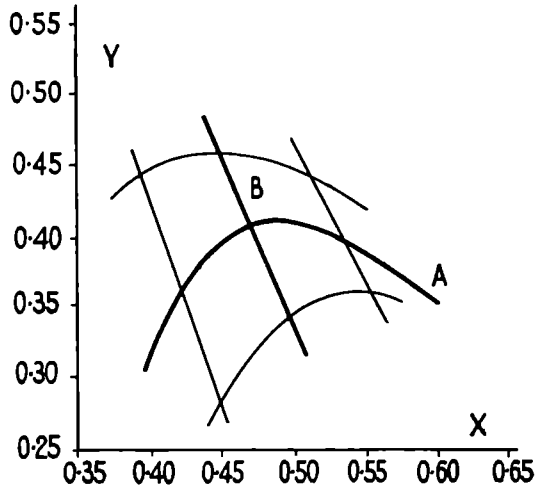


FIG. 52.

4. Thermonuclear reactions in other stars

Turning to the problem of thermonuclear energy-production in other stars, we must first get acquainted with the stellar classification as given by the so-called *Hertzsprung–Russell* diagram (Fig. 53). In this diagram the logarithms of the observed stellar luminosities (connected with 'stellar magnitudes' by the relation $\Delta M = -2{\cdot}5\Delta \log L$) are plotted against the logarithms of the observed surface temperatures (determining the so-called 'spectral class' of the star). The radius of a star is connected with the two quantities above by the obvious relation: $L = 4\pi R^2 \sigma T_s^4$, and is increasing towards the upper right corner of the diagram.

We see that the known stars can be classified into three major groups:

1. *The main sequence*, containing the stars that fall within a narrow band crossing the diagram from the region of very hot (blue)

† D. H. Menzel, ibid. (in the press, 1948).
‡ A. Unsöld, ibid. **100** (1944), 110.

and very luminous stars, to the region of cool (red) stars of rather small luminosity.

2. *The red giant group*, containing the stars of low surface temperature, high luminosities, and very large radii.
3. *The white dwarf group*, containing the stars of very high surface temperature, low luminosity, and very small radii.

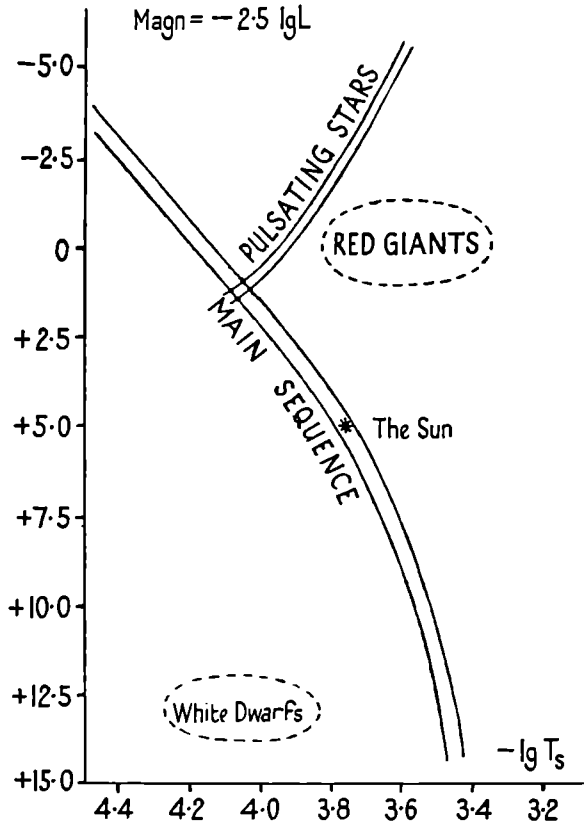


FIG. 53.

Since our sun itself belongs to the main sequence, it would be natural to suppose that all other stars of this group possess the same general structure and have the same source of energy as the sun, differing from it only with respect to mass. And, in fact, using the homologic transformations (33) (with $n = 18$ for $M > 0{\cdot}4\ M_{\odot}$, and $n = 3{\cdot}5$ for $M < 0{\cdot}4\ M_{\odot}$) it is possible to account for the observed luminosities and radii (or surface temperatures) of all the main-sequence stars entirely on the basis of their observed differences in mass.†

† G. Gamow, *Phys. Rev.* **65** (1944), 20 (Fig. 1).

The problem of the red giants, on the other hand, presents serious difficulties which have not yet been overcome. If we assume that these stars are built on the same plan as the sun, and the other stars of the main sequence, the calculations lead to only a few million degrees for the values of their central temperatures. Since neither the H—H reaction nor the carbon cycle would produce significant power at these low temperatures, it was suggested by Gamow and Teller† that the sources of energy for these stars lie in the transformations of light elements, Li, Be, and B. According to this assumption, red giants would represent the early stages of stellar evolution in which these light elements are being successively burned out, preliminary to the star's entrance into the main sequence. Since the reactions involving these elements are not cyclic, the duration of each evolutionary stage must be comparatively short, and a young star, contracting under the force of gravity from a primary diffuse mass, would stop in the corresponding regions of the Hertzsprung–Russell diagram only long enough to 'burn out' the small amounts of these elements which were originally present in the mass. A detailed study of this assumption, at first sight very attractive, indicated, however, that, although stars of such a description may exist in the sky, the majority of the red giants certainly require some other explanation.‡

A new bit of evidence concerning the nature of red giant stars is presented by an analysis made by Chandrasekhar,§ who has shown, on the basis of observational material, that red giants possess a much *higher central condensation* of material than normal stars of the main sequence. Thus, whereas in the normal point-source model 90 per cent. of the stellar mass occupies about one-half of the stellar radius, typical red super-giants like VV Ceph M or ϵ Aur I have 90 per cent. of their mass condensed within 0·21 and 0·24 R from their centres.

An attempt to explain the properties of red giant and supergiant stars by a change of model was made by Gamow and Keller,‖ who proposed that these stars may represent later stages of stellar

† G. Gamow and E. Teller, ibid. **55** (1939), 791; see also M. Greenfield, ibid. **60** (1941), 175.

‡ The difficulties of the Gamow–Teller hypothesis lay mainly in the fact that it fails to account for the characteristic distribution of the red giants in the frame of the Hertzsprung–Russell diagram, and also cannot explain their comparatively large number.

§ S. Chandrasekhar, *Introduction to the Study of Stellar Structure*, loc. cit.

‖ G. Gamow, *Phys. Rev.* **67** (1945), 120; G. Gamow and G. Keller, *Rev. Mod. Phys.* **17** (1945), 125; cf. also M. Harrison, *Ap. J.* **103** (1946), 193.

evolution that follow the depletion of hydrogen in the central convective zone. In fact, it is easy to see that, as soon as the material of this zone (which is being constantly stirred up by convective currents) becomes finally deprived of hydrogen, the structure of the star must change from the conventional point-source model to the so-called *shell-source model* in which the energy is produced along the spherical surface that separates the dehydrogenized material on the inside from the hydrogen-rich material on the outside. Whereas, under these conditions, the matter within the energy-producing shell must be considered to be in a state of isothermal equilibrium (since no energy is produced there), the outer portion of the star remains subject to the equations of radiative equilibrium. The temperature of the energy-producing shell, as well as that of the dehydrogenized material inside of it, must be maintained, of course, at a level for which the carbon cycle will operate, i.e. at about 2×10^7 ° C. As the process of the $H \rightarrow He$ transformation progresses, the energy-producing shell advances towards the outer surface of the star, and an ever-increasing fraction of the total mass of the star becomes included in the isothermal core. Detailed analysis of the process shows that the growth of the energy-producing shell leads to degeneracy of the material near the centre of the star, so that the resulting stellar model must consist of three regions: (1) the degenerate nucleus (similar to the white-dwarf configuration), (2) the isothermal core of dehydrogenized gas surrounding this nucleus, and (3), the outer envelope containing the normal concentration of hydrogen. Calculations on such composite stellar models are necessarily rather complicated, and in spite of the fact that they seem to indicate that the growth of the energy-producing shell in radius results in an unlimited increase of the outer radius of the star, it has not yet been possible to find a satisfactory correlation between the properties of such 'overgrown-shell' models and the observed characteristics of the red giant stars. It may be noticed here, however, that this point of view leads to an interesting interpretation of the origin of the white dwarf stars, which represent, as we shall see later, degenerate configurations of dehydrogenized stellar material. In fact, the proposed two-way process in which part of the material of the star becomes concentrated into the central degenerate core, while another part forms an ever-expanding envelope, must lead to the ultimate dispersion of the outer material, thus revealing the hot, central body to the eye of the

astronomer. This is in agreement with observational evidence which indicates that many stars of this type possess composite spectra in which high-temperature characteristics are mixed with those corresponding to a much lower temperature. According to the so-called *Nutty–Menzel* hypothesis, such composite spectra are due to the fact that we see the light of the hot central body shining through the comparatively cool outer envelope of the star. It may be added that the temperature of the degenerate central condensation which is left over after the dispersal of the envelope must be expected to be equal to the temperature of the isothermal core within which it was formed, i.e. to $2\times 10^7\,^\circ$ C. (since the temperature of the isothermal core is determined by the temperature of the energy-producing shell in which the carbon cycle is taking place), which is in perfect agreement with the estimated internal temperatures of the known white dwarfs.

We may add that the problem of internal structure and energy sources of red giants and supergiants is closely connected with the important problem of *stellar pulsations*. It is observed that the stars that fall within a certain narrow band in the Hertzsprung–Russell diagram ('pulsation-region' in Fig. 53) are in a state of regular pulsation, their luminosities and radii varying in time with a period that depends upon the total mass in each case. The dynamical theory of pulsating gaseous spheres developed by Eddington leads to the following relation between the period of pulsation Π and the central density ρ_0:

$$\Pi = \left[G\gamma\left(3-\frac{4}{\gamma}\right)\rho_0\right]^{-\frac{1}{2}}, \tag{37}$$

(where G is Newton's constant of gravitation and γ is the ratio of specific heats) which is in generally good agreement with the observed facts.† It can be shown also that, due to the dissipation of energy, a

† It may be remarked here that the observed stellar pulsations do not agree in every detail with the simple theory of pulsating gas spheres. First of all, whereas the simple theory requires a simple sine wave for the variation of the light emitted, the observed luminosity curves do not rise and fall symmetrically but rise sharply and fall slowly. Also, the simple theory predicts that the maximum luminosity is coincident with the minimum stellar radius, whereas the observations are that the maximum and minimum luminosities coincide with the maximum expansion and contraction velocities of the stellar surface (as measured by means of the Döppler effect). The asymmetry of the light curves and the 90° phase-shift of luminosity relative to radial motion suggest that the star does not pulsate as a whole, but that one has to do with waves of compression which originate from a regularly pulsating central body and travel through the outer envelope of the star. More detailed study of this process is bound to give us valuable information concerning the internal structure of these stars.

pulsation of a star which is excited by some external cause must die out in a few thousand years. Since this is certainly not the case for the observed pulsating stars, it is necessary to assume that some intrinsic instability exists which forces the pulsation to go on at the cost of the general energy production in the interior. To explain the observed distribution of pulsating stars in the frame of the Hertzsprung–Russell diagram, it must be assumed further that such an instability occurs only for a definite relation between the luminosity of the star and its radius.† One possible interpretation of the phenomenon of self-supporting stellar pulsations may lie in the notion of the so-called *superstability* in stellar structure, first introduced by Eddington. He indicated that, in such cases when the rate of energy-production near the centre of the star depends very strongly upon the temperature, the adiabatic temperature rise in the contracted phase may cause the liberation of such large amounts of energy in a short time that the body of the star will be 'swung out' to a maximum radius larger than the maximum radius preceding the contraction. The next contraction will then cause higher temperatures and will liberate still larger amounts of energy, resulting in a still larger subsequent expansion. Thus the pulsation of such a star will increase rapidly in amplitude and must terminate in the ultimate disruption of its entire structure. The problem of stellar 'superstability' was carefully investigated by Cowling,‡ who formulated a relation between the exponent n in the temperature dependence of the energy-production, and the ratio of specific heats γ of the stellar material for which this phenomenon should be expected to take place. For the value of $\gamma = 5/3$, which is typical for the sun and all other stars of not very large mass, the critical value of n is 450, i.e. much larger than that corresponding to the carbon cycle or any other possible nuclear reaction. However, for much more massive stars the situation becomes essentially different since, due to the ever increasing importance of radiation pressure, the value of γ must be compounded from the value 5/3 for a perfect gas and the value 4/3 for the radiation.

$$\gamma \simeq \frac{5\beta+4(1-\beta)}{3} = \frac{4+\beta}{3}. \tag{38}$$

† From Fig. 53 we find that the pulsation line is represented approximately by the equation: $L_{\text{puls}} = A \,.\, T_{\text{e(puls)}}^{12}$. Remembering that $L \sim R^2 T_{\text{e}}^4$ we find that the necessary condition for pulsation is $L_{\text{puls}} \sim R_{\text{puls}}^3$.

‡ T. G. Cowling, *M.N.* **94** (1934), 768; **96** (1935), 42.

For $n = 17$, superstability occurs when $\gamma = 1{\cdot}43$ which, according to (38), corresponds to $1-\beta = 0{\cdot}71$. Looking in the table given in the footnote on p. 280, we find that this corresponds to stellar masses larger than $(17/\mu^2)M_\odot$ ($100\,\mathrm{M}_\odot$ for $\mu = 0{\cdot}76$). This value is considerably larger than the masses of the pulsating stars, but, on the other hand, it must be remembered that it was arrived at on the basis of very rough calculations, and that a detailed analysis might bring it down. In any attempt to connect the phenomenon of stellar pulsations with the notion of superstability, one should, of course, account for the fact that these pulsations do not exceed a certain maximum amplitude; this may be due, for example, to the difference between homologically pulsating models for which the calculations are usually made, and the more complicated motions that take place in a pulsating gas-sphere. To sum up, we must say that the solution of the riddle of red giant stars lies most probably in their structural and dynamical aspects rather than in some peculiarity of their nuclear energy-sources.

The study of white dwarf stars brings in an entirely new aspect of the theory of matter: the problem of the *superdense state*. In fact, whereas the masses of the two best known stars of this type, the companion of Sirius and 40 Eridani B, are very close to our sun in mass ($0{\cdot}98\,M_\odot$ and $0{\cdot}45\,M_\odot$, respectively) their radii are only 0·020 and 0·018 of the solar radius corresponding to mean densities of $1{\cdot}7\times10^5$ and $1{\cdot}0\times10^5$ g./cm.^{3} As first suggested by Fowler,† matter compressed to such high densities must be considered as being in a state of complete ionization (regardless of temperature) since, indeed, the nuclei are brought so close together that 'there is no place' between them for the regular electronic orbits. Since at the densities of the order of magnitude 10^6 g./cm.^{3} the zero-point energy of free electrons is of the order of $1{\cdot}5\times10^{-7}$ erg,‡ we must conclude that an electron gas resulting from such 'pressure-ionization' will be in a state of almost complete degeneracy unless the temperature of the material is above $T = (1{\cdot}5\times10^{-7}/\frac{3}{2}k) = 7\times10^8$ ° C., which is much higher than the

† R. H. Fowler, *M.N.* **87** (1926), 114.

‡ The average volume available for each electron is, in this case

$$\frac{AM}{\rho Z} = \frac{2\times1{\cdot}66\times10^{-24}}{10^6} = 3{\cdot}3\times10^{-30}\ \mathrm{cm.}^3,$$

so that the average energy (in the non-relativistic case) may be estimated as

$$\frac{p^2}{2m} = \frac{3h^2}{10m}\left(\frac{3}{8\pi}\right)^{\frac{2}{3}}(3{\cdot}3\times10^{-30})^{-\frac{2}{3}} = 1{\cdot}5\times10^{-7}\ \mathrm{erg}.$$

estimated internal temperatures of the white dwarf stars (only 2×10^7 ° C.). Thus one can consider the structure of these stars as determined entirely by the temperature-independent equilibrium between the pressure of the degenerate electron gas and the forces of gravity. The dependence of pressure on density in a degenerate electron gas may be written in the form:

$$P_e = \frac{m^4c^5}{24\pi^2\hbar^3}[x(2x^2-3)(x^2+1)^{\frac{1}{2}}+3\sinh^{-1}x], \tag{39}$$

where

$$x^3 = \frac{3\pi^2\hbar^3}{m^3c^3}n_e \tag{40}$$

and n_e stands for the number of electrons per unit volume. For comparatively low densities this expression goes over into

$$P_e = \frac{3^{\frac{2}{3}}\pi^{\frac{4}{3}}}{5}\frac{\hbar^2}{m}n_e^{\frac{5}{3}} = 9{\cdot}86\times10^{12}\left(\frac{\rho}{\bar{\mu}}\right)^{\frac{5}{3}}, \tag{41}$$

corresponding to the *non-relativistic degenerate gas*, whereas for higher densities we obtain

$$P_e = \frac{3^{\frac{1}{3}}\pi^{\frac{2}{3}}}{4}\hbar c n_e^{\frac{4}{3}} = 1{\cdot}23\times10^{14}\left(\frac{\rho}{\bar{\mu}}\right)^{\frac{4}{3}}, \tag{42}$$

representing the case of complete *relativistic degeneracy*.

For a completely ionized element with atomic weight A and atomic number Z, the number of electrons is evidently

$$n_e = \frac{\rho Z}{AM} = \frac{\rho}{M\bar{\mu}_e}, \tag{43}$$

where $\bar{\mu}_e$ (the mean molecular weight per electron) is equal to *one* for hydrogen and *two* for helium (and approximately two for all heavier elements). Thus we see that the equilibrium in a white dwarf depends *only on the hydrogen content* X of its material (in contrast to normal stars where Y is also important). Combining the equation of state, (39), with the hydrostatic equation:

$$-\frac{dp_e}{dr} = \frac{G\rho M_r}{r} \tag{44}$$

one can calculate the external radii (as well as the internal density-distributions) for the degenerate white dwarf configurations of various total masses and hydrogen content. Such calculations have been performed by Chandrasekhar,† whose results are presented

† S. Chandrasekhar, *M.N.* **95** (1935), 208; see also *Introduction to the Study of Stellar Structure*, loc. cit.

graphically in Fig. 54. These calculations were carried out under the assumption, $\bar{\mu}_e = 1$; for other values of $\bar{\mu}_e$, M should be multiplied by $\bar{\mu}_e^{-2}$, R by $\bar{\mu}_e^{-1}$, and ρ by $\bar{\mu}_e$.

We notice first of all that for degenerate stellar configurations the radius of the star decreases with its mass and becomes zero for $M_{\text{crit}} = 5{\cdot}75\, M_\odot/\bar{\mu}_e^2$. Since it is reasonable to suppose that the degenerate configurations represent the ultimate state of gravitational contraction of normal stars which have used up their hydrogen, and are therefore deprived of any source of nuclear energy, we may con-

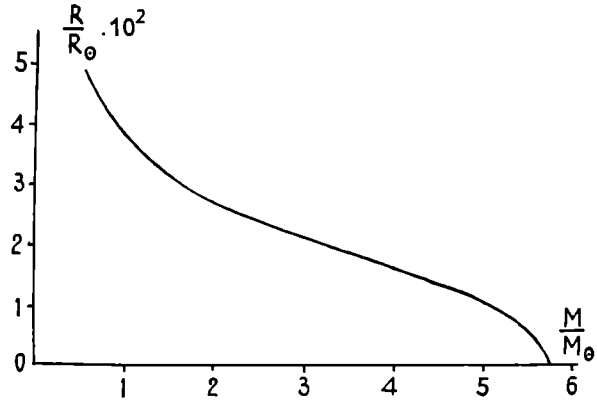


FIG. 54.

sider their material as consisting almost entirely of helium and put $\bar{\mu}_e = 2$. This gives for the critical mass the value $M_{\text{crit}} = 1{\cdot}44\, M_\odot$ which is consistent with the fact that the masses of the known white dwarf stars do not exceed the mass of the sun. Thus, whereas the contraction of the stars with comparatively small mass must lead ultimately to the formation of the degenerate white dwarfs (which after cooling down become lifeless 'black dwarfs'), the stars of larger masses should contract, theoretically speaking, into a mathematical point. It goes without saying that such a thing never actually happens since, whenever the radius of the star becomes too small, a number of other physical phenomena may take place. First of all, the ever-increasing rotational velocity of such a contracting star may ultimately lead to the formation of a sharp equatorial edge from which excess mass will be ejected into the surrounding space. Another reason for the ejection of stellar material can be found in the radiation pressure which increases parallel with the surface temperature of the contracting star and, at a certain stage of the contraction, may be expected to overbalance the forces of gravity holding the stellar

atmosphere together.† Finally, the extremely high temperatures obtaining in the central regions of contracting gas-spheres will lead to a peculiar type of nuclear process, which will be described in the following section, and which may result in a catastrophic collapse of the stellar body and the ejection of the major portion of its mass.

Returning to the sub-critical, degenerate configurations, we may compare the results of the calculations given in Fig. 54 with the observational data. Using the observed values of the M's and R's, we can estimate the corresponding value of $\bar{\mu}_e$ and, since this value depends only on X, deduce the hydrogen content of the star. The result of such calculations is rather surprising, at least in the case of the best known white dwarf, the companion of Sirius, for which one obtains $\mu = 1{\cdot}5$, which in turn means that about 50 per cent. of the material of this star is formed of hydrogen. If this hydrogen is assumed to be distributed more or less uniformly throughout the star, this result is contrary to what is expected from the evolutionary point of view, according to which the white dwarfs should represent the final stages of stellar evolution and therefore must be deprived completely of hydrogen, and it is in outright contradiction to our knowledge concerning the thermonuclear reactions. Indeed, since the internal temperatures of white dwarfs are known to be not very different from that of the sun, and since their central densities are several hundred thousand times larger, the energy-production due to the carbon cycle would exceed the energy-production of the sun by a factor of millions. Even if we suppose that, for some unknown reason, carbon and nitrogen are completely absent in the bodies of these stars, the $H+H \rightarrow H^2+e^+$ reaction alone would lead to an energy-production much higher than that in the sun!

A careful analysis of the temperature distribution in white dwarfs led Marshak‡ to the conclusion that the internal temperature of the companion of Sirius is $1{\cdot}5 \times 10^7$ °C. if it is formed entirely of Russell mixture of heavier elements (an unlikely assumption), and 7×10^6 °C. in the case when the star is made of pure helium. Even for the latter lower temperature, the hydrogen content may not exceed 0·1 per cent.

It is interesting to note that the conclusion concerning the negligible hydrogen content in the interior of white dwarfs can be reached independently of any knowledge of the temperature in the star if

† Cf. G. Gamow, *Phys. Rev.* **65** (1944), 20.
‡ R. E. Marshak, *Ap. J.* **92** (1940), 321.

the H—H-reaction is allowed. In fact, it can be shown that *even at zero temperature* hydrogen-containing materials, compressed to a density comparable to that in white dwarfs, will produce energy at a rate inconsistent with the low luminosities of these celestial bodies.† The point is that at these high densities the zero-point energy of protons is high enough to produce a high rate of mutual penetration leading to the formation of H^2-nuclei.

The difficulty presented by the companion of Sirius may be resolved by one of the following three assumptions:

1. The radius of the companion of Sirius is not 0·023 $R_\odot$ as deduced from observations, but rather only 0·008 $R_\odot$ which would follow from theoretical calculations for $M = 0{\cdot}98$ and $\bar{\mu}_c = 2$. It is hard to believe that the measurement of the radius of this star (based on the relativistic red shift of spectral lines) could be wrong by a factor of almost three.‡
2. The reaction $H^1 + H^1 \rightarrow H^2 + e^+$ is a strongly forbidden transformation such as would obtain if Fermi selection rules applied instead of the Gamow–Teller selection rules. We have seen, however, in Chapter IV, that the latter rule appears to be necessary to explain the β-transformations observed in the laboratory. Besides, this point of view would require also an explanation of the mysterious absence of carbon and nitrogen from these stars.
3. The hydrogen is not uniformly mixed, but occurs only on the surface and is completely absent from the hot interior.§ This point of view is subject to further observational confirmation.

Other, less likely, explanations of the discrepancy include the assumption that the companion of Sirius is formed almost entirely of He^3 which possesses the proper mean molecular weight, $\bar{\mu}_c = 1{\cdot}5$.‖ However, whereas comparatively large concentrations of He^3 might be expected in stars that obtained their energy from the H—H reaction and burned up their hydrogen supply in less than 3×10^7 years (the period of the $He^3 + He^4 \rightarrow Be^7 + \gamma$ reaction), it is hard to see how

† D. S. Kothari, *Nature* **142** (1938), 916; W. A. Wildhack, *Phys. Rev.* **57** (1940), 81.

‡ Nevertheless this explanation has been proved in 1948 to be correct. The point is that the spectrum of Sirius B is strongly blended with that of the main star, which affects the apparent positions of its absorbtion lines. If allowance is made for this blending, one arrives at the value of radius consistent with the degenerated model with zero hydrogen content.

§ C. L. Critchfield, *Ap. J.* **96** (1942), 1.

‖ G. Chertok, *Phys. Rev.* **65** (1944), 51.

any large amount of these nuclei could be produced in stars of solar, or larger, mass which are known to live almost exclusively on the carbon cycle. An alternative explanation lies in changing the fundamental relation, $P = f(\rho)$, for the degenerate electron gas as suggested, for example, by Eddington.† According to his theory, the pressure of a degenerate electron gas is expressed by the formula (41) no matter how high the density may be. For $M = 0{\cdot}98$ and $\bar{\mu}_e = 2$ (no hydrogen) Eddington finds that (41) stands in a satisfactory agreement with the observed radius of the companion of Sirius. It would be, however, a very hard decision to make to accept all the revolutionary changes proposed by Eddington in the present quantum theory solely on the basis of the white dwarf trouble.

It may be remarked in conclusion that the difficulty arising in the case of the companion of Sirius does not exist for the other white dwarf, 40 Eridani B, for which the values of M and R are reasonably well known. In this case, application of Chandrasekhar's results leads to a value $\bar{\mu}_e = 2$, corresponding to complete absence of hydrogen. Unfortunately we lack sufficiently exact knowledge of the M's and R's for other known white dwarfs which could help us settle the existing conflict with theory.

5. Urca-process and supernovae

In the preceding sections we have been considering thermonuclear reactions taking place between two different (or identical) nuclear species in material that is subject to sufficiently high temperatures. We shall now turn our attention to reactions between stable nuclei and the fast-moving particles of the electron gas. When the energy of an incident electron is sufficiently high, it can attach itself to a nucleus by ejecting a neutrino and forming the unstable (with respect to β-transformation) isobar of the preceding element:

$$X_Z^A + e^- \rightarrow Y_{Z-1}^A + \text{neutrino}. \qquad (45\,a)$$

Obviously the threshold of this reaction (corresponding to zero energy for the emitted neutrino) is given by the upper limit of the continuous β-spectrum in the spontaneous β-transformation:

$$Y_{Z-1}^A \rightarrow X_Z^A + e^- + \text{anti-neutrino} \qquad (45\,b)$$

of the product nucleus, if there is no γ-ray in the latter.

† A. Eddington, *Fundamental Theory*, Cambridge University Press, 1947, and previous publications.

If we should heat up the element X in an enclosure the walls of which are impenetrable by *any* of the particles participating in the reaction, there would exist for each value of the temperature a dynamic equilibrium between processes (45 *a*) and (45 *b*) with definite percentages of X- and Y-nuclei, electrons, neutrinos, and anti-neutrinos. Since, however, no walls, not even as thick as stellar bodies, can stop the neutrinos, they will escape continuously into the surrounding space and carry with them a certain amount of kinetic energy. As a result, the electron gas within the enclosure will cool down rapidly, and heat that may be flowing from outside into the region where this process is taking place will disappear without trace (so far as direct observation goes), being dissipated by the escaping stream of neutrinos. This type of process was first studied by Gamow and Schoenberg,† who called it the *Urca-process* because of the evident similarity between the traceless disappearance of heat by means of this nuclear reaction and the fate of the gamblers' money in the crowded playrooms of the famous Casino da Urca in Rio de Janiero.

To get a rough idea about the temperatures necessary for the Urca-process, and about the corresponding rates of energy-loss through neutrino emission, we must remember that, although in principle this process will take place at any temperature due to the fast electrons in the tail of the Maxwell distribution, it will run at full speed only when the mean kinetic energy of thermal motion, $\frac{3}{2}kT$, approaches the reaction-threshold, $W^{\max}$. Under these circumstances the atomic concentrations of elements X and Y become comparable to one another, so that the rate of energy-loss through neutrino emission will be of the order of magnitude:

$$W \simeq \tfrac{2}{3} Q\lambda \frac{c_Z}{AM}, \tag{46}$$

where Q is the energy of transformation and λ is the decay-constant of the process (45 *b*).

Let us consider three typical examples of Urca-processes. We take first the reactions:

$$\begin{aligned} &\mathrm{He}^3 + e^- \rightarrow \mathrm{H}^3 + \text{neutrino}, \\ &\mathrm{H}^3 \rightarrow \mathrm{He}^3 + e^- + \text{anti-neutrino}. \end{aligned} \tag{47}$$

The decay-constant of H^3 seems to be about $7 \cdot 10^{-10}$ (mean life of

† G. Gamow and M. Schoenberg, *Phys. Rev.* **59** (1941), 539.

about 30 years), whereas $W^{\max} = 15 \pm 3$ k.e.v. The energy 15 k.e.v. corresponds to a temperature of $1{\cdot}2 \times 10^8$ °C., i.e. only 6 times higher than that in the centre of the sun. According to (46) the rate of energy-loss at this temperature is $W = 2 \times 10^6 c_{He^3}$ erg/g.sec.

As our second example we choose the reactions:

$$\begin{aligned} &Fe^{56} + e^- \rightarrow Mn^{56} + \text{neutrino} \\ &Mn^{56} \rightarrow Fe^{56} + e^- + \text{anti-neutrino.} \end{aligned} \tag{48}$$

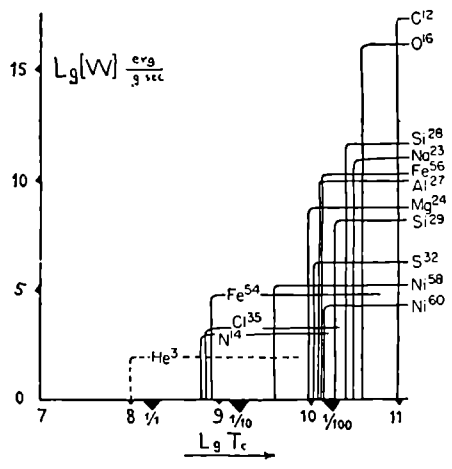


FIG. 55. A diagram showing relative importance of different elements for the energy-losses through neutrino emission. The fractions given on the abscissa axis represent the contraction of stellar radius [for $M = 5M(\odot)$] for which the corresponding central temperatures are attained.

The maximum energy of the β-spectrum, $W^{\max}$, for Mn^{56} is known to be 1·7 M.e.v., so that the temperature needed for this reaction is about 10^{10} °C. Since, on the other hand, the Fe^{56}-nuclei (the most abundant iron isotope) form about 13 per cent. of the Russell mixture or (assuming $c_{RM} = 0{\cdot}03$) 0·4 per cent. of the stellar material and, since in this case $\lambda = 7{\cdot}7 \times 10^{-5}$ sec.$^{-1}$ (mean life 2·5 hours) we have

$$W = 2 \times 10^{10} \text{ erg/g.sec.}$$

Finally, let us consider the reactions:

$$\begin{aligned} &O^{16} + e^- \rightarrow N^{16} + \text{neutrino,} \\ &N^{16} \rightarrow O^{16} + e^- + \text{antineutrino.} \end{aligned} \tag{49}$$

Here $W^{\max} = 6$ M.e.v., $\lambda = 7{\cdot}7 \times 10^{-2}$ sec.$^{-1}$ (mean life 9 sec.) and $c_0 = 0{\cdot}015$. This gives for the necessary temperature the value 4×10^{10} ° C., and for $W = 10^{15}$ erg/g.sec.

Since every isotope of every chemical element gives rise to an Urca-process, we have a large choice of different possibilities the relative importance of which is determined by the corresponding values of $W^{\max}$, λ, and c. In Fig. 55 we give a schematic presentation of different possible processes of that kind, showing the temperatures necessary and the corresponding rates of energy-loss (calculated for $c_{RM} = 0{\cdot}03$).

The exact expression for the rate of the Urca-process at different temperatures can be obtained† by using the regular theory of

† G. Gamow and M. Schoenberg, loc. cit.

β-transformation described in Chapter V. Consider a unit mass of stellar material containing n_e free electrons and n_z^0 nuclei of atomic number Z which can capture these electrons and go over into unstable nuclei of atomic number $Z-1$. Since the material is completely ionized, we have

$$n_e = \frac{Z\rho}{AM} \simeq \frac{\rho}{2M}. \tag{50}$$

Similarly we have

$$n_z^0 = \frac{c_Z\,\rho}{AM}. \tag{51}$$

If the electron gas is not degenerate (compare the discussion at the end of this section) and if the temperature is not high enough for the thermal electrons to have relativistic velocities, the number of electrons with energy between E and $E+dE$ is given by

$$n_e(E)\,dE = 2\pi^{-\frac{1}{2}} n_e (kT)^{-\frac{3}{2}} e^{-E/kT} E^{\frac{1}{2}}\,dE. \tag{52}$$

The mean life of an electron against capture by the nuclei through emission of a neutrino can be calculated to be

$$\tau(E) = \pi \log 2\,.\,\hbar^4 c^3/[g^2 n_Z (E - W^{\max})^2], \tag{53}$$

where g is Fermi's constant. The number of electrons captured by the nuclei in question, per unit volume per unit time, is:

$$N_- = \int_{W^{\max}}^{\infty} \frac{n_e(E)}{\tau(E)}\,dE. \tag{54}$$

Using (52) and (53) we get:

$$N_- = \frac{2g^2 n_Z\, n_e}{\pi^{\frac{3}{2}} \log 2 c^3 \hbar^4} (kT)^{-\frac{3}{2}} \int_{W^{\max}}^{\infty} e^{-E/kT} (E - W^{\max})^2 E^{\frac{1}{2}}\,dE, \tag{55}$$

or

$$N_- = \frac{2g^2 n_Z\, n_e}{\pi^{\frac{3}{2}} \log 2 c^3 \hbar^4} (kT)^2 J\left(\frac{W^{\max}}{kT}\right), \tag{56}$$

where the integral

$$J(x_0) = \int_{x_0}^{\infty} e^{-x} (x - x_0)^2 x^{\frac{1}{2}}\,dx \tag{57}$$

is written with respect to the arguments:

$$x = \frac{E}{kT}, \qquad x_0 = \frac{W^{\max}}{kT}. \tag{58}$$

Estimating this integral by the saddle-point method we get:

$$J(x_0) = 2{\cdot}15 (x_0 + \tfrac{5}{2})^{\frac{1}{2}} e^{-x_0}. \tag{59}$$

On the other hand, the number of electrons emitted by unstable nuclei into the energy-interval $E, E+dE$ is, according to the β-transformation formula (for non-relativistic energies):

$$dN^+ = \frac{g^2 m^{\frac{3}{2}} n_{Z-1}}{\sqrt{2}\,\pi^3 \hbar^7 c^3} (W^{\max} - E)^2 E^{\frac{1}{2}}\, dE, \tag{60}$$

and the total number of electrons per second per unit volume is:

$$N^+ = 0{\cdot}152 \frac{g^2 m^{\frac{3}{2}} n_{Z-1}}{\sqrt{2}\,\pi^3 \hbar^3 c^3} (W^{\max})^{\frac{7}{2}}. \tag{61}$$

In the case of equilibrium $N^+ = N_-$ which gives us:

$$\frac{n_{Z-1}}{n_Z} = \left[\frac{2g^2 n_e}{\pi^{\frac{3}{2}} \log 2 c^3 \hbar^7} (kT)^2 J\left(\frac{W^{\max}}{kT}\right)\right] \cdot \left[0{\cdot}152 \frac{g^2 m^{\frac{3}{2}}}{\sqrt{2}\,\pi^3 \hbar^7 c^3} (W^{\max})^{\frac{7}{2}}\right]^{-1}, \tag{62}$$

or since $n_{Z-1} + n_Z = n_Z^0$

$$n_{Z-1} = \left[1 + 6{\cdot}4 \times 10^{-3} \hbar^{-3} m^{\frac{3}{2}} (kT)^{-2} M \rho^{-1} \left(\frac{W^{\max}}{kT} + \frac{5}{2}\right)^{-\frac{1}{2}} (W^{\max})^{\frac{7}{2}} e^{W^{\max}/kT}\right]^{-1} \frac{c_Z \rho}{AM} \tag{63}$$

and a similar expression for n_Z. The total energy taken away by neutrinos ejected in the process of electron capture, per unit time per unit volume, can be calculated easily and is found to be:

$$W^{(1)} = \frac{8{\cdot}5 g^2 n_e n_Z}{\pi \log 2 \hbar^4 c^3} \left[\frac{W^{\max}}{kT} + \frac{7}{2}\right] (kT)^3 e^{-W^{\max}/kT}, \tag{64}$$

while the energy of anti-neutrinos accompanying electron-emissions is

$$W^{(2)} = \tfrac{2}{3} Q \lambda n_{Z-1}. \tag{65}$$

Comparing (63) and (64), and bearing in mind the expression for (65) we get:

$$W^{(1)} = 5{\cdot}5 \left(\frac{kT}{W^{\max}}\right) W^{(2)}, \tag{66}$$

so that for the total rate of energy-loss we obtain finally:

$$W = W^{(1)} + W^{(2)} \simeq \left[1 + 5{\cdot}5 \frac{kT}{W^{\max}}\right] \frac{3}{2} W^{\max} \lambda n_{Z-1}. \tag{67}$$

In the case when, due to very high temperature, the particles of the

electron gas have to be treated relativistically, and when also, $W^{\max} \gg mc^2$, similar calculations give:

$$n_{Z-1} = \left\{1+1{\cdot}2\times10^{-3}\hbar^{-3}c^{-3}M\rho^{-1}[12(kT)^2+6W^{\max}kT+(W^{\max})^2]\times\right.$$
$$\left.\times\left[1+\left(1+\frac{m_0c^2}{kT}\right)^2\right](W^{\max})^5e^{-W^{\max}/kT}\right\}^{-1}\frac{c_Z\rho}{AM} \quad (68)$$

and
$$W = \left[1+\frac{8kT}{W^{\max}}\right]\frac{W^{\max}}{2}\lambda n_{Z-1}. \quad (69)$$

Finally for the mixed case when we have a non-relativistic electron gas and a relativistic β-emission, we obtain

$$n_{Z-1} = [1+0{\cdot}98.10^{-3}\hbar^{-3}c^{-3}M\rho^{-1}(W^{\max})^{\frac{5}{2}}(kT)^{-\frac{3}{2}}e^{Q/kT}]^{-1}\frac{c_Z\rho}{M} \quad (70)$$

and
$$W \simeq \left[1+7{\cdot}1\frac{kT}{W^{\max}}\right]\tfrac{1}{2}n_{Z-1}\lambda W^{\max}. \quad (71)$$

Considering now the implications of the process described above in the theory of stellar energy and evolution, we must note first of all that it cannot be expected to play any important role in stars producing their energy by regular thermonuclear reactions. Thus, for example, the Urca-process in He^3 will proceed at a temperature of 2×10^{7}° C. at the approximate rate of

$$2\times10^{6}e^{-6}\times c_{He^3}\frac{\text{erg}}{\text{g.sec}} = 5{\cdot}10^{3}c_{He^3}\frac{\text{erg}}{\text{g.sec}}.$$

Since, according to the figures given in (29), the concentration of He^3 in stellar matter is expected to be

$$c_{He^3} = c_H\frac{3\times10^{7}}{1{\cdot}2\times10^{11}} = 10^{-5},$$

the energy-losses through neutrino emission will amount in this case to only $\simeq 10^{-2}$ erg/g.sec. Even if there exists an isotope of some element with an 'Urca temperature' as low as 2×10^{7}° C.

$$(W^{\max} = 2{\cdot}5 \text{ k.e.v.})$$

the corresponding decay-constant must be

$$\lambda = 7\times10^{-10}(\tfrac{1}{6})^{\frac{5}{2}}\text{ sec.}^{-1} \simeq 10^{-11}\text{ sec.}^{-1},$$

and the rate of energy-loss at 2×10^{7}° C. is of the order of magnitude $10^{3}c$ erg/g.sec. Since the concentration c of such an element would be probably rather small, and in any case smaller than the assumed

concentration (0·03) of the Russell mixture, the effect will be again negligible as compared with the rate of energy-production.

On the other hand, Urca-processes should be expected to play an important role in the later stages of stellar evolution, when, being ultimately deprived of its hydrogen content, a star begins to contract under the forces of gravity. It is easy to show, on the basis of the equilibrium equations corresponding to this case, that the central

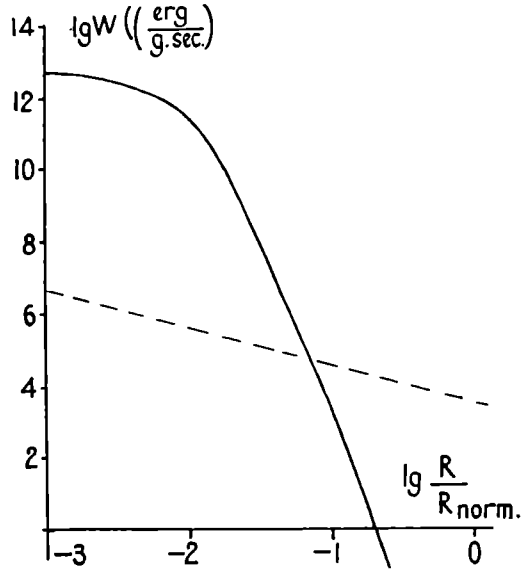


Fig. 56.

temperature of a gravitationally contracting star increases inversely proportionally to its radius, and that at a certain stage of contraction it is bound to attain the values necessary for Urca-processes in such elements as iron or nitrogen. In Fig. 56 we give, as an example, the rate of energy-loss by Fe-Urca-process in a contracting star of 5 solar masses. These data are calculated on the basis of the above-derived formulae (with the assumption $c_{\mathrm{Fe}} = 0{\cdot}4$ per cent.) and are plotted against the radius of the contracting star (relative to its radius on the main sequence). The broken line represents the liberation of gravitational energy (luminosity) for a contracting star calculated according to the standard stellar model. We see that for a contraction by a factor of about ten the energy-loss through neutrino-emission (in Fe alone) becomes larger than the energy-loss by radiation from the stellar surface, and that at hundred-fold contraction the former loss exceeds the latter by a factor 10^5. It follows that the

later stages of stellar evolution (contraction) must be considerably accelerated due to various Urca-processes that start in the star's interior.

If the increased rate of contraction caused by the Urca-processes becomes comparable to the rate of 'free-fall' of the stellar material towards the centre, we may expect the process to end in a catastrophic collapse of the star. It is easy to calculate that the time, Δt, of a free-fall collapse of a star of mass M_0 and initial radius R_0 is given by:

$$\Delta t \simeq \frac{R_0^{\frac{3}{2}}}{(GM_0)^{\frac{1}{2}}} \simeq \frac{1}{(G\bar{\rho})^{\frac{1}{2}}}. \tag{72}$$

For a star of mean density $\bar{\rho} = 100$ g./cm.^{3}, this gives $\Delta t \simeq 10^3$ sec. (i.e. about half an hour). On the other hand, the total gravitational energy liberated in a contraction to one half of the original radius is given by:

$$\Delta U \simeq GM_0^2 \frac{1}{2R_0} \simeq GM^{\frac{5}{3}}\bar{\rho}^{\frac{1}{3}}. \tag{73}$$

For a star of, say, 5 solar masses and an initial density 100 g./cm.^{3}, this gives 10^{50} erg. Thus the rate of energy-loss necessary for such a rapid collapse must be

$$\frac{10^{50}}{10^{34} \times 10^3} = 10^{13} \frac{\text{erg}}{\text{g.sec.}},$$

and is not at all inconsistent with the rates of energy-loss by Urca-processes discussed above. These considerations lead us to a possible explanation of the vast stellar explosions known as *supernova*. In these cosmic catastrophes a star increases its luminosity practically overnight by a factor of the order of 10^9, and this is followed by a gradual decrease of luminosity lasting for a period of a year or more. The total energy liberated in such explosions is of the order of magnitude 10^{50} erg, i.e. about equal to the total content of gravitational energy in the body of the star. Spectroscopic studies of supernovae indicate that these explosions are characterized by the ejection of hot masses moving at very high speed. Later, these ejected masses form a giant expanding nebulosity surrounding the remainder of the exploded star and is illuminated by its light. Thus, for example, the historically famous supernova recorded by Chinese astronomers in the year A.D. 1054 gave rise to a large and somewhat irregular mass of gas which is known at present as the *Crab Nebula*. This gaseous mass is expanding still, at a rate of 0·23″ per year (1,116 km./sec.)

which is in good agreement with its age (of about 900 years) and its present diameter. In the centre of this nebulosity one can observe a comparatively faint, but very hot, star† which presumably represents a degenerate configuration of the type discussed in the preceding section. However, this body is much hotter than the regular white dwarfs, its surface temperature being estimated at about 500,000° C. It is of interest to note also that, whereas the mass of the central star is presumably comparable to the mass of the sun, the mass of the expanding nebulosity is of the order of 10 solar masses.‡

On the basis of this information, we can form a rather convincing picture of what happens during stellar explosions of that type. Presumably we see here the result of some internal instability of the stellar structure which causes a rapid gravitational collapse of the entire stellar body (as evidenced by the amount of energy liberated). A part of the material of the original star concentrates near the centre while another, larger, part is ejected into the surrounding space. The observed increase of luminosity must be simply due to the fact that the gas masses ejected from the interior of the star are, so to speak, saturated with radiation which then becomes free to stream away. This phenomenon of rapid collapse is just what one would expect as the result of a sufficiently fast Urca-process being switched on in the late stages of contractive evolution. In fact, when the rate of energy-loss through neutrino-emission becomes comparable to the rate of energy-removal necessary for 'free-fall', the previously regular contraction must be expected to give way to an irregular collapse of the type described above. In particular, one would expect that, due to the axial rotation which becomes of particular importance in the later stages of contraction, the material of the polar regions will fall in towards the centre, whereas the equatorial material (being helped by the centrifugal force) will be thrown out into space. This last conclusion is in agreement with the observational evidence that the gaseous nebula ejected in the process of explosion seems to possess an axial symmetry. However, more study is needed to check the correctness of the point of view described above.

6. The origin of the chemical elements

We shall now turn our attention to the exciting problem concerned with the origin of the chemical elements, and try to

† W. Baade, *Ap. J.* **96** (1942), 188. ‡ R. Minkowski, ibid. p. 199.

understand the reasons for the widely varying proportions in which different nuclear species are encountered in nature. One of the most striking features of the chemical composition of the universe is its high degree of uniformity. In fact, careful chemical and spectroscopic studies of the composition of the earth, the meteorites, the sun, the stars, interstellar gas, and the distant galaxies indicate that, apart from some exceptions which can be explained in most cases by local conditions, the relative numbers of various nuclear species are very nearly the same everywhere in the universe. Notable among the local deviations mentioned above is the extremely low terrestrial abundance of hydrogen and helium which are otherwise the two most abundant elements in the universe (cf. p. 305). The almost complete absence of hydrogen, helium, and other rare gases from our planet (and presumably from all minor planets) can be explained easily by the nature of the processes which were responsible for the formation of the planetary system.†

Another example of local deviations is supplied by the observed differences between the relative carbon and nitrogen contents in the atmospheres of the hot Wolf–Rayet stars (the so-called CW- and NW-stars) which can be interpreted, presumably, as a result of the thermonuclear reactions (C–N cycle) responsible for their energy-production.‡ Some differences have been observed also in the relative abundances of the various isotopes of the same element. Thus, for example, helium samples obtained from atmospheric air were found to contain a larger proportion of the He^3-isotope than samples extracted from rocks.§ This is probably due to the reaction

$$N^{14}+n \rightarrow 3\ He^4+H^3$$

induced in atmospheric nitrogen by fast cosmic-ray neutrons, and subsequent decay of the H^3 into He^3. It remains to be seen whether the observed differences between H^1/H^2-ratios in the solar atmosphere, and in terrestrial hydrogen, and the apparent excess of the C^{13}-isotope in the atmospheres of certain cool stars, can be explained on a similar basis.

† Compare the new theory of the origin of the planetary system by C. F. v. Weizsäcker, *Zs. f. Astrophys.* **22** (1944), 319. This theory is reviewed in English by G. Gamow and J. H. Hyneck, *Ap. J.* **101** (1945), 249, and by S. Chandrasekhar, *Rev. Mod. Phys.* **18** (1946), 94.

‡ G. Gamow, *Ap. J.* **98** (1943), 500.

§ L. W. Alvarez and R. Cornog, *Phys. Rev.* **56** (1939), 379, 613; L. T. Aldrich and A. O. Nier, ibid. **70** (1946), 983.

Neglecting these, in most cases minor, deviations we can say that the chemical and isotopic constitution of the universe is remarkably constant. In Fig. 57 is presented a curve based on the work of Goldschmidt,† who has undertaken a most careful and painstaking analysis of all existing geochemical and astrochemical data. The most characteristic feature of the observed abundance is that, apart from more or less significant ups and downs, the curve shows a rapid exponential decrease of abundances for elements of the lighter atomic weights followed by an almost constant value for the heavier elements. It is difficult to escape the conviction that this striking difference between the two parts of the abundance curve is somehow characteristic of the processes which were responsible for the origin of the elements. On the other hand, the smaller details of the empirical curve must be related more or less directly to the individual stability of the various nuclear species, and may not be valid criteria for the validity of different theories of the formation of the atomic nuclei.

Before we embark on discussions concerning the exact nature of the processes responsible for the formation of various nuclear species, we may ask ourselves whether the observed relative abundance of elements require an explanation at all. In fact, if we should limit our considerations to stable nuclei, and accept the hypothesis of a permanent, non-evolving universe, the question of relative abundances could be answered by a dull but perfectly logical statement to the effect that some elements are very abundant while others are very rare 'because it was always that way'. However, the existence of unstable, gradually decaying elements forces us to look more deeply into the existing state of affairs and to assume that, at least, the nuclei of the natural radioactive substances *were formed 'once upon a time'* in the history of the universe. We can form some idea about the epoch during which the formation of these radioactive nuclei must have taken place from a study of their present relative abundance relative to stable nuclear species of comparable atomic weight. Thus, the fact that atoms of Th^{232} and U^{238}, with mean lifetimes of 18×10^9 and $4{\cdot}4\times10^9$ years, respectively, are about as abundant as the stable atoms of Bi or Hg, indicates that their formation must have taken place not much earlier than a few thousand million years ago. On the other hand, the comparative rarity of

† V. M. Goldschmidt, *Verteilungsgesetze der Elemente*, Oslo, 1938.

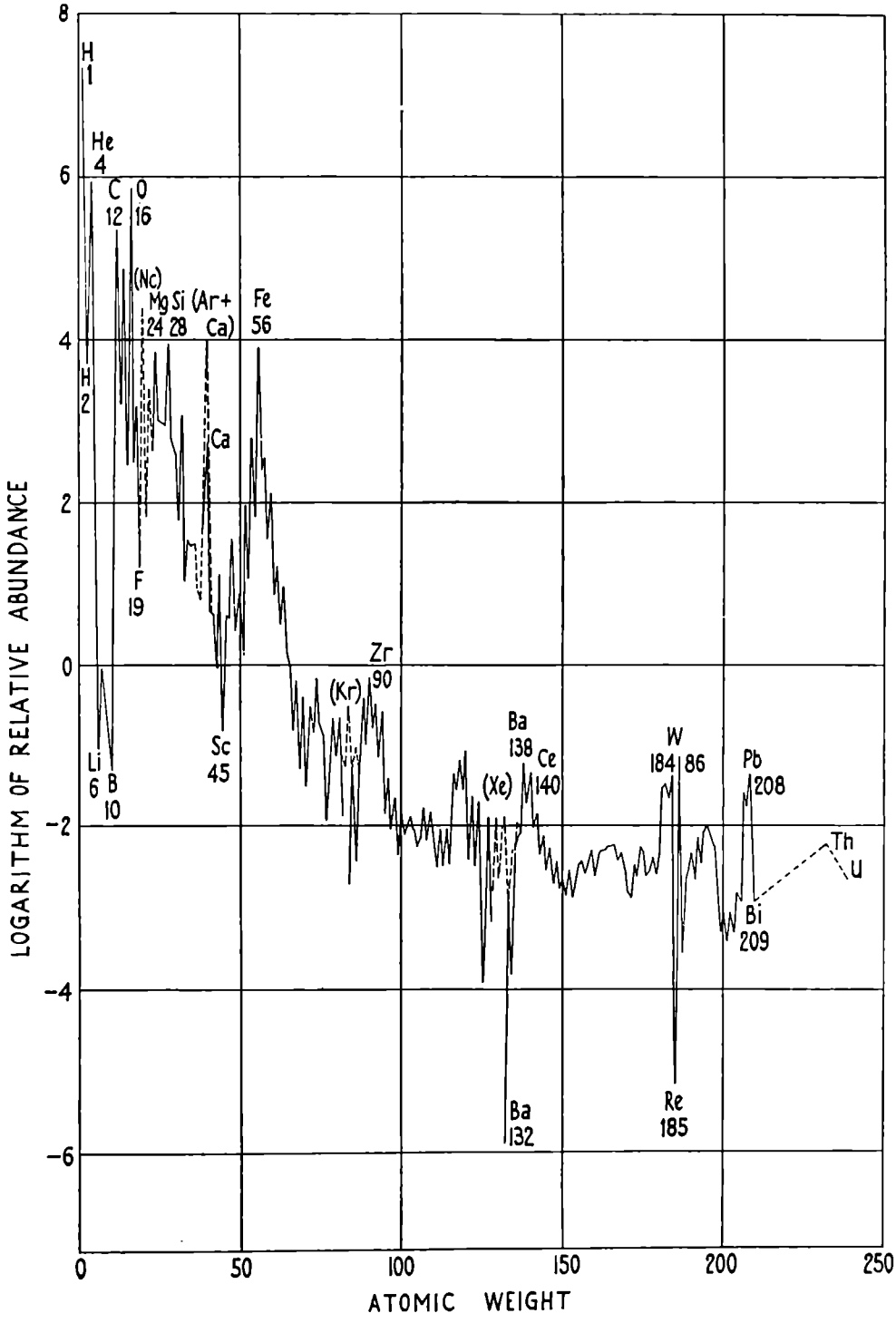


FIG. 57. Abundance of elements in the universe.

Actinouranium (U^{235}) and K^{40}, with mean lives of 7×10^8 and 4×10^8 years, respectively, as well as the almost complete† absence of Np^{237} and Pu^{239} (half-lives of $2{\cdot}25\times10^6$ and $2{\cdot}4\times10^4$ years), strongly suggests that the formation of these nuclei must have taken place at least a few thousand million years ago, thus giving them a chance to decay to a considerable extent.‡ Assuming for a moment that U^{235} and K^{40} were formed originally with abundances comparable to those corresponding to the main isotopes, and remembering that their present relative abundances are 7×10^{-3} and $2{\cdot}5\times10^{-5}$, we find that they must have been formed $3{\cdot}5\times10^9$ and $4{\cdot}5\times10^9$ years ago. During the same period of time the abundance of U^{238} would have decreased only by a factor two or three, whereas the abundance of Th^{232} would have changed hardly at all. Considerations of this kind lead us to the conclusion that *all radioactive nuclei found in nature must have been formed during a certain epoch a few thousand million years ago*, and it is quite natural to assume that the present proportions among various stable nuclei were established during the same epoch.

Since the possibility of thermonuclear reactions throughout the entire natural sequence of elements requires temperatures that are much higher than those encountered in the central regions of the stars, we must conclude that at the epoch when the elements were formed the thermal conditions in the universe were considerably more severe than they are at present. Furthermore, the striking uniformity of the chemical composition of the universe suggests strongly that these conditions of extremely high temperature must have been rather common for all points and masses of the universe.

The above conclusions concerning the past, based entirely on arguments pertaining to abundances of unstable nuclei in nature, find beautiful confirmation in the astronomical studies of the universe. In fact, the observation of the red-shift in the spectra of the distant galaxies§ can be interpreted in hardly any other way than by the assumption that our universe is at present in a state of rapid,

† Minute amounts of Pu^{239} have been found in uranium ores (10^{-14} with respect to uranium), apparently due to continuous production by neutrons arising from spontaneous uranium fission.

‡ The negligibly small amounts of these elements found by Seaborg and Perlman in natural uranium ores are due to continuous production from U^{238} by bombardment with cosmic-ray neutrons.

§ E. Hubble, *Proc. Nat. Acad.* **15** (1929), 168.

uniform expansion. Assuming as a first approximation that the rate of expansion has remained constant and equal to its present value of 560 km./sec. per megaparsec ($= 1{\cdot}8\times10^{-17}$ sec.$^{-1}$), we find that the expansion starts from a very dense phase about $5{\cdot}5\times10^{16}$ sec. or $1{\cdot}8\times10^{9}$ years ago. This figure for the age of the universe is in reasonable agreement with the results of other estimates based on the evolutionary features of massive stars, the rate of dissipation of stellar clusters, and the lead-uranium-ratio in the oldest volcanic rocks on our planet. It is about twice as long as the geological estimates of the age of the oceans (from salt-content, and sedimentation process) which evidently represent somewhat later formations. The astronomical picture of a highly compressed universe in the early stages of its expansion is exactly what one needs for the development of a theory for the origin of the elements, and the date of this epoch as given by purely astronomical evidence is in reasonable agreement with the figure derived above for the age of the radioactive elements. We should expect that in this highly compressed initial state the temperature of the material was very high. Unfortunately we cannot get a very exact idea about the mean temperature of the universe at different stages of its expansion by applying, for example, the temperature-density relation for the adiabatic expansion of a gas and comparing with the present heat content of the universe. The point is that a large part of the present heat content in the universe is due to various nuclear transformations in the interior of the stars, and there is no simple way of separating it from the heat that could have been left from the original compressed state. One can make, however, a simple estimate of the lengths of time which must have been taken by the universe in passing through different successive stages of expansion.† According to the general theory of relativity‡ the time-dependence of any linear dimension l in the expanding universe is given by the formula:

$$\frac{dl}{dt} = \sqrt{\left(\frac{8\pi G}{3}\rho l^2 - \frac{c^2 l^2}{R^2}\right)}, \tag{74}$$

where G is Newton's constant of gravitation, ρ the mean density and R the curvature of space (real for closed spaces; imaginary for open

† G. Gamow, *Phys. Rev.* **70** (1946), 572.

‡ Compare R. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford University Press, 1934).

spaces). It may be noted that the expression (74) represents a relativistic analogue of the simple classical formula

$$v = \sqrt{\left(2\frac{4\pi l^3}{3}\rho\frac{G}{l} - 2E\right)} \tag{75}$$

for the velocity for inertial expansion of a gravitating sphere of dust with the total energy E per unit mass. To use definite numbers, let us consider in the present state of the universe (assumed to be quite uniform) a cube containing 1 gramme of matter. Since the present mean density of the universe is $\rho_{\text{present}} \simeq 10^{-30}$ g./cm.^{3}, the side of our cube will be $l_{\text{present}} \simeq 10^{10}$ cm. Using the value given above for the rate of expansion ($1{\cdot}8\times10^{-17}$ sec.$^{-1}$) we find that

$$(dl/dt)_{\text{present}} = 1{\cdot}8\times10^{-7} \text{ cm./sec.}$$

Substituting numerical values we obtain

$$1{\cdot}8\times10^{-7} = \sqrt{\left(5{\cdot}7\times10^{-17} - \frac{c^2 l^2_{\text{present}}}{R^2}\right)}, \tag{76}$$

showing that at the present stage of expansion the first term under the radical (corresponding to gravitational potential energy) is negligibly small compared with the second one. For a numerical value of the (constant) curvature we obtain

$$R = 1{\cdot}7\times10^{27}\sqrt{(-1)} \text{ cm.} = 2\times10^9 \text{ imaginary light years.}$$

The formula (74) gives us also a more exact value for the beginning of the expansion. Writing $\rho = 1/l^3$ (according to the definition of l given above) and using for brevity, $A = cl_{\text{present}}/R$, we can integrate (74) from $t = 0$, $l = 0$ to $t = t_{\text{present}}$; $l = l_{\text{present}}$ obtaining

$$t_{\text{pres}} = \frac{l_{\text{pres}}(dl/dt)_{\text{pres}}}{A^2} - \frac{4\pi G}{3A^3}\left\{\ln\left[\frac{4\pi G}{3A} + l_{\text{pres}}\{A + (dl/dp)_{\text{pres}}\}\right] - \ln\frac{4\pi G}{3A}\right\}. \tag{77}$$

The first term in (77) corresponds to a linear expansion and the second to the gravitational deceleration at the beginning of the expansion. Numerically we get: $t_{\text{present}} = [1{\cdot}8\times10^9 - 2{\cdot}2\times10^7]$ years, which indicates that by taking into account the effect of gravitational deceleration we change the estimated age of the universe by only 1 per cent.

We can now apply our formula to estimate the time intervals required by the universe to pass through different stages of high density. For $\rho > 10^{-21}$ g./cm.^{3} the first term under the radical in

(74) is of primary importance and deductions about the rates of expansion become independent of the value of R. Using this formula, we easily find that the times when the mean density of the universe was 10^9, 10^6, 10^3, and 1 g./cm.^{3}, its density was dropping by an order of magnitude in time intervals of 0·04 sec., 1 sec., $\frac{1}{2}$ minute, and $\frac{1}{2}$ hour, respectively. These high rates of expansion are simply due to the fact that, in the early stages of expansion, the mass of the universe had to have extremely high velocities of recession in order to overcome the forces of mutual gravitational attraction.

The considerations above lead us to the important conclusion that *the early stage of expansion of the universe, characterized by very high densities and presumably correspondingly high temperatures suitable for the formation of the whole variety of nuclear species, could have lasted for only a very short period of time.*

The earliest attempts to give a theoretical explanation of the observed abundance-curve of the chemical elements were based on *the hypothesis of dynamical equilibrium* between the various nuclear species at a certain high density and temperature. Calculations of that kind have been carried out by Stern,† Weizsäcker,‡ Chandrasekhar and Henrich,§ Lattes and Wataghin,‖ and by Klein, Beskow, and Treffenberg.†† One can get a preliminary idea concerning the temperatures and densities which could have been responsible for the establishment of the present nuclear proportions by studying the percentage abundances of several isotopes of a given element (Weizsäcker, loc. cit.). In fact, let us consider the equations of equilibrium:

$$\begin{gathered} {}_ZX^A + {}_0n^1 \rightleftarrows {}_ZX^{A+1} \\ {}_ZX^{A+1} + {}_0n^1 \rightleftarrows {}_ZX^{A+2} \end{gathered} \tag{78}$$

between a neutron gas (which must have been present constantly at sufficiently high temperatures because of the neutron-evaporation) and the three isotopes A, $A+1$, and $A+2$ of some element of atomic number Z. According to the well-known formula of statistical equilibrium we can write:

$$\frac{n_z^A n_n}{n_z^{A+1}} = 2\frac{g_z^A}{g_z^{A+1}}\left(\frac{A}{A+1}\right)^{\frac{3}{2}}\frac{(2\pi MkT)^{\frac{3}{2}}}{h^3}e^{-E_{A+1}/kT} \tag{79}$$

† T. E. Stern, *M.N.* **93** (1933), 736.

‡ C. F. v. Weizsäcker, *Phys. Zs.* **39** (1938), 133.

§ S. Chandrasekhar and L. R. Henrich, *Ap. J.* **95** (1942), 288.

‖ C. Lattes and G. Wataghin, *Phys. Rev.* **69** (1946), 237.

†† O. Klein, G. Beskow, and L. Treffenberg, *Arkiv f. Math. Astr. och Fys.* **33B** (1946), 1.

and
$$\frac{n_Z^{A+1} n_n}{n_Z^{A+2}} = 2\frac{g_Z^{A+1}}{g_Z^{A+2}}\left(\frac{A+1}{A+2}\right)^{\frac{3}{2}}\frac{(2\pi MkT)^{\frac{3}{2}}}{h^3} e^{-E_{A+2}/kT}, \tag{80}$$

where n_Z^A, n_Z^{A-1}, n_Z^{A-2}, and n_n are the number-densities of the three isotopic nuclei and the neutrons, the g's represent statistical weights, E_{A-1} and E_{A-2} are the binding energies of the last neutron in the corresponding nuclei, and T is the absolute temperature of the mixture. Eliminating n_n from (79) and (80) we obtain:

$$kT = \frac{E_{A+2} - E_{A+1}}{\log_e\left[\frac{n_Z^A n_Z^{A+2}}{(n_Z^{A+1})^2}\frac{(g_Z^{A+1})^2}{g_Z^A g_Z^{A+2}}\left(\frac{[A+1]^2}{A[A+2]}\right)^{\frac{3}{2}}\right]}. \tag{81}$$

Then from observed abundance ratios, a value of n_n can be found. The results of such calculations carried out for different stable isotopic triples (Chandrasekhar and Henrich, loc. cit.) are shown in

TABLE XXV

Isotopes	*Obs. abundance*	*T° K. (calc.)*	*Calculated neutron density (g.cm.^{3})*
O^{16}	99·77		
O^{17}	0·04	$4{\cdot}2\ 10^9$	$5\ 10^2$
O^{18}	0·20		
Ne^{20}	90·00		
Ne^{21}	0·27	$2{\cdot}9\ 10^9$	$8\ 10^{-5}$
Ne^{22}	9·73		
Mg^{24}	77·4		
Mg^{25}	11·5	$10{\cdot}0\ 10^9$	$8\ 10^6$
Mg^{26}	11·1		
Si^{28}	89·6		
Si^{29}	6·2	$12{\cdot}9\ 10^9$	$2\ 10^7$
Si^{30}	4·2		
S^{32}	95·0		
S^{33}	0·74	$3{\cdot}3\ 10^9$	$2\ 10^{-5}$
S^{34}	4·2		

Table XXV. We note that all five isotopic triples shown in that table lead to consistently high temperatures with the mean value

$$T = (7 \pm 4) \times 10^9\ ^\circ\mathrm{K.},$$

whereas the neutron densities show considerably larger variations. (This can be, of course, partially due to the fact that the formulae (79) and (80) are considerably less sensitive to n_n than to T.)

In view of the above, at least partially, encouraging results, one can try to interpret the entire abundance-curve as the result of a

thermodynamical equilibrium at a certain high density and temperature. It can be done best by employing the powerful statistical thermodynamical method of Gibbs which leads to the result that the equilibrium concentrations n_i (particle number per unit volume) of the nuclear species made of N_i neutrons and Z_i protons is given by the formula:

$$n_i = (2s_i+1)\left(\frac{2\pi M_i kT}{h^2}\right)^{\frac{3}{2}} \exp\frac{\mu N_i+\lambda Z_i-E_i}{kT}, \tag{82}$$

where $2s_i+1$ is the multiplicity of the nucleus in question, M_i its

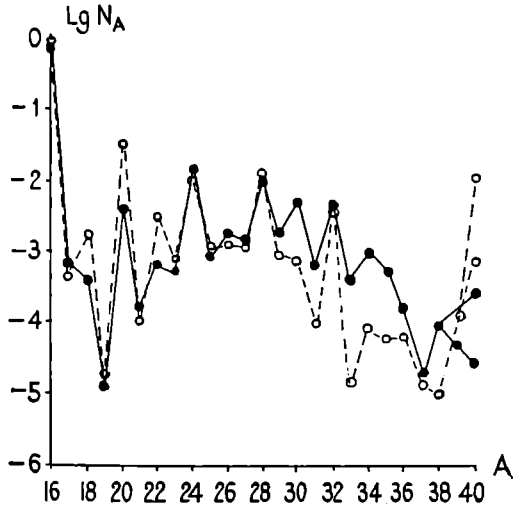


FIG. 58.

mass, E_i the total binding energy, and μ and λ the chemical potentials of a neutron and a proton, respectively. Using the experimental value of E_i the formula (82) can be fitted best to the empirical abundance curve, in the region of light elements, by setting $kT = 1$ M.e.v., $\mu = -7{\cdot}6$ M.e.v., and $\lambda = -11{\cdot}6$ M.e.v. (Klein, Beskow, and Treffenberg, loc. cit.). This corresponds to a temperature of $T = 10^{10}{}^{\circ}$ K., and to neutron- and proton-densities of $4{\cdot}10^8$ g./cm.^{3}, and $8{\cdot}10^6$ g./cm.^{3}, respectively. The large excess of neutrons over protons results, of course, from the fact that the high pressures involved favour the reaction $p+e^- \rightarrow n+$(neutrino). The result of such calculations is presented graphically in Fig. 58, which contains the calculated and the observed abundances for atomic weights from 16 to 40 (Lattes and Wataghin, loc. cit.).

The evident success of the equilibrium theory among the light elements of the periodic system turns into a catastrophe, however,

when the same method is applied to the heavier elements. This can be seen easily from Fig. 59, where the results of the equilibrium calculations are compared with the actual abundance. It is clear

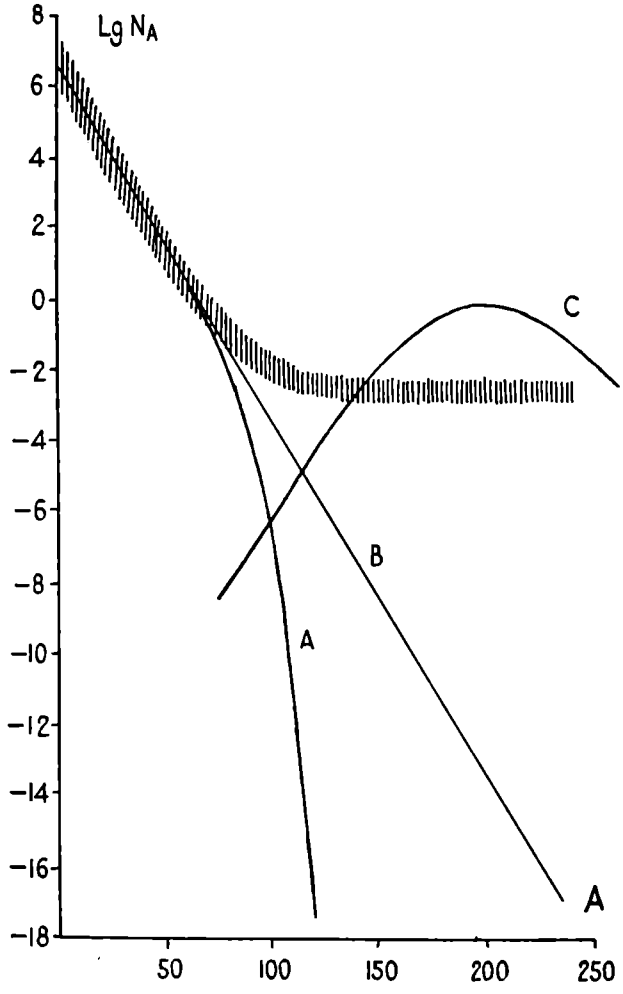


FIG. 59. Comparison of the observed abundance curve (shaded area) with three proposed theories.
Curve A: Equilibrium hypothesis.
Curve B: Aggregation hypothesis (with capture cross-section proportional to atomic weight).
Curve C: Fission-hypothesis.
The curves give only the general trend with individual variations being omitted.

that the striking discrepancy between the calculated and the observed curves cannot be removed by any reasonable modification of the calculations, and lies in the hypothesis of equilibrium itself. In fact, owing to the exponential dependence of the equilibrium concentration on the total binding energy of the nucleus and the approximate

linearity of the binding energy as a function of atomic weight, the equilibrium concentrations are bound to decrease exponentially for all atomic weights, just as for the lighter elements. An attempt was made (Chandrasekhar and Henrich, loc. cit.) to account for the discrepancy by assuming that different parts of the periodic system were formed at different stages of the expansion of the universe, and hence under different physical conditions. One could assume, for example, that the heavier elements were formed at higher temperatures (earlier stages of expansion) and that their relative abundances were 'frozen' at the lower temperatures when the transformation of lighter elements were still taking place. This hypothesis does not correspond to the physical situation, however, since at the temperatures involved the transformation of the nuclear species is mostly due to processes of neutron-evaporation and neutron-absorption which are practically independent of the atomic number of the nuclei involved.

Thus it seems necessary to abandon altogether the hypothesis of thermodynamical equilibrium, and to try to explain the observed abundance as a result of *some non-equilibrium process* which may have taken place in the early stages of expansion. Such an assumption appears particularly reasonable in view of the extremely short intervals of time allowed by the theory of the expanding universe for the formation of various atomic species. It can be argued, of course, that abandoning the equilibrium theory we shall lose the beautiful, detailed agreement between calculated and observed abundances in the region of the lighter elements. This may not be so conclusive an argument, however, since the peak-to-peak correspondence between the two curves in Fig. 58 represents essentially the correlation between abundance and intrinsic stability for individual nuclei and may be common to non-equilibrium—as well as equilibrium—theories.

An attempt to formulate a non-equilibrium theory of the origin of elements has been made by Gamow,† who suggested that the nuclei of various atomic weights might have originated as *aggregations of neutrons* formed in the early stages of the expansion of the universe. It is reasonable to suppose that in the very early stages of expansion matter consisted entirely of neutrons, since the high pressure obtaining in the compressed primordial material must have literally

† G. Gamow, *Phys. Rev.* **70** (1946), 572. More detailed calculations on this non-equilibrium process are being carried out by R. A. Alpher and will be published in due course.

'squeezed' all free electrons into the available protons.† As the rapid expansion of the universe took place, this superdense, superheated neutron gas-cooled adiabatically to some considerably lower temperature at which the formation of various neutral aggregates (polyneutrons) could have started.‡ The formation of such neutral aggregates is then followed by successive β-transformations through which the newly formed polyneutrons are changed into atomic nuclei of corresponding mass.

It must be remembered that a process of neutron-aggregation, as described above, has to compete with the natural decay of the neutron which should have reduced the number of neutrons practically to zero within the first few hours following the start of expansion. According to this view, the scarcity of heavier elements is due simply to the lack of time available for the formation of complex neutron-aggregates, and the slope of the abundance curve is determined by this time interval rather than by the temperature existing at that time. Detailed calculations of such a process of aggregation in a cool neutron gas are necessarily very complicated since one has to take into account the continual decrease in neutron-density (due both to the expansion of space and the natural decay of the neutrons), the variation of capture cross-sections with the rapidly decreasing temperature, as well as the irregular variations of these cross-sections from one element to another. One can, however, get a preliminary idea of the general features by introducing a number of simplifying assumptions. We may assume, for example, that the neutron-density (N_0) remains constant for a certain time τ, and drops to zero at the end of this period, and that the capture cross-section was independent of time and was the same for all nuclei. Under these assumptions we may write, evidently:

$$\frac{dN_i}{dt} = N_0 v(\sigma_{i-1} N_{i-1} - \sigma_i N_i), \quad (i = 1,2,3...) \tag{83}$$

† We leave it as an exercise for the reader to decide whether this highly compressed state of matter resulted from a previous contraction of the universe, all record of which is now completely lost, and whether the present expansion is simply a rebound from that contraction.

‡ The reader is given fair warning, however, that such aggregates of neutrons alone do not follow from the statistical theory of nuclear structure as presented in Chapter IV. On the contrary, one can show that such aggregates are not bound if the Majorana forces exist to the extent assumed there. On the other hand, that theory is by no means perfect, so that the conjectures made here, and in the following pages, simply express alternative views of the nature of nuclear forces.

where N_i is the number of nuclei with atomic weight i, σ_i is their capture cross-section, and v is the (thermal) velocity of the neutrons. The equations (83) can be integrated easily if $\sigma_i = \text{const.}$ and lead to

$$N_i = N_1 \frac{(N_1 \sigma v \tau)^{i-1}}{(i-1)!} e^{-N_1 \sigma v \tau}, \tag{84}$$

which gives a rapid decrease of relative abundance with atomic weight. Adjusting the formula (84) so as to give the correct slope in the region of the light elements, we find $N_1 \sigma v \tau = 30$. Since τ must be of the order of magnitude 10^3 sec. (decay-period of a neutron), and since the capture cross-section $\simeq 10^{-24}$ cm.^{2}, we get $N_1 v = 10^{22}$. Then if we assume that the neutron velocity was 10^4 cm./sec., the mean density of the universe becomes $\rho = 10^{18} \times 10^{-24} = 10^{-6}$ g./cm.^{3} According to formula (74) this state of the universe must have lasted for about 2 weeks.

It will be noted that the expression (84), which is plotted in Fig. 59 also, does not explain the observed shape of the abundance curve any better than the equilibrium theory. The possibility is not excluded, however, that the theoretical results can be improved considerably by taking into account variation of the capture cross-section with the size of the aggregate. Thus, for example, the observed curve of abundance can be represented very well if one permits the cross-sections to increase by a factor ten between the light and heavy elements (compare Appendix VI).

We now turn to another, no less, fantastic picture of the origin of the elements proposed recently by Göppert-Mayer and Teller.† In contrast to the theory described above, these authors assume that, in its most compressed state, the universe was filled with a *neutral nuclear fluid*‡ which consists of neutrons held together by cohesive forces. As expansion began this nuclear liquid must have broken up into a number of drops of various sizes. Consider the behaviour of such a drop, which may be 1 Ångström, 1 centimetre, or 1 kilometre in diameter. It is clear that for the reasons that apply to

† The authors are grateful to Dr. Marie Göppert-Mayer and Dr. E. Teller for communicating their results previous to their publication.

‡ The difference between this and the point of view discussed above evidently lies in different assumptions about the original temperature of the universe. According to Gamow this temperature was well above what could be called the 'critical temperature' of the nuclear fluid, so that one had essentially a highly compressed gas, whereas Göppert-Mayer and Teller assume that the temperature was below the critical temperature, so that the nuclear fluid is essentially a liquid.

ordinary nuclei having a neutron-excess, a large number of neutrons will undergo β-transformation with the emission of negative electrons. When internal equilibrium is finally established there will be a large, positively charged drop of nuclear liquid (neutron-proton mixture) with negative electrons partially dissolved in its large body,† partially forming a thin atmosphere just above its surface. More detailed calculations show that the thickness of this electronic atmosphere will be independent of the size of the drop, being of the order of magnitude of 10^{-11} cm. The important point concerning these 'overgrown atomic models' is that the system may be expected to possess a negative surface tension, because the mutual repulsion between electrons forming the 'atmosphere' will be larger than the surface tension due to nuclear forces. Thus, one can expect that the surface will become 'wavy' and that a number of small droplets will be separated from the surface of the large mother-drop. Following this 'budding-theory' of nuclear formation, Göppert-Mayer and Teller were able to show that the masses of the newly formed nuclei will be of the order of a few hundred proton-masses. It must be admitted, however, that this point of view does not explain the observed abundance-curve any better than any of the other theories, since it leads to a broad Gaussian distribution of relative abundance among the heavy elements (cf. Fig. 59), and offers no explanation whatsoever for the extremely high abundance of the lighter elements (compare Appendix VI).

† In fact, unlike the case for ordinary nuclei, the electrons can be completely immersed in the giant nuclear bodies, since, in this case, their Compton wave-length is smaller than the nuclear diameter.

XI

NUCLEAR CHAIN REACTIONS

1. Fast neutron chains

In the preceding chapter we have discussed thermonuclear reactions caused by general thermal agitation of matter, and we have seen that even the easiest reactions of this type require extremely high temperatures. There is, however, another type of self-sustaining reactions in which the transformation of each individual particle leads *directly* to similar transformations of one, or several, particles of the material. These so-called chain reactions take place when each elementary transformation gives rise to one or several 'active fragments' which induce new transformations of the same type as soon as they come into contact with other particles of the material in question. Depending upon whether such a reaction is developed due to the emission of a single active fragment, or to the emission of several such fragments, we speak of linear, or branching, chains respectively. Familiar examples of such reactions in the field of ordinary chemistry are: (1) the photo-induced linear chain in a mixture of hydrogen and chlorine ($Cl_2+h\nu \rightarrow 2\ Cl$; $Cl+H_2 \rightarrow HCl+H$; $H+Cl_2 \rightarrow HCl+Cl$: $Cl+H_2$ etc.), and (2) the thermally induced branching chains which probably take place in many ordinary explosives

$$(TNT+NO_3 \rightarrow H_2O\text{'s}+CO\text{'s}+\text{several } NO_3\text{'s, etc.}). \qquad (1)$$

It is clear that in the problem of nuclear chain reactions the role of the active fragments can be played only by neutrons that are emitted in the individual transformations. In fact, any charged fragment (proton, α-particle, etc.) is bound to lose all its original kinetic energy through electronic friction long before it has a chance to collide with another nucleus, and will have no chance, consequently, of penetrating the nuclear potential barrier when such a collision finally takes place. On the other hand, neutrons are practically unaffected by the electronic envelopes and they are able to penetrate into the nucleus even if their kinetic energy has been reduced practically to zero. Thus chain reactions are possible only in the case of such nuclear transformations which, being caused by the impact of a neutron, result in the emission of one or several neutrons of the next generation which are equally able to produce subsequent

transformations. The only known nuclear transformations satisfying the above conditions are the fission processes discussed in some detail in Chapters VI and IX.

We have seen that, although the fission process becomes exoergic for all nuclei heavier than silver, it becomes particularly efficient for the elements located near the end of the periodic system. In fact, for larger values of Z^2/A the activation energy for fission becomes smaller, whereas the reaction energy increases rapidly, resulting in larger fission cross-sections and larger numbers of neutrons per fission. Thus it is clear that the conditions for a self-sustaining chain reaction must be met for some sufficiently high value of Z^2/A. It is also clear that this condition will be satisfied by smaller values of Z^2/A in the case of odd A and even Z because energy of the captured neutron is greater in such nuclei.

In order to formulate exact conditions that have to be satisfied by a given nuclear species in order to make possible a self-sustaining chain reaction,† we shall assume that we are dealing with an infinite, homogeneous medium of this material for which we know‡ the cross-section for fission, σ_f, for radiative capture, σ_γ, for elastic scattering, σ_{el}, and for inelastic scattering $\sigma_i(E')$ (as a function of the energy E' of the emitted neutron) as well as the average number, of fission neutrons emitted, ν all expressed as functions of E, the energy of the incident neutron. It is easy to see that if we introduce into this material, N neutrons per unit volume, with an energy-distribution $f'(E)$, the energy-distribution of the next generation (defined as the neutron population produced by the first collision of all the neutrons of the previous generation) will be given by the formula:

$$f''(E) = \int f'(E') \frac{\sigma_{el}(E, E') + \sigma_i(E, E') + \nu\sigma_f(E, E')}{\int \{\sigma_{el}(E, E') + \sigma_i(E, E') + \sigma_f(E, E')\}\, dE + \sigma_\gamma(E')} dE' \tag{2}$$

in which it is to be understood that $\sigma_{el}(E, E')$ vanishes if $E \neq E'$ (i.e. it is essentially a δ-function).§ Hence the first and second terms represent the number scattered into the second generation elas-

† In common parlance such nuclear species are called 'fissionable'; it probably would be more rational to call such nuclei 'auto-fissionable', reserving the word 'fissionable' to apply to all nuclei heavier than silver.

‡ Compare Fig. 45 of Chapter IX.

§ We assume here that the elastic scattering does not result in loss of energy (due to recoil) because the above considerations are applicable only to very heavy elements.

tically and inelastically, respectively, whereas the last term represents the ν neutrons produced in the fission process. It can be expected that after a comparatively few generations (i.e. iterations of eq. (2)) the functions $f^{(i)}(E)$ will converge to a definite *relative* distribution in energy but such that the total number of neutrons per unit volume in a certain generation is k times that in the preceding generation,

$$f^{(i+1)}(E) = kf^{(i)}(E). \tag{3}$$

This defines the *multiplication factor*, k, which must be larger than or equal to unity if the chain reaction is to be sustained. It has been found empirically that, of all existing nuclear species, only U^{235} satisfies the above conditions, although one can build up other heavy nuclei not existing in nature that also satisfy those conditions. Thus, for example, another odd fissionable isotope of uranium may be made by the transmutation

$$\begin{aligned} Th^{232} + n &\rightarrow Th^{233} + \gamma\text{'s} \\ Th^{233} &\xrightarrow{\beta} Pa^{233} \\ Pa^{233} &\xrightarrow{\beta} U^{233}, \end{aligned} \tag{4}$$

and one can also produce a fissionable isotope of the new element plutonium through the transmutations:

$$\begin{aligned} U^{238} + n &\rightarrow U^{239} \\ U^{239} &\xrightarrow{\beta} Np^{239} \\ Np^{239} &\xrightarrow{\beta} Pu^{239}. \end{aligned} \tag{5}$$

For samples of fissionable material of finite size a certain fraction of the fission neutrons will escape through the surface, which, however, will not imperil the development of the chain if k is substantially larger than unity. One can introduce, in this case, the notion of the 'effective multiplication factor', k', which takes into account these neutron losses. The value of k' will depend upon the geometrical dimensions of the sample and will evidently drop below unity for certain critical dimensions for which the neutron losses through the surface become too large. These critical dimensions, known as the 'critical size', depend not only upon the properties of the material, e.g. the value of k, but also upon the shape of the sample.

It goes without saying that in systems of finite size the neutron distribution will be no longer uniform throughout the material due

to the loss of neutrons through the surface, and the exact calculation of the conditions for a reaction to be sustained becomes much more complicated. It can be easily understood, however, that in the case when k is not too close to unity the critical size will have linear dimensions of the order of magnitude of the free path of neutrons.

It is easy to see that if the sample is of exactly critical size it is in a state of indifferent equilibrium for which the neutron number remains constant at any arbitrary value. Whereas for k' larger than unity the neutron number will increase exponentially with time. In fact, if τ is the mean lifetime of a generation and N_0 is the number of neutrons at time zero, we obviously have:

$$N = N_0 e^{(k'-1)t/\tau}. \tag{6}$$

In pure fissionable material the generation time, estimated as the ratio of the free path to the velocity of fission neutrons: 4 cm. (corresponding to the geometrical cross-section of a heavy nucleus) divided by 10^9 cm./sec., becomes $\sim 3 \times 10^{-8}$ sec. Thus, unless k' differs from unity by less than a few millionth of a per cent., the reaction will develop into an explosion in a negligible fraction of a second; it goes without saying that such mechanical precision in preparing the sample of the fissionable material lies beyond practical possibility.

The above considerations, however, are changed completely if one remembers that a certain fraction of the fission neutrons are emitted with a considerable delay (Chap. VI), since, in this case, the number of prompt neutrons may be not enough to result in an exponentially increasing reaction rate, whereas the delayed neutrons may take considerable time to come into the picture. In the following table we give the data pertaining to the various delayed neutrons emitted from U^{235} and Pu^{239}.

The way in which the delayed neutrons help to reduce the rate of the runaway chain reactions can be understood on the basis of the following simple considerations. Let us assume, for simplicity, that there is only one single group of delayed neutrons emitted with a certain delay, δ, and representing a fraction, β, of the total number of neutrons emitted per fission. If we take a sample that is approximately critical with respect to the total number of neutrons emitted, it will be well under-critical with respect to the prompt neutrons alone. During the period of time preceding the appearance of the

TABLE XXVI†

Delayed neutron-groups from U^{235} and Pu^{239} fission products

Element and group-number	*Half-life (sec.)*	*Relative amount*	*Neutron energy* (k.v.)
U^{235} 0	0·006	0·02	—
U^{235} 1	0·43	0·116	420
U^{235} 2	1·52	0·330	620
U^{235} 3	4·51	0·291	430
U^{235} 4	22·00	0·228	560
U^{235} 5	55·6	0·035	250
Pu^{239} 1‡	1·1	0·301	—
Pu^{239} 2	5·2	0·371	—
Pu^{239} 3	22·5	0·288	—
Pu^{239} 4	55·2	0·040	—

delayed neutrons the reaction will proceed entirely on the basis of the prompt neutrons. If the reaction is started with N neutrons, the number of neutrons in the second, third, and subsequent generations will be given, respectively, by $N(1-\beta)$, $N(1-\beta)^2$,..., etc. The multiplication process is dying out and the total number of neutrons produced in the burst is obviously

$$N\sum_{i=0}^{\infty}(1-\beta)^i = \frac{N}{\beta}. \tag{7}$$

This burst will produce $(N/\beta)\beta = N$ delayed neutrons which will be emitted after the time-interval δ which is much greater than the time-interval required for the burst.

Suppose now that, for some reason, the multiplication k' pertaining to *all* neutrons becomes larger than unity, in such a way, however, that $k'(1-\beta) < 1$. In this case the total number of prompt neutrons produced in the first burst will be

$$N\sum_{n=0}^{\infty}[k'(1-\beta)]^n = \frac{N}{1-k+\beta}. \tag{8}$$

This produces $\dfrac{N\beta}{1-k+\beta}$ delayed neutrons which in their turn will produce a second burst of the strength $\dfrac{N\beta}{(1-k+\beta)^2}$ which is $\dfrac{\beta}{1-k+\beta}$ stronger than the previous one. If k' is reasonably close to unity $(k'-1+\beta)$ the multiplication factor per step, k'', is less than k'

† D. J. Hughes, J. Dabbs, A. Cahn, and D. Hall, *Phys. Rev.* (in the press, 1949); F. de Hoffmann, B. T. Feld, ibid.

‡ It seems very likely from the inspection of this table that the delayed neutron emitters are largely the same in the case of U^{235} and Pu^{239} fission.

without delayed neutrons and, what is much more important, the interval between successive bursts is of the order of magnitude of seconds instead of microseconds as in the prompt neutron chains, so that, even for substantial deviations from the critical size, the increase of the rate of reaction becomes very low. These simple considerations can be generalized easily to the actual case in which there are several groups of delayed neutrons of different mean delay.

2. Moderated neutron chains

In the preceding section we have discussed the case of a fast chain reaction in pure fissionable material where the neutrons produced in one-fission process usually induce another fission while still moving with high velocity. Remembering that the effective cross-section for fission increases with decreasing velocity of the neutrons, one can expect that it may be of some advantage (in the sense that a smaller amount of fissionable material will be needed) to slow down the original fission-neutrons to the thermal velocities before they have a chance to collide with another fissionable nucleus. This can be achieved by mixing the fissionable material with some other substance (the so-called *moderator*) of comparatively small atomic weight and low neutron-absorptivity that would slow the neutrons down efficiently without capturing them.

When a neutron undergoes an elastic collision with a nucleus of atomic weight A it transfers some of its energy to the recoiling nucleus. For scattering by the angle θ (in the centre of gravity system), the laws of conservation of energy and momentum lead to eq. (5) of Chapter VIII for the relation between initial and final energy of the neutron. Averaging the logarithm of the ratio of energies over all angles, we obtain

$$\overline{\ln\frac{E_2}{E_1}} = 1-\frac{(A-1)^2}{2A}\ln\frac{A+1}{A-1}. \tag{9}$$

The quantity on the right-hand side is usually denoted by ξ, and its value for the light elements appears in Table XXVII. It follows, for example, that in order to be slowed down from an initial energy of, say, 1 M.e.v. to thermal energy ($\frac{1}{30}$ e.v.) a neutron must undergo on the average 17 collisions in hydrogen, 86 collisions in beryllium, 108 collisions in carbon, or 470 collisions in iron. Among the light

TABLE XXVII

Energy-reduction factor for various Moderators

Hydrogen	1·00
Deuterium	0·752
Helium	0·426
Lithium	0·260
Beryllium	0·203
Boron	0·178
Carbon	0·158
Oxygen	0·120
Iron	0·036

elements there are two nuclear species, Li and B, for which the absorption cross-sections in the region of low energy (cf. eq. (11), Chap. VIII) are much larger than the cross-sections for elastic scattering;† it follows that these substances are completely excluded as possible moderators. In the other light elements for which the absorption cross-sections are very small as compared with the scattering, the neutron can be slowed down into the region of thermal energy provided the number of collisions required is not too large. Since the cross-section for neutron absorption increases with increasing atomic weight, in general, and the number of collisions necessary to slow the fission neutrons down to thermal velocities also increases the efficiency of moderators drops as the atomic weight increases.

Let us now describe in greater detail the processes of slowing down, diffusion, and subsequent capture of the neutrons in the moderated chain reaction. Consider the stationary case in which fast neutrons of given energy, E_0, are produced at a given rate and at some point within the infinite moderator, and are gradually slowed down by elastic collisions while diffusing away from that point. At a certain distance from the source we shall have a mixture of neutrons of all possible energies, since, owing to the statistical nature of the slowing-down process, some of the neutrons will manage to cover that distance with comparatively small energy-loss, whereas others may be slowed down quite considerably. The situation is represented schematically in Fig. 60, where the abscissa corresponds to the space coordinate while the ordinate represents the energies of the individual

† This is due to the fact that, whereas in the case of the other light elements the neutrons can be lost only through radiative capture, the absorption in Li and B is due to a break-up of the compound nucleus which is a much more probable process than that of the emission of a γ-ray.

neutrons. In a sense, we may speak of the 'diffusion of neutrons along the energy-axis', with the difference that, while diffusion along a space-axis involves motion in both directions, the energy-coordinate of the neutron can only decrease. The number of neutrons within a certain four-dimensional element formed by the spatial element, dv, and the energy-interval $(E, E-dE)$, may change for two reasons: (1) the difference in the number of neutrons of that energy entering and leaving the space element, and (2) the difference between the number of neutrons of that space element that are slowed down 'into' the energy-interval $(E, E-dE)$, and those that are slowed down 'out of it'.

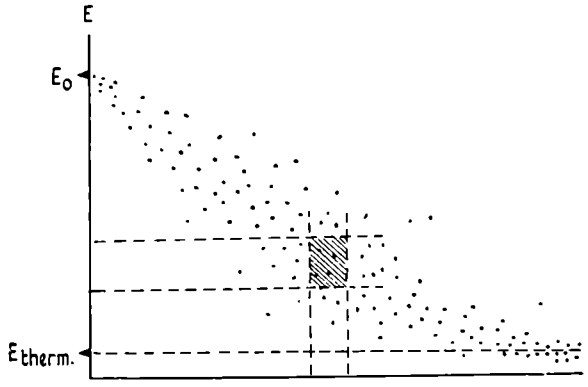


FIG. 60.

We shall introduce the quantity, $q(E, r)$, representing the number of neutrons (per unit volume) at the coordinate position, r, that are being slowed down (per unit time) 'across' the energy-value, E. It is clear that for the stationary case, the number $(\partial q/\partial E)(dE/dt)$ of neutrons which come into the energy-interval ΔE, by virtue of the slowing-down process, must be equal to the number of neutrons of that energy which escape per unit volume element by virtue of ordinary diffusion. According to the elementary theory of diffusion, the latter number is given by $(\lambda v/3)\nabla^2 q$, where λ is the free path and v the velocity of the neutrons in question. Thus we can write for the stationary state:

$$\frac{\lambda v}{3}\nabla^2 q = \frac{\partial q}{\partial E}\frac{dE}{dt}. \tag{10}$$

We can now relate the energy, E, of the neutron to the length of time elapsed since it was first emitted. In fact, since in each collision the ln of the neutron's energy decreases by ξ (cf. eq. (9)), and since the

neutron suffers, on the average, v/λ collisions per second, we can write, obviously:

$$-\frac{d(\ln E)}{dt} = \xi \frac{v}{\lambda}. \tag{11}$$

Integrating, we obtain,

$$t = \int_{E}^{E_0} \frac{\lambda}{\xi v} d(\ln E). \tag{12}$$

It is customary to express this (mean) length of time necessary for slowing the neutron from the original energy E_0 to the energy E, in units of length squared, multiplying it by $\lambda v/3$. The quantity:

$$\tau = \int_{E}^{E_0} \frac{\lambda^2}{3\xi} d(\ln E) \tag{13}$$

thus obtained is known as the 'age' of the neutron. Substituting it into the original equation (10) we get:

$$\nabla^2 q = \frac{\partial q}{\partial \tau}, \tag{14}$$

which is the well-known *Fermi's age equation.*

In the derivation of the above formulae we have assumed that the number of collisions experienced by a neutron is a large number, and also that the mean free path between collisions does not vary much with the energy of the neutron. These assumptions, which are necessary for the treatment of the slowing down as a continuous process, are fairly well fulfilled for comparatively heavy moderators, such as carbon, but may lead to considerable errors in the case of hydrogen or deuterium.

The eq. (14) has the solution

$$q = Q \frac{e^{-r^2/4\tau}}{(4\pi\tau)^{\frac{3}{2}}}, \tag{15}$$

which is readily checked by substitution, and for which the integral over all space is just Q. Therefore Q represents the number of neutrons emitted per second at the point $r = 0$. For a given moderator the 'age' of neutrons that have just become thermal may be calculated from eq. (13). Thus the mean distance from the source at which neutrons become thermalized may be calculated from eq. (15), and we shall take for this distance the value of r for which the exponential becomes e^{-1}, i.e. $r_{\text{th}} = 2\sqrt{\tau_{\text{th}}}$. Beyond this distance from the source the neutrons diffuse in the ordinary way without further

loss of energy. Properly speaking, fissions can be produced by neutrons at any stage of their slowing-down process. Since, however, according to eq. (11) the time spent in the low-energy regions is much greater than in the high-energy regions (factor $E^{-\frac{3}{2}}$) and since, in addition, the fission cross-section increases with decreasing energy, an overwhelming fraction of the induced fissions will take place after the neutrons have been thermalized. If the moderator has no strong regions of absorption in the energy-range between fission energy and thermal energy of neutrons, which must be true of any good moderator, the multiplication factor for an infinite medium, k'', will be given essentially by

$$k'' = \frac{\nu\sigma_f}{\sigma_f + \sigma_{\text{abs}}}, \tag{16}$$

where all the cross-sections refer to thermal neutrons and σ_{abs} includes the effect of the moderator as well as the absorption by the material itself.†

In the development of the detailed theory of the moderated chain reaction it is necessary to use the 'age theory' combined with the theory of the diffusion of the thermalized neutrons. The situation is considerably simplified in the case when the fissionable material and the moderator are distributed homogeneously over a region (size of sample) that is very large compared with the slowing-down distance, r_{th}, as determined above. Then one can, in good approximation, describe the distribution of the neutron density, n, by means of the usual differential equation for diffusion:

$$\frac{\lambda v}{3}\nabla^2 n - (Nv\sigma_{\text{abs}})n + Q(\mathbf{r}) = \frac{\partial n}{\partial t}, \tag{17}$$

where the first term gives the familiar effect of non-uniform density, with the diffusion coefficient, $\lambda v/3$ (λ = mean path and v an average velocity of the neutrons). The second term represents the loss of

† Of particular practical importance is the possibility of a chain reaction in natural uranium which contains only 0·7 per cent. of the 'fissionable' isotope, U^{235}, mixed with the 'non-fissionable' U^{238} which possesses a strong neutron absorption line. This results in such a strong absorption of neutrons as they are being slowed down through the resonance region that no moderator is good enough to raise the value of k'' above unity. As is well known, the situation can be considerably improved by distributing the uranium in lumps, embedded in the moderator, so that most of the neutrons are in pure moderator at the time they are going through the resonant-energy region. However, the requirements on the moderator in such a system are so stringent that only a few, exceptionally good ones, such as deuterium and carbon can be used as moderators for natural uranium.

neutrons per unit time due to an effective absorption per unit volume of $N\sigma_{\text{abs}}$ and $Q(\mathbf{r})$ is the number of thermal neutrons created, per unit time and unit volume, at the point $\mathbf{r}$. Since the source of thermal neutrons, $Q(\mathbf{r})$, is primarily fission neutrons that have been slowed down, the value of $Q(\mathbf{r})$ will depend upon the thermal neutron density in a region, essentially of the radius r_{th}, surrounding the point $\mathbf{r}$. In good approximation, the average over this region can be replaced by

$$Q(r) = (Nv\sigma_f)n(\mathbf{r}), \tag{18}$$

and eq. (17) becomes a homogeneous partial differential equation for $n(\mathbf{r})$. For the stationary case, the right-hand side of (17) vanishes and the equation becomes a characteristic value problem that has to be solved subject to the boundary conditions appropriate to the shape and size of the sample. In general, the solution of the equation for which n is everywhere positive will determine a certain minimum size for this system, and this is the critical size. For large values of k'' (cf. eq. (16)) the critical size will have linear dimensions of the order of a few slowing-down radii, r_{th}, whereas for small values of k'' the critical size becomes very large, as in the case of the natural uranium 'piles'.

APPENDIX I

HEAVY AND LIGHT MESONS

IT has been discovered by Lattes, Muirhead, Occhialini, and Powell† that, in addition to the ordinary, cosmic ray meson of mass ~ 200 electron masses (called the μ-meson) there exists a heavier meson which has been called the π-meson. Their experimental technique involves the development and analysis of tracks caused by the mesons in photographic emulsions. Their analyses show (1) that the π-meson has a mass $313 \pm 16m$ (m is the electron mass), and (2) the π-meson disintegrates into a charged meson of mass $\sim 200m$ and with kinetic energy about 4 mev, thus indicating the simultaneous emission of a neutral meson of mass $\sim 90m$ (the μ^0-meson). Presumably, the daughter meson of mass 200 is the cosmic ray, μ-meson. This discovery is corroborated by the production of π-mesons, $313m$, by high-energy (380 mev) α-bombardment with the Berkeley cyclotron.‡ The cyclotron bombardment appears to produce the mesons of mass $200m$ simultaneously with the heavier ones. This is apparently due to the decay of π-mesons stopped in the frame of the cyclotron. Both the π- and the μ-mesons occur with either sign of electric charge.

The significance of the various kinds of mesons relative to the theory of atomic nuclei is unknown as yet, but current conjecture runs as follows: the π-mesons are supposed to be the field particles in Yukawa's theory of nuclear forces and thus to interact strongly with nucleons; if this meson obeys Bose statistics and if there is an electrically neutral state for it (π^0) this supposition fits very well with the discovery by Breit§ and others that the nuclear scattering of protons by protons can be accounted for by a Yukawa type potential if produced by a meson of mass $326m$. On the other hand, the μ-mesons are supposed to have practically no interaction with nucleons in order to account for the great penetrating power of such mesons in the cosmic radiation. In fact Marshak and Bethe‖ anticipated the discovery of the π-meson by proposing just such a relationship between two kinds of mesons on theoretical grounds. The primary cosmic radiation (containing energetic nucleons) quickly produces π-mesons upon entering the atmosphere through the action of their nuclear fields; these π-mesons then decay into μ-mesons which can penetrate the atmosphere with ease.

† *Nature*, **159** (1947), 694.

‡ E. Gardner and C. M. G. Lattes, *Science*, **107** (1948).

§ L. E. Hoisington, S. S. Share and G. Breit, *Phys. Rev.* **56** (1939), 884.

‖ R. E. Marshak and H. A. Bethe, *Phys. Rev.* **72** (1947), 506. See also: C. M. G. Lattes, G. P. S. Occhialini, and C. F. Powell, *Nature*, **160** (1947), 453, 486; *Proc. Phys. Soc.* **61** (1948), 173; V. Goldschmidt, D. T. King, H. Muirhead, and D. M. Ritson, ibid. **61** (1948), 183.

APPENDIX II

FURTHER DEVELOPMENTS IN β-DECAY

RECENT refinements and improvements in measuring the energy-distribution in the continuous β-spectra have raised new questions regarding the exact form of the theory. In particular, the positron and the electron spectra emitted by Cu^{64} have been measured quite precisely down to kinetic energies of 10 k.e.v. by Cook and Langer.† It appears that the upper part of the spectra (from 270 k.e.v. to 657 k.e.v. for the positrons, and from 190 k.e.v. to 571 k.e.v. for the electrons) are accounted for very well by the Fermi theory. The lower parts of these spectra, however, do not fit well with the theory, but rather more β-particles (of both signs) are found than are predicted. Further, the positrons are in greater excess than the electrons (at the same energy) and the ratio increases as the energy decreases. This confirms a previous measurement of the ratio by Backus.‡ The significance of these new measurements is not known, at present, but precise data of this type may be expected to have an important part in the final construction of the theory.

A second source of information regarding β-spectra that should assist in deciding upon the theory of β-decay is the angular correlation between the direction of the emitted electron and the direction of the recoil nucleus.§ Even in the allowed transitions which may be considered as originating through interactions with S-waves there will be an angular correlation between electron and neutrino waves determined through the 'small' components of these waves. If θ is the angle included between the paths of electron and neutrino, p is the momentum and E the energy of the electron, the predicted correlation factors for the various theories are

Scalar	$1-(cp/E)\cos\theta$
Polar Vector	$1+(cp/E)\cos\theta$
Tensor	$1+\frac{1}{3}(cp/E)\cos\theta$
Axial Vector	$1-\frac{1}{3}(cp/E)\cos\theta$
Pseudoscalar	$1-(cp/E)\cos\theta$

From these, the expected recoil of the nuclei can be determined. The corresponding factors for the forbidden transitions have been determined.|| Experiments for the determination of these factors are currently in progress.

† C. S. Cook and L. M. Langer, *Phys. Rev.* **73** (1948), 601.
‡ J. Backus, ibid. **68** (1945), 59.
§ F. Bloch and C. Møller, *Nature*, **136** (1935), 912.
|| D. R. Hamilton, *Phys. Rev.* **71** (1947), 456.

APPENDIX III

MASS OF THE NEUTRINO

A ROUGH indication of the shape of the continuous β-spectrum of H^3 is given by the experiments of C. E. Nielsen,† who has counted the number of ions in the cloud-chamber tracks produced by tritium-containing water vapour. The results of these measurements are compared in Fig. 61 with the theoretical curves calculated for the neutrino masses: $\mu = 0$ and $\mu = \frac{1}{30}$.

In spite of a large experimental uncertainty arising from the small total number of measured tracks, the results fit much better with the vanishing or, at least, extremely small neutrino mass.

† C. E. Nielsen, *Phys. Rev.* (in the press, 1949).

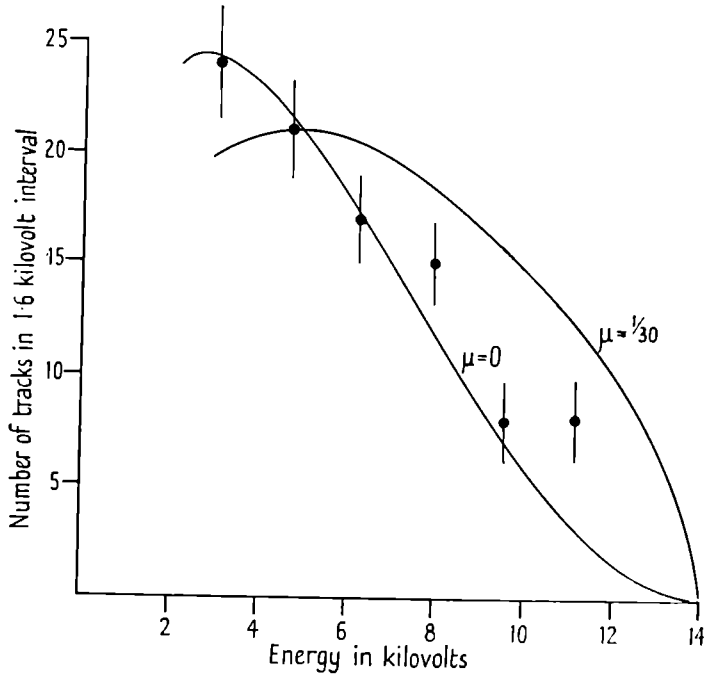


FIG. 61.

APPENDIX IV

NEW CALCULATIONS OF SPONTANEOUS FISSION

A DETAILED mathematical study of the fission process in the nuclear droplet-model was carried out recently by Frankel and Metropolis† who used for this purpose the electronic computer known as the 'Eniac'. Assuming the value 1·5 for the ratio of coulomb energy to the surface tension energy in the U^{235} nucleus (this value, usually denoted by $2x$, was originally calculated by Bohr and Wheeler) they have obtained the values 2×10^{20} sec.$^{-1}$ for the vibration frequency of that nucleus, and the expression $G = 10^{-7 \cdot 85 \Delta E}$ for the penetrability of the potential barrier (here ΔE is the energy-deficit at the saddle point in M.e.v.). The mean life of the U^{238}-nucleus with respect to spontaneous fission comes out to be 10^{26} years, in good agreement with the observed figure.

† S. Frankel and N. Metropolis, ibid. **72** (1947), 914.

APPENDIX V

NUCLEAR REACTIONS AT VERY HIGH ENERGY

THE 184-inch synchro-cyclotron at the Berkeley Radiation Laboratory has been operating at such a frequency as to produce nuclear beams in which the energy is about 95 M.e.v. per nucleon (deuterons at 190 M.e.v., or α-particles at 380 M.e.v.). The production of mesons from the α-particles beam has been discussed in Appendix I. When the deuteron beam is directed upon a heavy target, mesons do not appear, but a narrow beam of very high energy-neutrons emerges from the target. The angular spread of the neutron beam averages about 6° around the direction of the incident deuteron beam. This spread in angle is just what one would obtain if the proton were suddenly removed from a deuteron (of energy $E_d = 190$ M.e.v.) without disturbing the neutron. When neutron and proton are outside their range of forces in the ground state of the deuteron the average kinetic energy of the neutron is $\frac{1}{2}\epsilon$ ($\epsilon = 2{\cdot}18$ M.e.v., the binding energy of H^2). Hence, if the proton is suddenly removed, the trajectory of the neutron may be expected to lie at an average angle, $\sqrt{(\epsilon/190)} = 6°$, relative to the trajectories of the deuterons. The mechanism by which the proton is suddenly removed from the deuteron is, of course, the collision of the proton with a target nucleus while the neutron escapes collision. This process is called *stripping* and has been studied in detail by Serber.† The interpretation given leads also to the observed spread in energy of the neutrons, which should be

$$E_n = \tfrac{1}{2}(E_d^{\frac{1}{2}} \pm \epsilon^{\frac{1}{2}})^2 \simeq 95 \pm 20 \text{ M.e.v.}$$

Serber shows that the cross-section for stripping should be, approximately,

$$\sigma_{\text{str}} = \frac{\pi\hbar R}{4\sqrt{(M\epsilon)}},$$

where R is the effective radius of the target nucleus. The same cross-section applies to the production of protons (when neutron strikes and proton does not) but the emergent protons are caught in the magnetic field of the cyclotron and have to be detected inside the machine. The cross-section for the process in which neither neutron nor proton reaches the nucleus but the deuteron is simply dissociated by the Coulomb field is considerably smaller.

The neutrons produced by 'stripping' have been used to study nuclear reactions and scattering. For example, the 95 ± 20 M.e.v. neutrons have been used‡ to determine nuclear radii by the method cited in Chapter I, as applied by Sherr for 25 M.e.v. neutrons. The results are that the effective radii for 95 M.e.v. neutrons are a little smaller than for 25 M.e.v. neutrons among the heavy nuclei and considerably smaller among the light nuclei. This indicates that nuclear matter becomes somewhat transparent as the energy of the neutrons is increased. In other words, the cross-section for collisions between nucleons decreases with the energy of collision. Thus, nuclear reactions produced by the neutrons in very light nuclei should be caused mostly by a single collision in the nucleus. The activation of C^{12} by $(n, 2n)C^{11}$ supports

† R. Serber, *Phys. Rev.* **72** (1947), 1008.

‡ Cork, McMillan, Peterson, and Sewell, unpublished.

this hypothesis because the cross-section increases with energy up to 30 M.e.v. and then remains fairly constant up to 90 M.e.v. If, on the other hand, the concept of the compound nucleus were applicable, one would expect that at the higher energies the compound, C^{13*}, nucleus could disintegrate in a large variety of ways, thus reducing the probability of formation of C^{11} through competition.

The average momentum transfer in a single collision between a very fast neutron and a target neutron should be of the order of $\hbar/a$, where a is the range of nuclear forces. Actually, since nuclei are degenerate quantum mechanical systems, the recoiling nucleon must have a higher energy in order to reach empty orbits in the nucleus and Serber† estimates that the average energy of recoil is 25 M.e.v. The free path of the 95 M.e.v. neutron is also estimated at 4×10^{-13} cm. in the nuclear fluid. Consequently, the effect of a collision with a very high energy-neutron depends upon just where the bombarding particle strikes the nucleus. If it strikes near the edge (of the projected area) the probability is great that the single recoiling nucleon escapes or, if it recoils towards the centre, it transmits 25 M.e.v. excitation to the whole nucleus. The latter then disintegrates according to the 'evaporation' model discussed in Chapter IX. If the bombarding particle strikes near the centre it may make several collisions and result in an excited nucleus of correspondingly higher temperature. Now, the production of a particular type of disintegration may be expected to be most probable at a certain temperature, but the temperature may range from that corresponding to 25 M.e.v. excitation to that corresponding to the full energy of the neutron. The probability of attaining a certain temperature will not change rapidly as the neutron energy is increased, as a result of this mechanism, and in this way one can account for the observation that the yields of particular products from the bombardment of heavy nuclei do not change very much as the neutron energy is increased.

† R. Serber, *Phys. Rev.* **72** (1947), 1114.

APPENDIX VI

THE ORIGIN OF ELEMENTS

In their recent work, Alpher, Bethe, and Gamow† arrive at a satisfactory explanation of the observed abundance curve by integrating the system of equations (83) with σ_i's taken to be equal to the measured values of capture-cross-sections for fast neutrons. According to the latest measurements by Hughes,‡ these cross-sections increase exponentially by a factor of about one thousand when atomic weight increases up to one hundred, and remain more or less constant for heavier elements. The results of computations taking into account these capture-cross-sections are shown in Fig. 62. For very large $\rho\Delta t$ the process reaches the saturation and the number of various nuclei becomes inversely proportional to the corresponding capture-cross-sections, whereas for very small $\rho\Delta t$ heavier nuclei do not have time to build up to the saturation point. We see from the figure that for

$$\rho\Delta t = 1{\cdot}3\times 10^{-6}\frac{\text{g.}}{\text{cm.}^3}\text{sec.} \quad \left(n_n\,\Delta t = 0{\cdot}8\times 10^{18}\frac{\text{sec.}}{\text{cm.}^3}\right)$$

the calculated curve stands in a very good agreement with the observational data. The obtained agreement is amplified by the fact that the observed abundances show the abnormally high values for the isotopes containing the completed shells of neutrons or protons; in fact, it is known that such nuclei possess abnormally small capture-cross-sections which would cause the accumulation of the material at these particular atomic weights. Since the building-up process must have been accomplished within a time period comparable with the decay period of neutrons, we have $\Delta t \cong 30$ min. $\cong 10^3$ sec., from which follows that during that period the density of matter must have been of the order of magnitude 10^{-9} g./cm.^{3} On the other hand, since the temperature must have been of the order of $10^{9\,\circ}$K., the mass-density of radiation aT^4/c^2 was comparable with the density of water. Thus we come to an important conclusion that *at that time the expansion of the Universe was governed entirely by radiation and not by matter.* In this case the relativistic formula for the expansion can be written in the form:

$$\frac{dl}{dt} = \sqrt{\left(\frac{8\pi G}{3}\,\frac{aT^4}{c^2}\,l^2\right)}, \tag{1}$$

where l is an arbitrary distance in the expanding space, and the constant term containing the radius of curvature is neglected because of the high density value. Remembering that for the adiabatic expansion of the radiation: $T \sim 1/l$, we can integrate (1) into the form:

$$T = \sqrt[4]{\left(\frac{3c^2}{32\pi Ga}\right)}\,\frac{1}{t^{\frac{1}{2}}} = \frac{2{\cdot}14\times 10^{10}}{t^{\frac{1}{2}}}\,^{\circ}\text{K}. \tag{2}$$

† R. A. Alpher, H. A. Bethe, and G. Gamow, *Phys. Rev.* **73,** 803 (1948), compare also: R. A. Alpher, ibid. **74,** 1577 (1948); and R. A. Alpher and R. C. Herman, ibid. **74,** 1737 (1948).

‡ D. T. Hughes, ibid. **70,** 106A (1946), compare also: M. G. Mescheryakov, *C.R. Acad. Sci., USSR,* **48,** 555 (1945); J. H. E. Griffiths, *Proc. Roy. Soc.* **170,** 513 (1939); H. von Halban and L. Kowarski, *Nature,* **142,** 392 (1938).

For the mass-density of radiation we have:

$$\rho_{\text{rad}} = \frac{3}{32\pi G}\frac{1}{t^2} = \frac{4{\cdot}5\times 10^5}{t^2}\ \frac{\text{g.}}{\text{cm.}^3}. \tag{3}$$

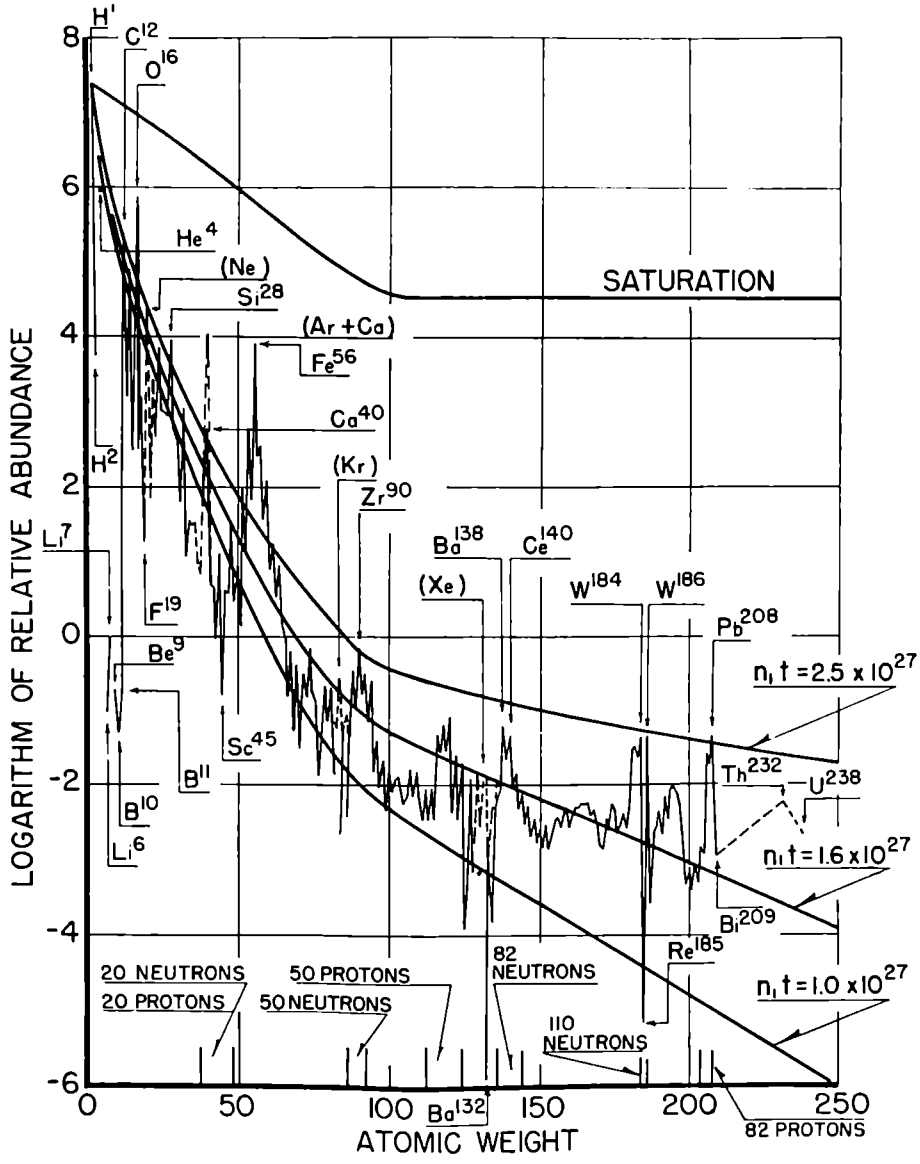


FIG. 62.

For the density of matter we must evidently write:

$$\rho_{\text{mat}} = \frac{\rho_0}{t^{\frac{3}{2}}}, \tag{4}$$

where ρ_0 is to be determined from the conditions of nuclear building-up process. It can be done in the simplest way by considering the building-up of deuterons by proton-neutron collisions. Writing $X(t)$ for the concentration of neutrons

(with $X(0) = 1$), and $Y(t)$ for the concentration of protons (with $Y(0) = 0$) we obtain the equations:

$$\frac{dX}{dt} = -\lambda X - \frac{XY}{m}\rho v\sigma, \qquad \frac{dY}{dt} = +\lambda X - \frac{XY}{m}\rho v\sigma, \tag{5}$$

where v is the thermal velocity and σ the capture-cross-section which (for the energies in question) can be sufficiently accurately represented by the formula:

$$\sigma = \frac{2^{\frac{1}{2}}\pi e^2\hbar\epsilon^{\frac{1}{2}}}{m^3c^5}\frac{1}{E^{\frac{1}{2}}}, \tag{6}$$

$\epsilon = 2{\cdot}19$ M.e.v. being the binding energy of the deuteron. Expressing $V$and E through the temperature, and using (2) we can rewrite (5) as:

$$\frac{dX}{d\tau} = -X - \alpha\frac{XY}{\tau}, \qquad \frac{dY}{d\tau} = +X - \alpha\frac{XY}{\tau}, \tag{7}$$

where $\tau = \lambda t$ and $$\alpha = \frac{2^{13/2}\pi^{5/4}e^2\hbar\epsilon^{5/2}G^{1/4}a^{1/4}}{3^{5/2}m^{9/2}c^{11/2}k}\rho_0. \tag{8}$$

In order that the equation (7) should yield $Y \cong 0{\cdot}5$ for $\tau \to \infty$ (since hydrogen is known to form about 50 per cent. of all matter) the coefficient α must be set equal to 0·5. Assuming we find from equation (8) that $\rho_0 = 7{\cdot}2 \times 10^{-3}$ which fixes the dependence of material density on the age of the Universe.

It was shown by Gamow† that the knowledge of the temperature and density in expanding Universe leads to very interesting cosmogonical consequences. In fact, once we have ρ_{rad} and ρ_{mat} as the functions of time we can follow the physical processes taking place during the further expansion of the Universe, and in particular calculate the masses and sizes of the condensations of that primordial gas which must have originated sooner or later according to Jeans's principle of gravitational instability. Jeans's classical formula‡ gives the diameter D of the condensations which will be formed in the gas of the temperature T and density ρ in the form:

$$D^2 = \frac{10\pi}{9m G\rho}\tfrac{3}{2}kT. \tag{9}$$

Using the expressions (2) and (4) we get:

$$M = \rho D^3 = \frac{2^{31/8}5^{7/4}\pi^{5/4}e\hbar^{5/4}\epsilon^{5/4}}{3^{17/8}m^{15/4}c^{5/4}G^{7/4}} \tag{10}$$

(where a was expressed through other fundamental constants). It is interesting to notice that the time-factor cancels out in the calculation of M, so that the mass of the condensations come out the same, independent of the epoch when they were formed. It seems, however, reasonable to assume that the effect of gravitational instability became important only when the mass-density of radiation became comparable with the density of matter, since it is hard to imagine a 'gravitational condensation of pure radiation'. Using (3) and (4), we find that $\rho_{\text{rad}} = \rho_{\text{mat}} = 3{\cdot}10^{-26}\frac{\text{g.}}{\text{cm.}^3}$ for $t = 3{\cdot}9 \times 10^{15}$ sec. $= 1{\cdot}3 \times 10^8$ years, at which point $T = 340°\text{K}$. For this value of t we obtain:

$$D = \frac{2^{45/8}5^{1/4}\pi^{7/4}e^3\hbar^{3/4}\epsilon^{15/4}}{3^{27/8}m^{29/4}c^{35/4}G^{5/4}}. \tag{11}$$

† G. Gamow, *Phys. Rev.*, 15 Aug. 1948; *Nature*, **162**, 680 (1948).

‡ J. Jeans, *Astronomy and Cosmogany* (Cambridge University Press, 1929).

Substituting numerical values we have:

$$M = 5{\cdot}5 \times 10^{40} \text{ g.} = 2{\cdot}7 \times 10^{7} \text{ sun-masses}$$
$$D = 1{\cdot}3 \times 10^{22} \text{ cm.} = 13{,}000 \text{ light-years} \qquad (12)$$

which must represent the masses and the diameters of the original galaxies.

The above given estimate of galactic masses falls short by a factor of about one hundred from the mass-values of galaxies obtained from astronomical data. But it must be remembered that the simple Jeans's formula used in these calculations does not take into account the effect of radiation-pressure, and also is applicable only to the gravitational condensations in the non-expanding space. The effect of additional radiation pressure (which is quite important according to the previous considerations) and the tearing force of expansion will lead to considerably larger condensation masses. The detailed study of this question will require, however, the extension of Jeans's classical arguments for the case of gas and radiation mixture in the expanding space. At the present stage one should be satisfied with the fact that through such comparatively simple and rather natural considerations *the masses and sizes comparable to those of stellar galaxies can be expressed in terms of fundamental constants, and the basic quantities of nuclear physics.*

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PRINTED IN GREAT BRITAIN
AT THE UNIVERSITY PRESS, OXFORD
BY CHARLES BATEY, PRINTER TO THE UNIVERSITY