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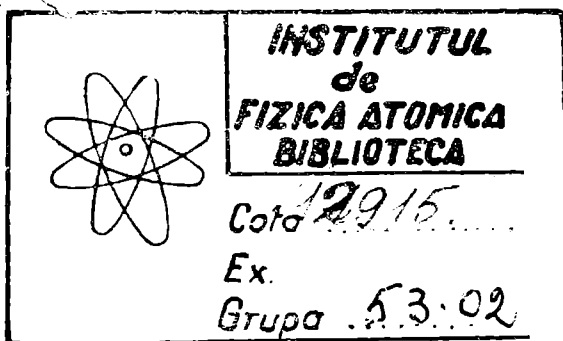
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DIFFERENTIAL EQUATIONS
OF PHYSICS

By
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Translated by
WALTER NEF

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CHAPTER I

THE DIFFERENTIAL EQUATION AS AN EXPRESSION OF A LAW OF NATURE

1. Law of causality

Any differential equation expresses a relation between derivatives or between derivatives and given functions of the variables. It thus establishes a relation between the increments of certain quantities and these quantities themselves. This property of a differential equation makes it the natural expression of the principle of causality which is the foundation of exact natural science. The ancient Greeks established laws of nature in which certain relations between numbers (harmony of spheres) or certain shapes of bodies played a privileged rôle. The law was supposed to state something about a process as a whole, or about the complete shape of a body. In more recent times (Galileo, Newton, etc.) a different concept has been adopted. We do not try to establish a relation between all phases of a process immediately, but only between one phase and the next. A law of this type may express, for example, how a certain state will develop in the immediate future, or it may describe the influence of the state of a certain particle on the particles in the immediate neighbourhood. Thus we have a procedure for the description of a law of nature in terms of small (mathematically speaking, infinitesimal) differences of time and space. The increments with which the law is concerned appear as

derivatives, i.e., as the limits of the quotient of the increments of the variables which describe the process over the increment of space or time in which this development takes place. A law of nature of this form is the expression of the relation between one state and the neighbouring (in time or space) states and therefore represents a special form of the principle of causality.

2. Ordinary and partial differential equations

A differential equation is called ordinary if there is only one independent variable, and partial if there are several independent variables. In an ordinary differential equation only ordinary derivatives appear, while a partial differential equation contains partial derivatives. The difference between these two types of differential equations appears more significant from the standpoint of physics, and is related to the deepest physical problems. There exist two fundamentally different concepts of physical processes upon which physical theories are based. According to the first point of view, matter consists of single particles which move in space without undergoing any changes. The position of each particle is determinable as a function of time, the only independent variable of all processes. This point of view provides the foundation of Newtonian mechanics, of atomism, in which the motion of elementary particles is considered as the sole basis of all physical processes. The ordinary differential equation is the mathematical expression of laws of this kind, since there is only one independent variable, namely the time.

On the other hand we have the field theory in physics, employed particularly in the domains of electromag-

netism and optics. In this theory, all processes are determined by field quantities which have a well defined value at each point of space. This value is usually a function of the time. We thus have four independent variables (the three space coordinates and the time). The laws which are based on this concept are expressed by partial differential equations.

Ordinary and partial differential equations are thus the mathematical expressions of the two fundamental points of view, the synthesis of which in quantum theory is one of the major problems of contemporary physics.

Ordinary differential equations may occur, however, in physical problems which have nothing to do with the atomistic point of view, e.g., in the theory of electrical oscillations or the bending of beams. But in these cases the ordinary differential equation is not a direct expression of any fundamental law, but rather an approximation which neglects the influence of all but one of the variables.

3. Initial and boundary conditions

✓ The differential equation alone does not express a specific physical problem. It expresses the general law under consideration, but not the specific case. A specific case is defined by initial or boundary conditions. Any integration implies arbitrary constants and, moreover, the integration of a partial differential equation involves arbitrary functions. In order to formulate the problem completely in mathematical terms and to solve the differential equation, there must be given as many physical boundary conditions as there are arbitrary

functions or constants in the integrated equation. In the following, we shall always be careful about the necessary number of conditions.

4. Scalars and vectors

The mathematical quantities which are used to describe a physical quantity or a process are of different kinds. They differ principally in the number of numerical values which are required for their determination. A quantity which can be completely determined by only one number is called a scalar (e.g., temperature, density, time.) Before we can specify this single number, we must, of course, choose a system of units.

The position of a point in space, however, can not be determined by a single number, but must be given by three of them, since the space has three dimensions. The manner in which the three numbers are determined is not important. They may be the coordinates of the point in a rectangular coordinate system, or the distance of the point from a fixed origin and two other numbers which determine the direction from the origin to the point. In any case, we must use three numbers in order to determine the position of a point in space. There exist many quantities which can be characterized by a line segment in three-dimensional space as, for instance, the forces in problems of graphical statics. Any such quantity which is determined by three numbers is called a vector.

There exist many complicated physical quantities which require more than three numbers for their determination (e.g., the stress and the strain in an elastic body, each of which is defined by six numbers). Quan-

tities of this kind are called tensors. In general tensor theory, vectors appear as tensors of the first order, while scalars are tensors of the zero-th order. In modern physics, other quantities exist which can be represented as segments in a four-dimensional space. They are called four-dimensional vectors. Furthermore, there exist four-dimensional tensors, etc. We shall be concerned here only with scalars and vectors (3-dimensional).

The limited space which is at our disposal in this book does not allow us to enter into the deeper special problems of physics, so that even the most elementary cases of wave mechanics and the theory of relativity cannot be presented. As tools we shall use only the fundamental theorems of differential and integral calculus. This introduction strives for a close connection between the physical concepts and their mathematical formulation. As a basis for further studies we suggest the book: *Die Differential- und Integralgleichungen der Mechanik und Physik* by Ph. Frank and R. von Mises.¹

¹2nd ed., Photo-reprint, M. S. Rosenberg, New York, 1945.

CHAPTER II

THE ORDINARY DIFFERENTIAL EQUATIONS OF THE MECHANICS OF PARTICLES

1. The motion of a particle

The elementary unit of mechanics is the particle, the physical behaviour of which is completely determined by its mass m which is an invariable scalar quantity, and by its position in space which depends on the time and which may be described by the rectangular coordinates x, y, z . The quantities x, y, z are the components of a vector \mathbf{r} , which is drawn from the origin of the coordinate system to the point x, y, z . The only independent variable is the time t .

The motion of a particle is described by two quantities which represent the change in the position of the particle in a very short interval of time by means of a limiting process. To begin with, let the point move along the x -axis, so that $y = 0$ and $z = 0$ at every instant. If we designate the x -coordinate at the time t by x and that at time t_1 by x_1 , then the ratio (difference of position)/(difference of time) $= (x_1 - x)/(t_1 - t)$ is called the "average velocity" in the interval $(t_1 - t)$, and the limit dx/dt the "instantaneous velocity" at the time t or just the "speed" v . The speed is usually not a constant, but depends on the time. If one forms the ratio (difference of speed)/(difference of

time) and then proceeds to the limit for $(t_1 - t) \rightarrow 0$, one obtains the "acceleration" $b = dv/dt = d^2x/dt^2$.

If the motion does not have a fixed direction, so that y and z can also be different from 0, we obtain three components of the velocity by the same limit processes

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt} \quad (2.1)$$

and three components of acceleration

$$b_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}, \quad b_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}, \quad b_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}. \quad (2.2)$$

Thus velocity and acceleration are also vectors, and as such will be designated by the bold-face letters \mathbf{v} and $\mathbf{b} = d\mathbf{v}/dt$.

The fundamental law of Newton constitutes a relation between the "acceleration" vector and the "force" vector \mathbf{P} which acts on the particle. The law is expressed by the differential equation,

$$m \frac{d\mathbf{v}}{dt} = \mathbf{P} \quad (2.3)$$

or in component form by

$$m \frac{d^2x}{dt^2} \left(= m \frac{dv_x}{dt} \right) = P_x, \quad m \frac{d^2y}{dt^2} = P_y, \quad (2.3a)$$

$$m \frac{d^2z}{dt^2} = P_z.$$

In this book it is not possible to discuss all the physical and philosophical problems which are connected with the conceptions of "force" and "mass". Our

only guide is the intuition we develop from everyday experiences in moving heavy masses, stretching elastic ropes, etc., by muscular exertion. The nature of a force as a vector having direction as well as magnitude is apparent in these actions. Any force causes a change of velocity and consequently a change of position. These changes are expressed as derivatives. The law, therefore, has the form of a differential equation.

2. Free fall

In order to pass from the general law of nature to a specific case, we take a body which is under the influence of gravity only, such that the sole force acting on it is its weight G . This weight acts in a certain direction, "downwards". We take the negative y -axis in this direction. Then the three components of the force are

$$P_x = 0, \quad P_y = -G, \quad P_z = 0, \quad (2.4)$$

and we have the differential equations of the problem

$$m \frac{d^2x}{dt^2} = 0, \quad m \frac{d^2y}{dt^2} = -G, \quad m \frac{d^2z}{dt^2} = 0. \quad (2.5)$$

Our task is to determine the three quantities x , y , z as functions of t . For this purpose we require additional information so that the special case under consideration becomes separated from all the others enabling us to determine the constants of integration.

We have three differential equations of the second order, i.e., equations in which second derivatives occur. The two integrations which transform the second derivatives into the functions yield two arbitrary constants, so that we have six constants in all. For their determina-

tion we need six given values. This may be expressed physically as follows; the natural law is only concerned with the acceleration, and while a particle may have a definite acceleration, it can have any velocity or position. The fact that velocity and position are not determined but characterize the special case and therefore must be given in order to deal with this case, is expressed mathematically by the order of the differential equations. If, however, at any moment the three velocity components and the three position coordinates are given, their future development is subject to the differential equation and we can calculate their values at any instant, assuming of course that we are able to integrate the differential equation.

As an example we shall take as the "initial values" at the time $t = 0$:

$$\begin{aligned}x &= 0, & y &= h, & z &= a, \\ \frac{dx}{dt} &= 0, & \frac{dy}{dt} &= 0, & \frac{dz}{dt} &= 0.\end{aligned}\tag{2.6}$$

This means physically that at the moment $t = 0$ a particle at the point $0, h, a$, which has no initial velocity is allowed to fall under the sole influence of its weight. This is "free fall".

The three differential equations (2.5) are simultaneous, i.e., they contain three unknowns and are valid at the same time. In this case they are very simple, since each equation contains only one unknown. Since the same is true for the initial conditions, the mathematical problem is reduced to the integration of three independent differential equations of the second order, for each

of which two initial conditions are given. The general solution is also elementary in our case. With the arbitrary constants $C_1 \cdots C_6$, we obtain

$$x = C_1 t + C_2, \quad y = -\frac{gt^2}{2} + C_3 t + C_4, \quad (2.7)$$

$$z = C_5 t + C_6,$$

where $g = G/m$ is the acceleration of free fall. These equations are equivalent to the differential equations (2.5). From them we obtain the solution of the special case if we form $dx/dt \cdots$, then introduce in all expressions $t = 0$ and the corresponding values according to (2.6), and finally solve the six linear equations with respect to the six unknowns $C_1 \cdots C_6$. A problem of this kind can be very difficult to evaluate. In our case, however, it is elementary. It is sufficient to give the result:

$$x = 0, \quad y = -\frac{gt^2}{2} + h, \quad z = a \quad (2.8)$$

These expressions and their derivatives with respect to t give us the position and the velocity of the point at any time, and all questions concerning the trajectory can be answered numerically. Often the main task of the physicist begins here, namely the physical interpretation of the mathematical results. We can not treat this third part of the problem here, but must be satisfied with the two preliminary parts: 1. Translation of the physical problem into the mathematical form ((2.5) and (2.6)), 2. Transformation into the mathematical solu-

tion (2.8). Two other simple examples are given to illustrate the procedure.

3. Trajectory of a projectile

A particle is thrown from a height h at an angle α , and with an initial speed v_0 . How does the motion develop?

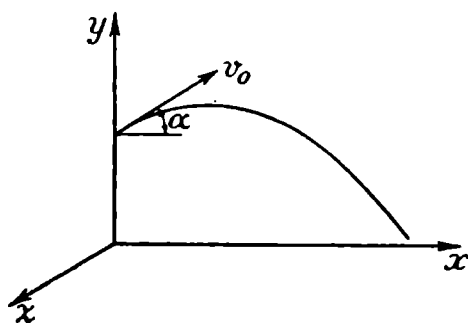


Fig. 1. Trajectory of a Projectile

The differential equations remain the same (2.5) as above. We note, incidentally, that the ratio G/m has the same value for all bodies. This value g is the "acceleration due to gravity". The vertical line through the initial position of the particle may again be chosen as y -axis, and we let the projection of the direction of the initial velocity on the horizontal plane determine the direction of the x -axis.

Then we have for the initial position ($t = 0$).

$$x = 0, \quad y = h, \quad z = 0,$$

and for the initial speed,

$$\frac{dx}{dt} = v_0 \cos \alpha, \quad \frac{dy}{dt} = v_0 \sin \alpha, \quad \frac{dz}{dt} = 0.$$

Without regard to the initial conditions, the solution is again (2.7). By use of the initial conditions we obtain

$$\begin{aligned} x &= (v_0 \cos \alpha)t, \\ y &= -\frac{gt^2}{2} + (v_0 \sin \alpha)t + h \quad z = 0. \end{aligned} \tag{2.9}$$

The position and the velocity at any moment t follow immediately from these formulae. The same is true for the times corresponding to prescribed values of the coordinates, such as the time corresponding to $y = 0$. Furthermore, by elimination of t one obtains a relation between x and y , describing the "trajectory" of the particle; from this relation the range, altitude, and similar quantities can easily be determined. The equation $z = 0$ shows that the whole motion takes place in a plane.

4. Friction

A particle which moves along a straight line with an initial speed v_0 is subjected to a force due to friction which is proportional to the speed of the body and therefore equal to cv . How does this motion develop?

The direction of the motion may be chosen as the x -axis. We have only one differential equation. But this time it is a bit more difficult to solve. It is the following:

$$m \frac{d^2x}{dt^2} = m \frac{dv}{dt} = -cv. \tag{2.10}$$

In addition we have the initial conditions that at $t = 0$, $v = v_0$ and $x = 0$. We obtain the solution of the differential equation for v by the method of "separation of variables". This means that the equation is trans-

formed in such a way that on one side there are only quantities depending on v , while those depending on t are all on the other side. Doing this and integrating we obtain

$$\frac{dv}{v} = - \frac{c}{m} dt \quad (2.10a)$$

or

$$\ln v = - \frac{c}{m} t + \text{const} \quad (2.10b)$$

or

$$v = C_1 e^{-ct/m}, \quad (2.11)$$

The constant of integration C_1 is a factor here because we have written (2.10b) in exponential form. The initial condition for v gives $C_1 = v_0$. For further integration we write

$$\frac{dx}{dt} = v_0 e^{-ct/m}$$

From this it follows that

$$x = - \frac{mv_0 e^{-ct/m}}{c} + C_2$$

and from the initial condition for x ,

$$0 = - \frac{mv_0}{c} + C_2 \quad \text{or} \quad C_2 = \frac{mv_0}{c}.$$

Thus the result is

$$x = \frac{mv_0}{c} (1 - e^{-ct/m}); \quad (2.12)$$

Discussing this we see that the velocity never becomes zero, while the body approaches the point $x = mv_0/c$ asymptotically.

5. The motion of a planet

Our differential equation, however, is not always of such a simple type that it can be integrated by such easy methods as simple integration or separation of variables, etc. As an example we have the classical problem in the mechanics of particles—the motion of a planet under the influence of the gravitational force of the sun. If M and m are the masses of the sun and the planet respectively, Newton's law of gravitation states that (with the notations of Fig. 2) $P = fMm/r^2$ (f = gravitational constant). Suppose now that we know the position and the velocity at a certain instant, and let us choose our coordinate system so that the x, y -plane contains the sun (as origin) and the direction of the planetary motion at the instant $t = 0$. Then for $t = 0$ we have $z = 0$ and $dz/dt = 0$, and we obtain the differential equations

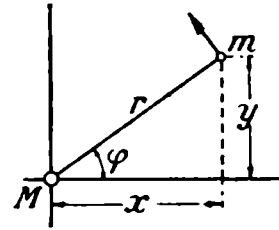


Fig. 2.

Sun and planet

$$\begin{aligned}\frac{d^2x}{dt^2} &= -\frac{fM}{r^2} \cos \varphi = -\frac{fMx}{r^3}, \\ \frac{d^2y}{dt^2} &= -\frac{fM}{r^2} \sin \varphi = -\frac{fMy}{r^3},\end{aligned}\tag{2.13}$$

while d^2z/dt^2 is proportional to z , and by the initial conditions is always equal to zero. Thus $r^2 = x^2 + y^2$,

and we have two simultaneous differential equations and four initial conditions. Both contain x and y in a rather complicated manner, however, and an elementary solution is not possible. There still exist fundamental methods for a solution, which we now intend to deduce from the general equation and which we shall later apply to the special case of the planetary motion.

6. Energy

We start with the general vector equation

$$m \frac{d\mathbf{v}}{dt} = \mathbf{P} \quad (2.14)$$

and form the scalar product of either of its sides with the vector \mathbf{v} . The scalar product $\mathbf{A} \cdot \mathbf{B}$ of two vectors \mathbf{A} and \mathbf{B} is the quantity $AB \cos \alpha$, where A and B are the lengths (magnitudes) of the two vectors. Thus the scalar product is the product of the length of one of the vectors and the projection of the second vector on to the first (Fig. 3). In rectangular coordinates this is simply

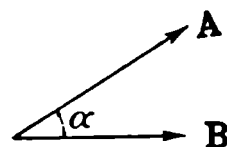


Fig. 3. Scalar product of two vectors

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z. \quad (2.15)$$

From the elements of differential calculus, it follows that

$$m \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = \frac{d}{dt} \left(\frac{mv^2}{2} \right).$$

Since $\mathbf{v} = d\mathbf{r}/dt$, multiplication of

$$m \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = \mathbf{P} \cdot \mathbf{v}$$

by dt gives the general equation

$$d\left(\frac{mv^2}{2}\right) = \mathbf{P} \cdot d\mathbf{r}. \quad (2.16)$$

This equation expresses the energy theorem of mechanics; "The increment of the 'kinetic energy' $mv^2/2$ is equal to the 'work' done by the force \mathbf{P} on the corresponding element $d\mathbf{r}$ of the trajectory." This equation can be integrated immediately, if \mathbf{P} depends only on the position (and not on the velocity). In this case the integral $-\int \mathbf{P} \cdot d\mathbf{r}$ (which contains an arbitrary constant) is called the "potential energy" of the mass m at the point which is given by the upper limit of the integral. This potential energy is the work which must be done against the force \mathbf{P} in order to move the particle from the arbitrary initial point to the point \mathbf{r} under consideration. The energy theorem now appears in this simple form; "The sum of the potential and kinetic energies is constant during the entire motion".

With this integral we have derived a relation in which the time does not occur and in which there are no second derivatives, but only the first derivative $\mathbf{v} = d\mathbf{r}/dt$ appears.

In our example of the planetary motion we can obtain the same result by multiplying the first equation (2.13) by dx/dt , the second by dy/dt and adding both, corresponding to (2.15). We obtain

$$d\left[\frac{m}{2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}\right] = -\frac{fMm}{r^3}(x\,dx + y\,dy). \quad (2.17)$$

The expression $(dx/dt)^2 + (dy/dt)^2$ is the square of the velocity (which is formed from the two velocity components) and may be designated by v^2 .

This equation is integrable. We take the values corresponding to $t = 0$ (which may be characterized by a subscript 0) as the lower limits. Then we obtain

$$\begin{aligned}
 & \frac{m}{2} (v^2 - v_0^2) \\
 &= -fMm \left[\int_{x_0}^x \frac{x \, dx}{(x^2 + y^2)^{3/2}} + \int_{y_0}^y \frac{y \, dy}{(x^2 + y^2)^{3/2}} \right]^{1)} \\
 &= fMm \left[\frac{1}{(x^2 + y^2)^{\frac{1}{2}}} - \frac{1}{(x_0^2 + y_0^2)^{\frac{1}{2}}} \right] \\
 &= fMm \left(\frac{1}{r} - \frac{1}{r_0} \right)
 \end{aligned}$$

or

$$\frac{m}{2} v^2 - f \frac{Mm}{r} = \frac{m}{2} v_0^2 - f \frac{Mm}{r_0} \quad (2.18)$$

Thus for each position we know the magnitude of the velocity which the planet must have there, but not the direction. Also we cannot obtain any information in this way about how the point moves with time.

It is important to note that (2.18) contains only first derivatives [$v^2 = (dx/dt)^2 + (dy/dt)^2$] and no second derivatives, and thus is only of the first order.

7. Moment of Momentum

We can find a first integral of the Newton equation with only first derivatives in a similar way to the above

¹⁾In the first integral y has the constant value y_0 , and in the second x has the constant value x_0 .

by multiplying the differential equation

$$m \frac{d\mathbf{v}}{dt} = \mathbf{P}$$

by \mathbf{r} (vector product). The origin of the vector \mathbf{r} , although arbitrary, must be fixed.

The vector product $\mathbf{A} \times \mathbf{B}$ is, so far as magnitude is concerned, the area of the parallelogram shown in Fig. 4, i.e., $AB \sin \alpha$. The orientation of this plane surface in space can be characterized by two angles.

Therefore the vector-product can be determined by three numbers, similar to a vector. The simplest way to choose these numbers is to

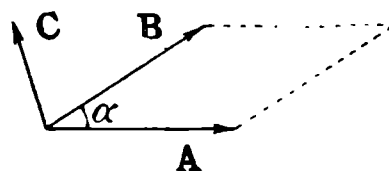


Fig. 4. The vector product of two vectors

take the areas of the projections of the parallelogram on three perpendicular planes. Since it can be determined by three numbers, the vector product can be represented by a vector. (However, it is not a vector). For this representation one takes a vector \mathbf{C} perpendicular to the parallelogram and with a length numerically equal to the area of the parallelogram. Furthermore, the orientation of \mathbf{C} shall be such that \mathbf{A} , \mathbf{B} , \mathbf{C} in this order form a right-handed coordinate system. Then we have $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$. The three rectangular components of \mathbf{C} are

$$[\mathbf{A} \times \mathbf{B}]_z = A_x B_y - A_y B_x ;$$

$$[\mathbf{A} \times \mathbf{B}]_x = A_y B_z - A_z B_y ;$$

$$[\mathbf{A} \times \mathbf{B}]_y = A_z B_x - A_x B_z .$$

We obtain

$$m \frac{d\mathbf{v}}{dt} \times \mathbf{r} = \mathbf{P} \times \mathbf{r}. \quad (2.19)$$

The expression on the right is called the “moment” of the force \mathbf{P} with respect to the origin. Fig. 5 shows us the relation between this definition of “moment” and the usual one which uses the perpendicular from the origin to the direction of the force, for in Fig. 5 $h = r \sin \alpha$. The expression on the left-hand side can be transformed by the use of

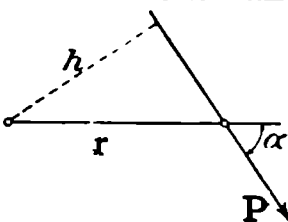


Fig. 5. The moment

$$\frac{d}{dt} (\mathbf{v} \times \mathbf{r}) = \frac{d\mathbf{v}}{dt} \times \mathbf{r} + \mathbf{v} \times \frac{d\mathbf{r}}{dt}.$$

Since $d\mathbf{r}/dt = \mathbf{v}$, and the vector product of any vector with itself is equal to 0 ($\alpha = 0$), it follows from (2.19) that

$$\frac{d}{dt} (m\mathbf{v} \times \mathbf{r}) = \mathbf{P} \times \mathbf{r}. \quad (2.20)$$

The quantity $m\mathbf{v} \times \mathbf{r}$ is called the “moment of momentum” of the particle with respect to the origin. The derivative of the moment of momentum with respect to the time is equal to the moment of the force (for any position of the origin). This theorem can be used for integration whenever the origin can be so chosen that the moment of the force is a simple expression, for example zero. This can always be done for all “central forces”. These are forces which at any position are always directed towards a fixed point. If this fixed

point is used as the origin, then the moment of the force is equal to zero in the whole space. Our problem of the motion of a planet may serve as an example if the sun is taken as the origin. This can also be seen formally if in our equation (2.13) we write the vector multiplication in coordinate form. Multiplying the upper equation by y , and the lower one by x and subtracting, we find the right-hand sides will cancel out.

This means physically that the moment is equal to zero. We obtain then

$$m \left\{ \frac{d^2x}{dt^2} y - \frac{d^2y}{dt^2} x \right\} = 0$$

or by a simple transformation

$$m \frac{d}{dt} \left\{ \frac{dx}{dt} y - \frac{dy}{dt} x \right\} = 0.$$

From this we obtain for the moment of momentum

$$m \left\{ \frac{dx}{dt} y - \frac{dy}{dt} x \right\} = \text{const.} \quad (2.21)$$

Thus we again obtain an integral of the equations of motion which is only of first order. The vector product in this equation is the area of the parallelogram formed by the velocity vector and the radius vector of the sun. This is twice the area of the triangle which is covered in unit time by the moving radius vector. This is the second law of Kepler. We have thus shown this law to be a consequence of the general theorem on moments. It states "The moving radius vector sweeps equal areas in equal times".

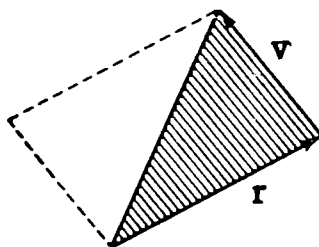


Fig. 6. The vector product as an area

We were able to transform the equations (2.13) into the simpler equations (2.18) and (2.21) with the aid of the two general theorems. From these we could obtain the other laws of Kepler by further integrations. These, however, are not of a general nature. They hold only for the special case and therefore we shall not consider them.

8. Systems of particles

We proceed from the problem of the mechanics of a single particle to examples with several particles. The particles are acted upon by exterior forces and by interior forces which represent the mutual influences of the particles of the system on each other. The interior forces occur only as couples of two forces, equal in magnitude but opposite in direction. Not only are the two forces \mathbf{J}_{21} and \mathbf{J}_{12} exerted by the point 1 on point 2 and vice versa, equal and opposite but the same is true for the moments of these forces with respect to an arbitrary point. The motion of such a system of n points can be described by n vector equations or by $3n$ equations for the components of the forces and the accelerations. They are

$$\begin{aligned}
 m_1 \frac{d\mathbf{v}_1}{dt} &= \mathbf{P}_1 + \mathbf{J}_{12} + \mathbf{J}_{13} + \cdots + \mathbf{J}_{1n} \\
 m_2 \frac{d\mathbf{v}_2}{dt} &= \mathbf{P}_2 + \mathbf{J}_{21} + \mathbf{J}_{23} + \cdots + \mathbf{J}_{2n} \\
 &\dots\dots\dots \\
 m_n \frac{d\mathbf{v}_n}{dt} &= \mathbf{P}_n + \mathbf{J}_{n1} + \mathbf{J}_{n2} + \cdots + \mathbf{J}_{n,n-1},
 \end{aligned} \tag{2.22}$$

The signification of the \mathbf{J}_{lm} and the relations between them follow from Fig. 7. To the exterior forces corre-

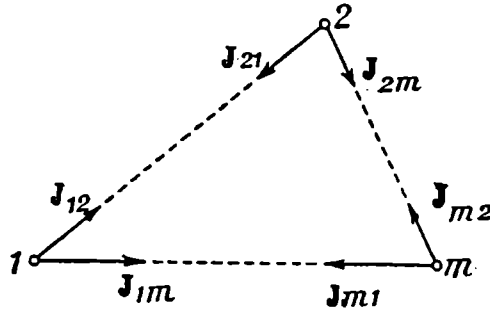


Fig. 7. Interior forces

pond opposite forces of the same strength, but since they act on points exterior to the system we need not consider them. The limitation of a mechanical system in space, and consequently the distinction between interior and exterior forces is arbitrary and suggested by practical considerations.

In order to integrate the system (2.22) we must try to eliminate the interior forces. The integrals without interior forces express the main theorems of mechanics of systems of particles.

9. Momentum theorem (motion of the center of gravity)

If we add all equations (2.22), the \mathbf{J}_{lm} cancel one another because they exist in couples of equal but opposite forces. The following vector equation remains

$$\frac{d}{dt} (m_1 \mathbf{v}_1 + \cdots m_n \mathbf{v}_n) = \mathbf{P}_1 + \cdots + \mathbf{P}_n. \quad (2.23)$$

The product $m_1 \mathbf{v}_1$ is called the “momentum” of the particle 1. The vector sum of all exterior forces is called

the “resultant” of the exterior forces, and the vector sum of all momenta the “total momentum” of the system. The “momentum theorem” of mechanics is contained in equation (2.23) which, for a system of particles, corresponds to Newton’s equation for a single particle. “The derivative of the total momentum with respect to the time is equal in magnitude and direction to the resultant of the exterior forces.” Let us introduce an ideal point having the same mass and the same momentum as the total system. This point is called the center of gravity. The same equation holds for this point as for a particle on which all the exterior forces act at the same time. ✓

10. Moment of momentum theorem

The moments of the interior forces also exist in couples of equal but opposite moments. If then we form the vector product of the equations (2.22) with the radius vectors $\mathbf{r}_1, \mathbf{r}_2, \dots$ of each point from an arbitrary point of origin and add the equations, we will again obtain a vector equation without interior forces, namely

$$\begin{aligned} m_1 \left[\frac{d\mathbf{v}_1}{dt} \times \mathbf{r}_1 \right] + \dots + m_n \left[\frac{d\mathbf{v}_n}{dt} \times \mathbf{r}_n \right] \\ = \mathbf{P}_1 \times \mathbf{r}_1 + \dots + \mathbf{P}_n \times \mathbf{r}_n . \end{aligned}$$

If we use here the same transformations as in Section 7, we obtain

$$\begin{aligned} \frac{d}{dt} [m_1 \mathbf{v}_1 \times \mathbf{r}_1 + \dots + m_n \mathbf{v}_n \times \mathbf{r}_n] \\ = \mathbf{P}_1 \times \mathbf{r}_1 + \dots + \mathbf{P}_n \times \mathbf{r}_n . \end{aligned} \tag{2.24}$$

“The derivative with respect to the time of the total moment of momentum of a system of points is equal to the resulting moment of the exterior forces”. This is the “moment of momentum theorem.”

The following is a simple but important special case. A rigid system of an arbitrary number of particles is allowed to rotate about a fixed axis (a system is rigid if the distance between all particles of the system is invariable). Of the three equations (2.24) only the one containing the components in the plane perpendicular to the axis is important here. Let the distance of the n th point from the axis be r_n and the angular velocity of the rotation ω . The velocity of the n th point is then $r_n\omega$. Since the velocity vector in this motion is always perpendicular to r , we have $|\mathbf{v} \times \mathbf{r}| = \omega r^2$. From (2.24) the simplified equation

$$\frac{d}{dt} \left\{ \omega \sum m_n r_n^2 \right\} = M,$$

follows, if M is the moment of the exterior forces with respect to the axis. The sum $\sum m_n r_n^2$ is called the moment of inertia J . The equation now becomes similar to the Newton equation (2.3a)

$$J \frac{d\omega}{dt} = M \quad (2.25)$$

If on the other hand we take the angle of rotation as a variable, the derivative with respect to the time of which is equal to the angular velocity, then

$$J \frac{d^2\varphi}{dt^2} = M. \quad (2.25a)$$

The two theorems (2.23) and (2.24) are general consequences of Newton's principles, namely of equation (2.3) for a particle, and the law that to each force there corresponds an equal and opposite reaction. The theorems therefore are generally valid.

11. Energy theorem

The integral of equation (2.22) which corresponds to our previous theorem is, however, not quite so general. If we multiply each one of the equations (2.22) by the corresponding velocity (scalar product) and add, we obtain expressions originating from the interior forces such as $\mathbf{J}_{12} \cdot \mathbf{v}_1 + \mathbf{J}_{21} \cdot \mathbf{v}_2$, etc. By projecting the vectors \mathbf{v}_1 and \mathbf{v}_2 on the direction of \mathbf{J}_{12} and \mathbf{J}_{21} we see that this sum is equal to zero if the two projections are equal and opposite, that is, if during the motion the distance of the two points is constant. The components of \mathbf{v}_1 and \mathbf{v}_2 perpendicular to the line m_1m_2 and the direction of this line may change. Thus we obtain an energy theorem independent of the interior forces only if the distances of the points from one another are constant, i.e. if the system is rigid. In this case we obtain the relation corresponding to (2.16) above,

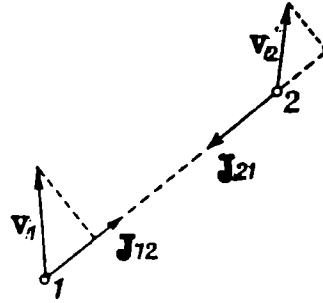


Fig. 8. Work done by the interior forces

$$d\left(\frac{m_1 v_1^2}{2} + \cdots + \frac{m_n v_n^2}{2}\right) = \mathbf{P}_1 \cdot d\mathbf{r}_1 + \cdots + \mathbf{P}_n \cdot d\mathbf{r}_n . \quad (2.26)$$

“The increment of the total kinetic energy of a rigid mechanical system is equal to the total work of the exterior forces.” If the forces are functions of the positions only, this equation is again easily integrated as in (2.16). If the system is not rigid, the work of the interior forces must be taken into consideration. Thus this theorem often becomes impractical for calculation.

Most problems of mechanics are solved with the aid of one or several of the three theorems mentioned. In certain cases they simplify (e.g., for rigid bodies, for pure rotation, etc.). It is not feasible to consider further examples here, since we would not gain any further experience concerning the construction and integration of differential equations, but only learn more about physical questions. One problem, however, is an exception, since a more direct approach gives us the solution and since it is probably the most important one in practice—the theory of oscillations.

12. Equation of oscillation

Oscillations originate wherever there is an exterior force tending to bring a displaced system back to its

restoring force is $P_x = -cx$. The motion is then characterized by the differential equation

$$m \frac{d^2x}{dt^2} = -cx. \quad (2.27)$$

For the exact determination of a single case we again require two given values of initial position and velocity. If, for example, at the beginning the particle is displaced from the equilibrium position and then released without any initial velocity, we have for $t = 0$, $x = x_0$ and $dx/dt = 0$. If on the other hand the particle at its equilibrium position receives a blow at the time $t = 0$, we have at this instant: $x = 0$, $dx/dt = v_0$. In this example it is possible to find the essential properties of the motion from the differential equation alone without the aid of the initial conditions.

The differential equation (2.27) is homogeneous and linear, that is, the unknown x or one of its derivatives is contained in the same power (the first) in every term. Differential equations of this type can be integrated directly (for any order and any number of unknowns) by using exponential functions. A solution of a differential equation of this type is given by

integration (in our case two). We also can determine all the α 's. For by introducing (2.28) in (2.27) we obtain

$$m\alpha^2 Ae^{\alpha t} = -cAe^{\alpha t},$$

and this is an algebraic equation for α since we can divide both sides by the exponential factor. The degree of the algebraic equation is determined by the order of the differential equation, and we thus obtain just the necessary number of solutions. We have in our case

$$\alpha = \pm i \left(\frac{c}{m} \right)^{\frac{1}{2}}$$

and it follows that

$$x = Ae^{i(c/m)^{1/2}t} + Be^{-i(c/m)^{1/2}t}. \quad (2.29)$$

Using the well known formula

$$e^{\pm i y} = \cos y \pm i \sin y \quad (2.30)$$

we can write (2.29) in the form

$$\begin{aligned} x &= (A + B) \cos \left(\frac{c}{m} \right)^{\frac{1}{2}} t + i(A - B) \sin \left(\frac{c}{m} \right)^{\frac{1}{2}} t \\ &= C \cos \left(\frac{c}{m} \right)^{\frac{1}{2}} t + D \sin \left(\frac{c}{m} \right)^{\frac{1}{2}} t, \end{aligned} \quad (2.31)$$

Since the solution of the problem must be real, C and D are two real constants which can easily be determined from the initial conditions.

The trigonometric functions represent oscillations. The "frequency" of these oscillations (number of oscillations in 2π units of time) is $\nu = (c/m)^{\frac{1}{2}}$, and the

“period” is $T = 2\pi(m/c)^{\frac{1}{2}}$. Both results are independent of the initial conditions and the constants of integration. On the other hand the amplitude A and phase constant φ (Fig. 10) are determined solely by the initial

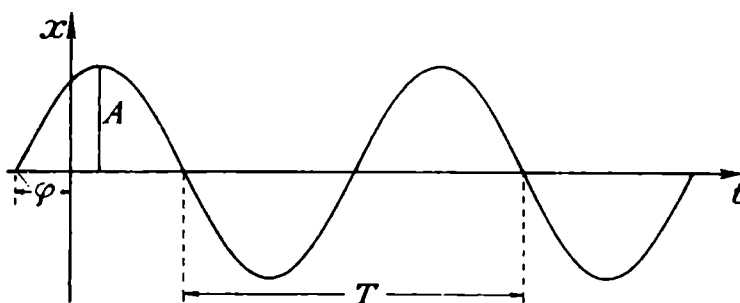


Fig. 10. Oscillation

conditions. In the majority of problems of oscillation theory the most important thing to know is the frequency, so that the initial conditions are not very important. We now generalize the problem in three ways.

13. Damped Oscillations

We assume that in addition to the restoring force which pulls the particle back to the equilibrium position there is also a force acting due to friction. This resisting force may be proportional to the speed dx/dt (factor k). This, however, is not always the exact form of a resisting force, but in most cases can be used as an approximation. The differential equation is now the following

$$m \frac{d^2x}{dt^2} = -cx - k \frac{dx}{dt} \quad (2.32)$$

This is again homogeneous and linear and integrable in the form $x = Ae^{at}$.

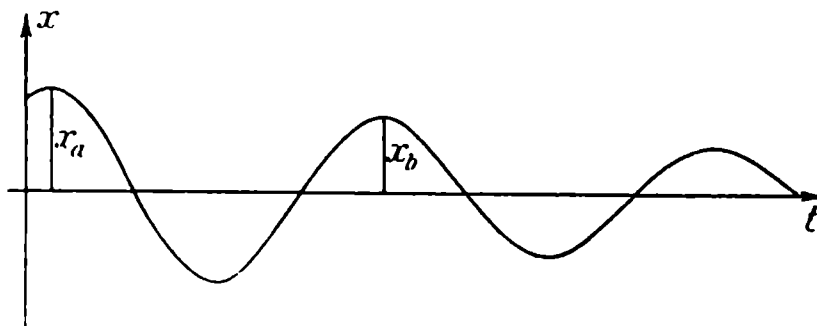


Fig. 11. Damped oscillation

We now have

$$m\alpha^2 + k\alpha + c = 0$$

and

$$\alpha = -\frac{k}{2m} \pm i \left(\frac{c}{m} - \frac{k^2}{4m^2} \right)^{\frac{1}{2}}. \quad (2.33)$$

Using (2.30) again we obtain solutions of the form

$$e^{-kt/2m} \times \frac{\cos}{\sin} \left[\left(\frac{c}{m} - \frac{k^2}{4m^2} \right)^{\frac{1}{2}} t \right].$$

These are damped oscillations of “frequency”

$$\left(\frac{c}{m} - \frac{k^2}{4m^2} \right)^{\frac{1}{2}}$$

and the logarithmic decrement $k/2m$. The latter term means that the values of x which correspond to the instants t and $t + T$ (the values, for example, marked x_a and x_b in Fig. 11) have the ratio $e^{-kT/2m}$, so that the ratio of their logarithms is equal to $kT/2m$.

The influence of the damping constant on the frequency is small when the restoring force is much larger than the frictional force. If the friction is great, how-

ever, the motion may not be periodical. This is true if $k^2/4m^2 > c/m$. In this case the oscillation is aperiodically damped.

14. Forced oscillation

Now in addition to the restoring and the friction force there may exist another force which is not a function of x but of the time t . For example, an exterior force tending to move the particle. We designate it by $P(t)$. The equation governing the motion is now

$$m \frac{d^2x}{dt^2} = -cx - k \frac{dx}{dt} + P(t) \quad (2.34)$$

and is no longer homogeneous. It represents a “forced” and not a “free” oscillation. In order to find its general solution we first look for any special function $x_1(t)$ which satisfies the equation (2.34). This is only a particular solution and does not contain any constant of integration. If we add the general solution $x_2(t)$ of the homogeneous equation

$$m \frac{d^2x_2}{dt^2} = -cx_2 - k \frac{dx_2}{dt},$$

which we already know, then the sum $x_1 + x_2$ is also a solution of (2.34), and moreover it is the complete one. In our case the particular solution x_1 is often called the “forced oscillation” while x_2 is called “free oscillation”. The latter is superposed on the former in a way which is determined by the initial conditions. The remaining task, to find a particular solution, is solved by the so called method of “variation of parameters”, and it is

assumed that the reader is familiar with this method from elementary courses.

We shall restrict our considerations to the case where the exterior force is periodical. This case is the most important one in practice and can be handled easily. The differential equation is

$$m \frac{d^2 x}{dt^2} + k \frac{dx}{dt} + cx = P_0 \cos \omega t, \quad (2.35)$$

where P_0 is the amplitude, and ω the frequency of the exterior force. We want to know the amplitude, the frequency, and the phase of the forced oscillation x_1 which is any particular solution without a constant of integration. If k were equal to 0, we could use a function proportional to $\cos \omega t$ for x_1 . For then each term would contain the factor $\cos \omega t$, which consequently would cancel out. Since k is different from zero, x_1 will be more complicated. There will be a difference of phase between the force and the oscillation. The most convenient practical method to find a particular solution of (2.35) is the following. By the addition of $iP_0 \sin \omega t$, the right hand term of the equation becomes equal to $P_0 e^{i\omega t}$. The solution of the complex equation

$$m \frac{d^2 x_1}{dt^2} + k \frac{dx_1}{dt} + cx_1 = P_0 e^{i\omega t} \quad (2.36)$$

is now a complex function x_1 , the real part of which is the solution of (2.35) we are looking for, while we are not interested in the imaginary part. This transition from trigonometric functions to complex exponential

functions is often used to simplify the integration of differential equations. If in (2.36) we put $x_1 = Ae^{i\omega t}$, the factor $e^{i\omega t}$ cancels out in all terms and the remaining equation is

$$A(-m\omega^2 + ik\omega + c) = P_0$$

or

$$A = \frac{P_0}{-m\omega^2 + c + ik\omega} = \frac{P_0(-m\omega^2 + c - ik\omega)}{(-m\omega^2 + c)^2 + k^2\omega^2}.$$

We shall write A in the form $A = Ce^{i\varphi}$, where

$$C = \frac{P_0}{(c - m\omega^2)^2 + k^2\omega^2} \quad \text{and} \quad \tan \varphi = \frac{k}{m\omega^2 - c}.$$

Then our particular solution of (2.35) is

$$x_1 = \text{real part of } [Ce^{i(\omega t - \varphi)}] = C \cos(\omega t + \varphi), \quad (2.37)$$

From this we immediately obtain the amplitude and the difference of phase between the forced oscillation and the exterior force. The most important result of (2.37) is the possibility of "resonance", that is large amplitudes in the solution if k is small and $\omega = (c/m)^{\frac{1}{2}}$, the frequency of the free oscillation.¹

15. Coupled oscillations

Two or more oscillating systems can be coupled. We shall demonstrate the setting up and the integration of

¹The determination of the free oscillation has been given in Section 13.

the corresponding differential equations for the most simple example of two particles, the first of which is pulled into a certain position of equilibrium by a spring (as above), while the other particle is connected with the first one by yet another spring. We assume no friction or exterior forces. We designate the displacements of the two particles from their respective equilibrium positions by x and y . The force exerted on the first particle by the first spring is of the form c_1x , while the second spring acts on the second particle with a force $c_2(y - x)$, as the extension of the second spring is given by the difference of x and y . The differential equations are now

$$\begin{aligned} m_1 \frac{d^2x}{dt^2} &= -c_1x + c_2(y - x) = -(c_1 + c_2)x + c_2y, \\ m_2 \frac{d^2y}{dt^2} &= -c_2(y - x). \end{aligned} \tag{2.38}$$

These are again two simultaneous and homogeneous equations. Equations of this kind can be reduced to a single equation by elimination of one of the unknowns. We can, for example, solve the second equation in (2.38) with respect to x and then introduce the expression for x so obtained into the first equation. Then we will have

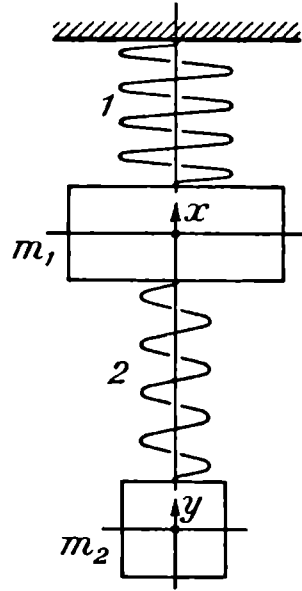


Fig. 12.
Coupled oscillations

an equation of fourth order which can be integrated by the method we used previously. There are four constants of integration corresponding to the two initial positions and the two initial velocities of the two points. However we are particularly interested in the consequences of the differential equations themselves.

We can also apply the method of exponential functions without the above-mentioned elimination, and this is usually the simpler way. We try the following expressions

$$x = A_1 e^{at}, \quad y = A_2 e^{at} \quad (2.39)$$

and obtain from (2.38) the expressions

$$\begin{aligned} A_1(m_1\alpha^2 + c_1 + c_2) - A_2c_2 &= 0, \\ -A_1c_2 + A_2(m_2\alpha^2 + c_2) &= 0. \end{aligned} \quad (2.40)$$

These are two homogeneous linear algebraic equations for A_1 and A_2 . Whenever the equations are independent of one another, the only solution is $A_1 = A_2 = 0$. If the equations are dependent on one another, they only determine the ratio A_2/A_1 while the absolute values of the unknowns are still undetermined. This corresponds to the free choice of one integration constant. Since for $A_1 = A_2 = 0$ we would have $x = y = 0$, the latter case (the two equations dependent) is the one which actually occurs. That means that the determinant of the system of equations is equal to zero, which for our example means that

$$\begin{vmatrix} m_1\alpha^2 + c_1 + c_2 & -c_2 \\ -c_2 & m_2\alpha^2 + c_2 \end{vmatrix} = 0. \quad (2.41)$$

This relation can also be obtained by direct calculation of A_2/A_1 from both equations. However, to set the determination equal to zero is more convenient when there are more than two unknowns.

Equation (2.41)

$$(m_1\alpha^2 + c_1 + c_2)(m_2\alpha^2 + c_2) - c_2^2 = 0$$

gives us two values for α^2 corresponding to the two frequencies. To each frequency there corresponds an amplitude and a phase constant. But for the term $-c_2^2$ on the left-hand side, the two frequencies furnished by this equation would equal the frequencies of the masses m_1 and m_2 on the springs c_1 and $c_1 + c_2$, respectively. The quantity c_2 is thus characteristic for the coupling by spring 2. The two afore-mentioned frequencies are changed by the coupling. In acoustics this is called "being out of tune". The equation for the frequencies in the general case is of a higher degree, and usually not of the second degree for α^2 as in our example.

16. Coupling by friction

The coupling also can be obtained by friction or similar forces. A simple mechanical example is the following one. Two disks are restored to the equilibrium position by springs which permit rotational oscillations (Fig. 13). Between the two disks there may be a small amount of a viscous liquid, the frictional force

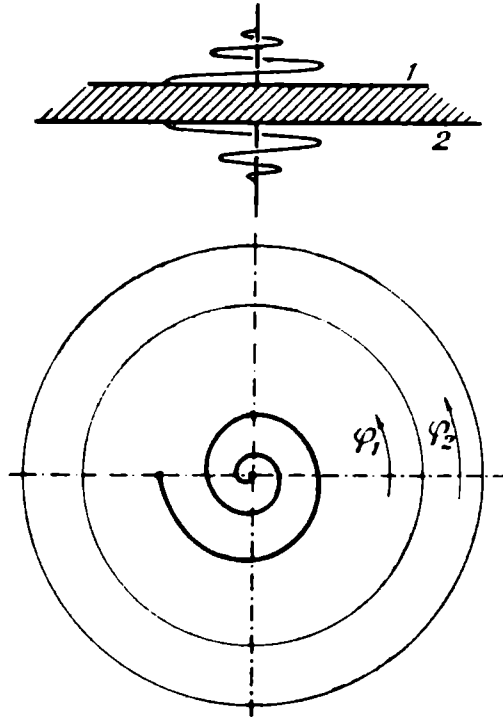


Fig. 13. Rotational oscillations coupled by friction

of which is proportional to the relative angular velocity of the disks. This example has technical significance in connection with the performance of anti-rolling tanks in ships.

The differential equations are, according to (2.32),

$$\begin{aligned} J_1 \frac{d^2 \varphi_1}{dt^2} &= -c_1 \varphi_1 - k \left(\frac{d\varphi_1}{dt} - \frac{d\varphi_2}{dt} \right), \\ J_2 \frac{d^2 \varphi_2}{dt^2} &= -c_2 \varphi_2 - k \left(\frac{d\varphi_2}{dt} - \frac{d\varphi_1}{dt} \right), \end{aligned} \quad (2.42)$$

where φ_1 and φ_2 are the angular displacements of the disks, and J_1 and J_2 are the moments of inertia. Since we

are interested only in the frequency and in the damping of the oscillations which follow from (2.42), we do not require any initial conditions.

With the aid of

$$\varphi_1 = A_1 e^{\alpha t}, \quad \varphi_2 = A_2 e^{\alpha t},$$

we obtain

$$A_1(J_1\alpha^2 + k\alpha + c_1) + A_2(-k\alpha) = 0,$$

$$A_1(-k\alpha) + A_2(J_2\alpha^2 + k\alpha + c_2) = 0.$$

Since the principal determinant must be equal to 0, we have the following equation of the fourth degree for α :

$$(J_1\alpha^2 + k\alpha + c_1)(J_2\alpha^2 + k\alpha + c_2) - k^2\alpha^2 = 0. \quad (2.43)$$

In the general case it is difficult to solve this equation which gives us four (complex) values for α . Since this solution constitutes a different problem in each case, we are not able to say anything general about it. Each of the complex values of α represents an oscillation, the frequency of which is given by the imaginary part of α , while the damping is given by the real part [compare (2.33)]. Since the four solutions form two couples of conjugate imaginary numbers, we only have two values for the frequency. Two or four of the solutions can be real. The corresponding oscillation is not periodic in this case.

17. Stability

The problem of stability is another one of the many cases in which the equations of motion of a mechanical

system are homogeneous and linear differential equations. If a system is in stationary equilibrium (for example a rigid body (airplane) in a motion that does not change with time), this motion satisfies as many conditions of equilibrium as there are degrees of freedom. A rigid body for example, can move in three directions and rotate about three axes. It has six degrees of freedom. Its stationary motion satisfies the six conditions of equilibrium which express the balance between forces and moments. We now assume that the equilibrium is slightly disturbed so that one or several variables of the motion have values different from those in the equilibrium. We shall assume in addition that the disturbance is so small that all forces and moments can be represented as linear functions of the disturbances (in the same way as a curve in a small neighbourhood of a point can be represented by the tangent). As a consequence of the disturbance, the motion is now accelerated; the equilibrium is disturbed. The linear and angular accelerations also are linear functions of the variables which describe the disturbance, and so the differential equations are linear in these variables. The differential equations are also homogeneous, for the components of the forces and moments which are free from disturbance correspond to the equilibrium condition and therefore cancel one another. Thus problems of stability are described by equations of the same type as the equation of oscillation, and therefore can be solved by using exponential functions Ae^{at} . This method is called "method of small oscillations", although in special cases the result may not be periodical.

For α we obtain as many values as correspond to the number and the orders of the differential equations. If the real parts of all quantities α are negative, all oscillations are decreasing as the time proceeds, and this happens the faster the greater the absolute values of these negative quantities α are. The original motion is stable in this case. If, on the other hand, one or several of the quantities α have a positive real part, then the disturbance increases. The original motion will not be restored and is instable. Any problem of stability can be solved on this basis. Nevertheless the numerical difficulties can be very great particularly for the determination of the quantities α as solutions of an algebraic equation of a higher degree.

CHAPTER III

THE SIMPLEST PARTIAL DIFFERENTIAL EXPRESSIONS

1. Gradient

In all field equations, the unknowns (which are scalars, vectors, or tensors) depend on the time and on the three space coordinates. It is clear that in equations describing physical reality, the derivatives can not be contained in arbitrary combinations. Thus only those expressions which have a meaning independent of the coordinate system can appear in equations of this kind. The coordinate system is arbitrarily chosen and has as little meaning for the natural phenomena which are described by the differential equations as has the frame of the glasses for the house which we see through them. If, for example, u is a scalar and x a space coordinate, then $\partial u / \partial x$ by itself cannot have a physical meaning. Since the coordinate system is arbitrary, what at one instant is the x -axis can at some other instant be the y -axis. Similarly the vector \mathbf{P} appeared in the differential equations which we have already encountered, but never the component P_x alone. The component only appeared in component equations together with the corresponding equations for the other components.

Our task is to find differential expressions which have a physical meaning and, being independent of the coordinate system, are defined in any system. The first expression of this kind corresponds to the derivative

itself and describes the variations of a scalar field at a certain point (x, y, z) of space. This variation is evidently not a scalar as in the case of differentiation with respect to the time. It is different in different directions. There exist, however, relations between the derivatives in different directions. We assume the concepts of partial and total derivatives to be known. Let us start from the point (x, y, z) and move along the vector $d\mathbf{r}$, which

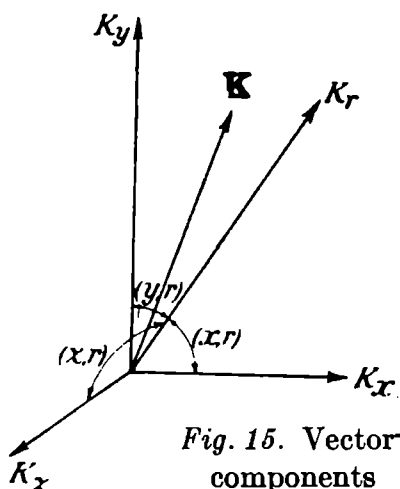


Fig. 15. Vector components

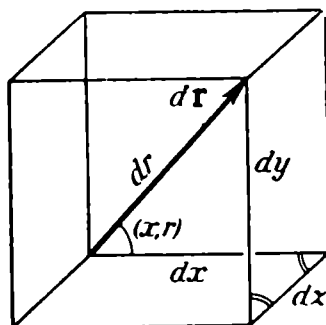


Fig. 14. Element of path and components

may have the length dr and the components dx, dy, dz (Fig. 14). Then if Φ is any scalar function, we have

$$\frac{\partial \Phi}{\partial r} dr = \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz$$

or

(3.1)

$$\frac{\partial \Phi}{\partial r} = \frac{\partial \Phi}{\partial x} \cos(x, r) + \frac{\partial \Phi}{\partial y} \cos(y, r) + \frac{\partial \Phi}{\partial z} \cos(z, r).$$

Thus the derivative in any direction can be expressed by the three quantities $\partial\Phi/\partial x$, $\partial\Phi/\partial y$, $\partial\Phi/\partial z$, and this expression has the same form as the expression of a component K_r of a vector \mathbf{K} by means of three perpendicular components K_x, K_y, K_z . The direction of the vector \mathbf{K}

is the direction for which the component in this direction has a maximum absolute attainable value. This length is equal to the Pythagoric sum of the three components. In the same way we can easily calculate (we assume this to be known) that in a certain direction \mathbf{r}_0 , $\partial\Phi/\partial r$ has a maximum value and is equal to zero in all directions perpendicular to this. In the direction opposite to \mathbf{r}_0 , we have a minimum of the same absolute value as the maximum. Thus a vector in the three-dimensional space corresponds to the derivative in the case of one variable. This vector corresponds in length and in direction to the maximum rate of increase of the scalar Φ from the point under consideration. Its component in any direction is equal to the partial derivative in this direction. This vector is called the "gradient" (grad).

The vector grad Φ has the components $\partial\Phi/\partial x$, $\partial\Phi/\partial y$, $\partial\Phi/\partial z$ and in general the component $\partial\Phi/\partial r$ in the direction \mathbf{r} .

$$|\text{grad } \Phi| = \left(\left(\frac{\partial\Phi}{\partial x} \right)^2 + \left(\frac{\partial\Phi}{\partial y} \right)^2 + \left(\frac{\partial\Phi}{\partial z} \right)^2 \right)^{\frac{1}{2}}. \quad (3.2)$$

The variation of a scalar with respect to the space coordinates can be contained in any physical equation in the form of the vector grad (or ∇ , pronounced "del") only. As an example, we may choose a heat flow \mathbf{q} which is proportional to grad T everywhere (T = temperature). If T is a function of position, then grad T is the vector which gives the increment of T in every direction and at every point. The heat flows in the direction opposite to the gradient (from higher to lower temperature). The factor of proportionality, λ , is called the conductivity of the material. The equation is

$$\mathbf{q} = -\lambda \text{ grad } T. \quad (3.3)$$

For any direction there is a certain heat flow; the greatest flow is in the direction opposite to $\text{grad } T$. No heat flows in the directions perpendicular to $\text{grad } T$, for which T is constant.

2. Cylindrical and spherical coordinates

The vector $\text{grad } \Phi$ has a significance independent of the coordinate system. It represents the properties of variation of the scalar Φ at a certain point. All we have to do to find the component of this vector in any direction is to take two points in this direction and to divide the corresponding difference of Φ by the distance between the points. We may use any coordinate system

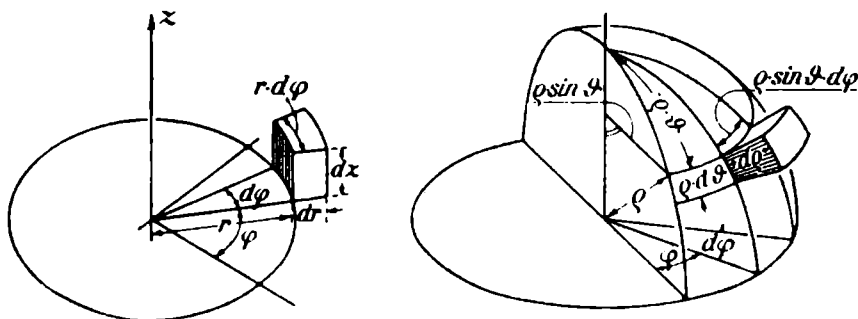


Fig. 16. Cylindrical and spherical coordinates

for this purpose. Consider, for example, cylindrical coordinates. In this system, the elements of length are

$$dr, \quad r d\varphi, \quad dz.$$

The corresponding components of the gradient are

$$\frac{\partial \Phi}{\partial r}, \quad \frac{1}{r} \frac{\partial \Phi}{\partial \varphi}, \quad \frac{\partial \Phi}{\partial z}. \quad (3.4)$$

In spherical coordinates, the length elements are

$$d\rho, \quad \rho d\vartheta, \quad \rho \sin \vartheta d\varphi,$$

and the components of the gradient

$$\frac{\partial \Phi}{\partial \rho}, \quad \frac{1}{\rho} \frac{\partial \Phi}{\partial \vartheta}, \quad \frac{1}{\rho \sin \vartheta} \frac{\partial \Phi}{\partial \varphi}. \quad (3.5)$$

3. Vector fields

Besides scalar fields, there exist vector fields defined by a vector (e.g. \mathbf{q}) which is variable in the space. Such a vector is not always the gradient of a scalar field, as in the case of the heat flow \mathbf{q} . The condition under which a vector field is a gradient field will be established below [(3.23)].

As an example of a vector field we shall use the flow of a fluid. At every point where there is any fluid, the velocity has a certain magnitude and direction (which can vary with time). Other examples are electric and magnetic fields. It is possible to denote these vectors by some points in a diagram, but a better representation can be obtained by drawing only the directions of the vectors and not the lengths. In this case we obtain continuous lines which at every point have the direction of the vector to be represented. Best known of all are the lines in a magnetic field, which can be seen as the pattern formed by iron filing placed in the magnetic field. They are called "lines of force" in this case. The corresponding lines in a fluid are called "streamlines". Any section of the field whose surface is composed of lines of force or streamlines is called a tube of force or a stream tube. The component of the field vector in the direction

perpendicular to the surface of such a tube is always equal to zero.

4. Divergence

Assume now a vector field in which there is a closed surface dividing the space into two parts—the interior and the exterior of the surface. For each element of this surface, the product “element of surface df times the normal component of the field vector (exterior normal)” has a value which is independent of the coordinate system. It can be represented by the scalar product of the field vector with another vector whose length is equal to the area of the element and the direction of which is that of the direction of the exterior normal to the element. The integral of this

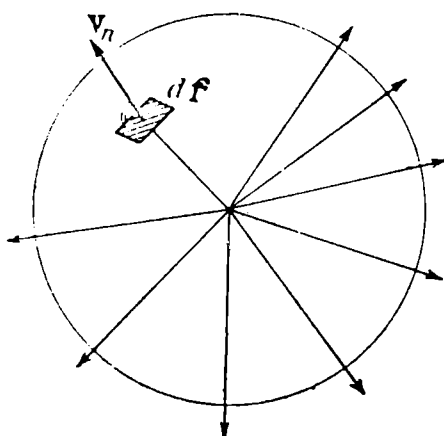


Fig. 17. Source, element of an arbitrary surface

product over the whole closed surface has a physical significance. If the field vector is the velocity of a fluid flow, then the product is the volume of liquid which flows through the element from the interior to the exterior in unit time. Thus the integral represents the

volume of fluid leaving the interior. At points on the surface where \mathbf{v} is directed into the interior the integrand becomes negative, of course. The integral thus represents the total strength of all sources in the interior. If the latter is not equal to zero, this can be caused either by streamlines originating in the interior or by streamlines along which the flow rate increases or decreases. In the

hydrodynamical example, the first case corresponds to a flow with a source in the interior (Fig. 17), while the second case corresponds to the stationary flow of a gas

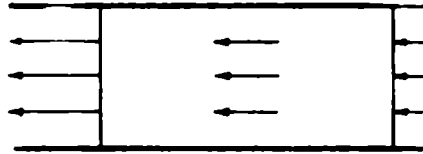


Fig. 18. Flow with variable density

in which the density decreases in the direction of the flow (Fig. 18). For, in the latter case, the same mass of gas enters the region bounded by the two walls and two cross sections (Fig. 18) in unit time as it leaves it. Thus the product "density times velocity times area of entrance or exit" must have the same value for both cross sections. Since the density is different on the two sections, the products "velocity times area of cross section" are different, so that the integral, which in this case is equal to the difference of the two products, is different from zero.

In order to find the mathematical expression for the "magnitude of a source" at a point in space, we have to take a closed surface which contains an infinitely small neighbourhood of this point. In this way the integral becomes infinitesimal of third order (the surface itself is of second order, but the difference between the field vector at the point of entrance and point of exit also becomes infinitesimal). Thus if we divide by the volume

V of the enclosed region, we shall have a finite limit. We call this limit the “divergence” (div) of the field,

$$\operatorname{div} \mathbf{A} = \lim_{V \rightarrow 0} \frac{1}{V} \int \mathbf{A}_n \, df. \quad (3.6)$$

The divergence is a scalar.

5. Mathematical expression of the divergence

The calculation of the divergence in various coordinate systems is now simple. In rectangular systems we use an infinitesimal cube in the field \mathbf{A} , the edges of

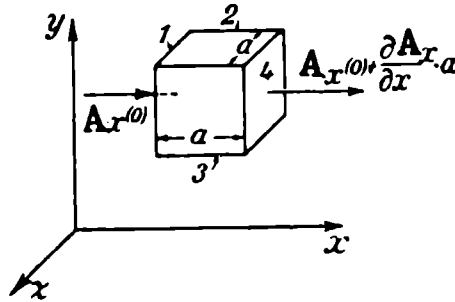


Fig. 19. Derivation of the formula for the divergence

which are of length a parallel to the axes of the system. In this case the normal components of the vector on the surface elements of the cube are either A_x or A_y or A_z . The positive direction is the one which points perpendicularly into the interior for the planes 1, 2, 3, and to the exterior for the other planes. Therefore we have $A_n = -A_x$ on 1 and $A_n = A_x$ on 4. The integral taken over 1 is equal to $-A_{x0} \cdot a^2$ if A_{x0} is the average of A_x on 1. The difference between A_x on 4 and A_x on 1 is so small that we can assume linear dependence. For constant y and z we then have

$$\mathbf{A}_x(a) = \mathbf{A}_x(0) + \frac{\partial \mathbf{A}_x}{\partial x} a$$

and the value of the integral over 4 is

$$\left\{ \mathbf{A}_{x0} + \left(\frac{\partial \mathbf{A}_x}{\partial x} \right)_0 a \right\} a^2,$$

if $(\partial \mathbf{A}_x / \partial x)_0$ is the average value of the derivative over 1. The same consideration is valid for the other planes. Since the average values go over into the values themselves in the limiting process when $a \rightarrow 0$, we obtain

$$\begin{aligned} \operatorname{div} \mathbf{A} &= \lim_{a \rightarrow 0} \frac{1}{a^3} \int \mathbf{A}_n df \\ &= \lim_{a \rightarrow 0} \frac{1}{a^3} \left[\mathbf{A}_x + \frac{\partial \mathbf{A}_x}{\partial x} a - \mathbf{A}_x + \mathbf{A}_y \right. \\ &\quad \left. + \frac{\partial \mathbf{A}_y}{\partial y} a - \mathbf{A}_y + \mathbf{A}_z + \frac{\partial \mathbf{A}_z}{\partial z} a - \mathbf{A}_z \right] a^2, \\ \operatorname{div} \mathbf{A} &= \frac{\partial \mathbf{A}_x}{\partial x} + \frac{\partial \mathbf{A}_y}{\partial y} + \frac{\partial \mathbf{A}_z}{\partial z}. \end{aligned} \quad (3.7)$$

Similar calculations are easy for any other coordinate system if one constructs the volume element with the elements of coordinates. Fig. 20 shows an element in

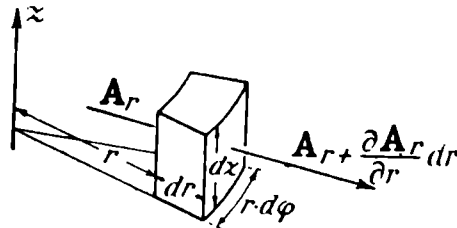


Fig. 20. Calculation of the divergence in cylindrical coordinates

(Note: dr , dz , $d\varphi$ in Fig. 20 should be replaced by Δr , Δz , $\Delta\varphi$)

cylindrical coordinates. The difference here is that the three coordinates are not equivalent. The finite but small dimensions of the element may be Δr , $r\Delta\varphi$ and Δz . The integral over the different parts of the surfaces are again (if \mathbf{A}_r , etc. again are the averages)

$$\begin{aligned}
 1) & -\mathbf{A}_r r \Delta\varphi \Delta z & 4) & \left(\mathbf{A}_r + \frac{\partial \mathbf{A}_r}{\partial r} \Delta r \right) (r + \Delta r) \Delta\varphi \Delta z \\
 2) & -\mathbf{A}_\varphi \Delta r \Delta z & 5) & \left(\mathbf{A}_\varphi + \frac{\partial \mathbf{A}_\varphi}{\partial \varphi} \Delta\varphi \right) \Delta r \Delta z \\
 3) & -\mathbf{A}_z r \Delta r \Delta\varphi & 6) & \left(\mathbf{A}_z + \frac{\partial \mathbf{A}_z}{\partial z} \Delta z \right) r \Delta r \Delta\varphi.
 \end{aligned}$$

The volume element is $r\Delta r\Delta\varphi\Delta z$. By addition and the proper limiting process we obtain

$$\begin{aligned}
 \operatorname{div} \mathbf{A} &= \frac{\partial \mathbf{A}_r}{\partial r} + \frac{\mathbf{A}_r}{r} + \frac{1}{r} \frac{\partial \mathbf{A}_\varphi}{\partial \varphi} + \frac{\partial \mathbf{A}_z}{\partial z} \\
 &= \frac{1}{r} \frac{\partial}{\partial r} (r \mathbf{A}_r) + \frac{1}{r} \frac{\partial \mathbf{A}_\varphi}{\partial \varphi} + \frac{\partial \mathbf{A}_z}{\partial z}.
 \end{aligned} \tag{3.8}$$

In a similar way we obtain for spherical coordinates

$$\begin{aligned}
 \operatorname{div} \mathbf{A} &= \frac{\partial \mathbf{A}_\rho}{\partial \rho} + \frac{2\mathbf{A}_\rho}{\rho} + \frac{1}{\rho} \frac{\partial \mathbf{A}_\vartheta}{\partial \vartheta} + \frac{\cot \vartheta}{\rho} \mathbf{A}_\vartheta \\
 &\quad + \frac{1}{\rho \sin \vartheta} \frac{\partial \mathbf{A}_\varphi}{\partial \varphi} \\
 &= \frac{1}{\rho^2} \frac{\partial}{\partial \rho} (\rho^2 \mathbf{A}_\rho) + \frac{1}{\rho \sin \vartheta} \frac{\partial}{\partial \vartheta} (\mathbf{A}_\vartheta \sin \vartheta) \\
 &\quad + \frac{1}{\rho \sin \vartheta} \frac{\partial \mathbf{A}_\varphi}{\partial \varphi}.
 \end{aligned} \tag{3.9}$$

6. Examples and the theorem of Gauss

As an illustration of the application of the divergence in a physical example we shall use the electric and magnetic fields. The first one originates from charges only, the charges being its sources. The relation between the density of charge ρ and field \mathbf{E} is

$$\operatorname{div} \mathbf{E} = \rho. \quad (3.10)$$

On the other hand, there exist no magnetic charges, so that there are no sources, positive or negative, in the magnetic field \mathbf{H} . We have here

$$\operatorname{div} \mathbf{H} = 0. \quad (3.11)$$

Let us now form the integral $\int \operatorname{div} \mathbf{A} dV$ over a finite region. This region can be divided into an infinite number of infinitesimal parts, for each of which the surface integrals can be evaluated

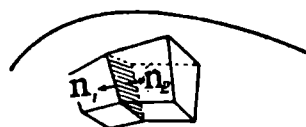


Fig. 21. Derivation of the theorem of Gauss

as above. Each surface element which is in the interior of the original region appears in two integrals (of the two adjoining parts), and \mathbf{A}_n has opposite signs in these two cases. Those integrals therefore cancel out one another and we obtain the "theorem of Gauss"

$$\int \operatorname{div} \mathbf{A} dV = \int \mathbf{A}_n df. \quad (3.12)$$

Here the first integral is to be taken over the interior, and the second one over the surface of the region.

7. Curl

Let C be a closed plane curve in the vector field. Along this curve we shall form the integral of the scalar product of the field vector and the element ds of the curve. Naturally we have to determine (arbitrarily) the direction in which we shall integrate along the curve. This integral is independent of the coordinate system.

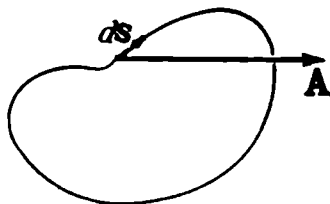


Fig. 22. Line element and vector

If \mathbf{A}_s is the vector component in the direction of the tangent to the curve and ds the length of ds , of an element of the curve, the value of the integral is $\oint \mathbf{A}_s ds$. If the curve is now subjected to a limiting process where its length tends to 0, the value of the integral becomes small of the second order. For on one hand the length of the curve becomes small, and on the other hand the differences of the field vector on different parts of the curve are small. Dividing the integral by the area f of the plane surface enclosed by the curve, we shall therefore obtain a finite limit. This limit is called the “curl” of the vector \mathbf{A} :

$$\text{curl } \mathbf{A} = \lim_{f \rightarrow 0} \frac{1}{f} \oint \mathbf{A}_s ds. \quad (3.13)$$

Now let us take any surface (not necessarily plane) which is limited by a finite curve C , and let us divide this surface into small parts, which can be considered as plane. For each one of these parts we shall form the integral as discussed above. If we add all these integrals, those parts which correspond to arcs of curves in the interior of the surface cancel out one another. Only the

integral over C remains and we obtain

$$\int \text{curl } \mathbf{A} \, df = \int \mathbf{A}_s \, ds, \quad (3.14)$$

since the line integral around any one of the infinitesimal parts is equal to $\text{curl } \mathbf{A} \, df$ corresponding to (3.13). This is the “theorem of Stokes”. The nature of the curl is determined by the orientation of a plane in space, thus the curl can not be a scalar. A simple consideration shows that the values corresponding to different orientations of the curve are determined by three numbers. We shall take a triangle (Fig. 24) which may have an

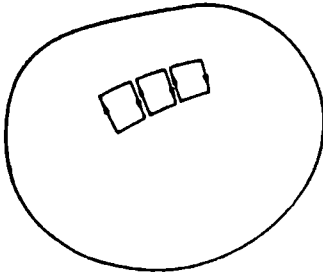


Fig. 23. Derivation of the theorem of Stokes

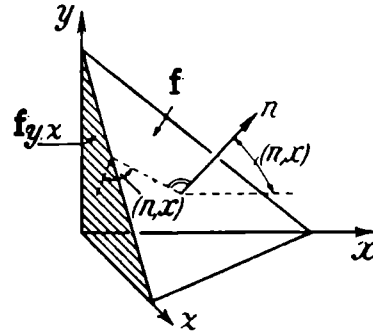


Fig. 24. Geometrical representation of the curl

arbitrary orientation in a x, y, z system as a surface element. The areas of its projections on the three coordinate-planes are $f_{yx} = f \cos (n, x) \dots$, where n is the normal to the plane of the triangle. For infinitesimal dimensions, we have:

$$f \cdot \text{curl}_{\text{triangle}} \mathbf{A} = \int \mathbf{A}_s \, ds \text{ over the three sides of the triangle,}$$

$$f_{yx} \cdot \text{curl}_{yx} \mathbf{A} = \int \mathbf{A}_s \, ds \text{ over one side of the triangle and two axes.}$$

If we form $f_{zx} \text{curl}_{zx} \mathbf{A}$ and $f_{xy} \text{curl}_{xy} \mathbf{A}$ in the same way and add the three expressions together, we have taken every side of the triangle once, while the axes occur twice each and in opposite directions. Thus we have

$$f \text{curl}_{\text{triangle}} \mathbf{A} = f_{xy} \text{curl}_{xy} \mathbf{A} + f_{yz} \text{curl}_{yz} \mathbf{A} + f_{zx} \text{curl}_{zx} \mathbf{A}$$

or

$$\begin{aligned} \text{curl}_{\text{triangle}} \mathbf{A} &= (\text{curl}_{xy} \mathbf{A}) \cos(nz) + (\text{curl}_{yz} \mathbf{A}) \cos(nx) \\ &+ (\text{curl}_{zx} \mathbf{A}) \cos(ny). \end{aligned} \quad (3.15)$$

so that the value of the curl is known for any direction if we know the three components of $\text{curl}_{xy} \mathbf{A} \dots$. The curl is determined by three values, like a vector. Nevertheless the curl is not a vector since its orientation in space is not given by a line but by the orientation of a plane. It can, however, be represented by a vector, the length of which is equal to the absolute value of the curl, and the direction of which is normal to the plane of rotation. For this purpose we require a relation between the vector and the direction of rotation. Usually one uses the first three fingers of the right hand, keeping them perpendicular to one another like the axes of a coordinate system. Then the definition that the vector in the direction of the first finger corresponds to a

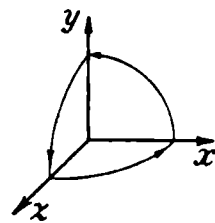


Fig. 25. Correspondence between the direction of a vector and the direction of rotation in a plane

rotation from the second to the third finger is used. Henceforth, we shall write $\text{curl}_z \mathbf{A}$ not $\text{curl}_{xy} \mathbf{A}$ as in (3.15). Equation (3.15) then becomes identical in form with (3.1) which is valid for the calculation of vector components.

8. Physical significance of the curl

To illustrate the physical significance of the curl let us consider a rotating rigid circular disk of radius r . We shall form the integral $\oint \mathbf{v}_t \cdot d\mathbf{s}$ along the edge of the disk. The velocity \mathbf{v} has the tangential direction and its absolute value is ωr , where ω is the angular velocity. Thus the integral is equal to $\omega r 2r\pi$ and the expression $f^{-1} \oint \mathbf{v}_t \cdot d\mathbf{s}$ which corresponds to the curl is equal to 2ω . This means that the angular velocity is of the same nature as a curl. For another interpretation of the curl we can use the example of the mechanical work done in moving a particle along a closed curve. If \mathbf{A} is a field of force, $\oint \mathbf{A} \cdot d\mathbf{s}$ is the work which the force does if the point upon which the force is acting moves along the curve. This definition is particularly useful for explaining the relation between electric current and magnetic field. We must keep in mind that this work depends on the orientation of the curve. The vector character appears only in this dependence on the orientation; the work itself is a scalar.

9. Mathematical expression of the curl

We shall first calculate the curl in rectangular coordinates. As a closed curve we use a rectangle which lies in the field. We designate the average of \mathbf{A}_z for the first side of the rectangle by \mathbf{A}_{z0} . Then this average on side

3 has the value $\mathbf{A}_{z0} + \Delta y \partial \mathbf{A}_z / \partial y$, if Δy and Δz are small enough. We find \mathbf{A}_{y0} and $\mathbf{A}_{y0} + \Delta z \partial \mathbf{A}_y / \partial z$ for the average of the tangential components on sides 2 and 4. The integral over the whole curve in the direction corresponding to a rotation from the y - to the z -axis is thus,

$$\int \mathbf{A}_s ds = + \mathbf{A}_{y0} \Delta y + \left(\mathbf{A}_{z0} + \frac{\partial \mathbf{A}_z}{\partial y} \Delta y \right) \Delta z - \left(\mathbf{A}_{y0} + \frac{\partial \mathbf{A}_y}{\partial z} \Delta z \right) \Delta y - \mathbf{A}_{z0} \Delta z.$$

Dividing by $\Delta z \Delta y$ and proceeding to the limit, we obtain

$$\text{curl}_x \mathbf{A} = \frac{\partial \mathbf{A}_z}{\partial y} - \frac{\partial \mathbf{A}_y}{\partial z} \quad (3.16)$$

(According to our rule concerning signs, the positive x -axis is directed toward us in Fig. 26). By interchanging the subscripts we obtain

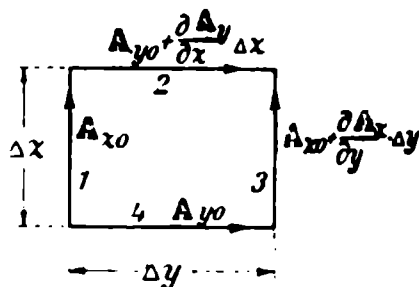


Fig. 26. Derivation of the formula for the curl

$$\text{curl}_y \mathbf{A} = \frac{\partial \mathbf{A}_x}{\partial z} - \frac{\partial \mathbf{A}_z}{\partial x} \quad (3.16a)$$

$$\text{curl}_z \mathbf{A} = \frac{\partial \mathbf{A}_y}{\partial x} - \frac{\partial \mathbf{A}_x}{\partial y}.$$

For the calculation of the curl in any other coordinate system, we must again select the curve according to the

system. As an example, in cylindrical coordinates Fig. 27 shows that

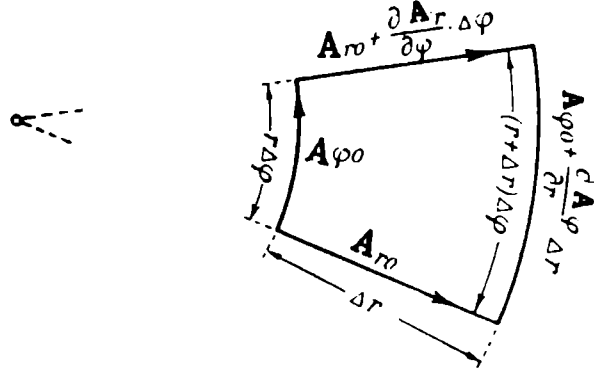


Fig. 27. Calculation of the curl in cylindrical coordinates

$$\begin{aligned} \int \mathbf{A} \cdot d\mathbf{s} &= \mathbf{A}_{r0} \Delta r + \left(\mathbf{A}_{\varphi 0} + \frac{\partial \mathbf{A}_{\varphi}}{\partial r} \Delta r \right) (r + \Delta r) \Delta \varphi \\ &\quad - \left(\mathbf{A}_{r0} + \frac{\partial \mathbf{A}_r}{\partial \varphi} \Delta \varphi \right) \Delta r - \mathbf{A}_{\varphi 0} r \Delta \varphi \end{aligned}$$

and

$$\begin{aligned} \text{curl}_r \mathbf{A} &= \frac{\partial \mathbf{A}_{\varphi}}{\partial r} + \frac{\mathbf{A}_{\varphi}}{r} - \frac{1}{r} \frac{\partial \mathbf{A}_r}{\partial \varphi} \\ &= \frac{1}{r} \frac{\partial (r \mathbf{A}_{\varphi})}{\partial r} - \frac{1}{r} \frac{\partial \mathbf{A}_r}{\partial \varphi}. \end{aligned} \quad (3.17)$$

The other components are

$$\text{curl}_{\varphi} \mathbf{A} = \frac{\partial \mathbf{A}_r}{\partial z} - \frac{\partial \mathbf{A}_z}{\partial r} \quad (3.17a)$$

$$\text{curl}_z \mathbf{A} = \frac{1}{r} \frac{\partial \mathbf{A}_z}{\partial \varphi} - \frac{\partial \mathbf{A}_{\varphi}}{\partial z}.$$

In spherical coordinates we have

$$\begin{aligned}\operatorname{curl}_\rho \mathbf{A} &= \frac{1}{\rho \sin \vartheta} \left(\frac{\partial \mathbf{A}_\vartheta}{\partial \varphi} - \frac{\partial}{\partial \vartheta} (\mathbf{A}_\varphi \sin \vartheta) \right) \\ \operatorname{curl}_\varphi \mathbf{A} &= \frac{1}{\rho} \frac{\partial \mathbf{A}_\rho}{\partial \vartheta} - \frac{1}{\rho} \frac{\partial (\rho \mathbf{A}_\vartheta)}{\partial \rho} \\ \operatorname{curl}_\vartheta \mathbf{A} &= \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho \mathbf{A}_\varphi) - \frac{1}{\sin \vartheta} \frac{\partial \mathbf{A}_\rho}{\partial \varphi} \right].\end{aligned}\tag{3.18}$$

As an example we shall deduce the ρ -component. The element of surface perpendicular to the radius ρ is bounded by two arcs of meridians with lengths $\rho \Delta \vartheta$ and two arcs of parallels with lengths $\rho \sin \vartheta \Delta \varphi$ and $\rho \sin (\vartheta + \Delta \vartheta) \Delta \varphi$ (Fig. 28). We have

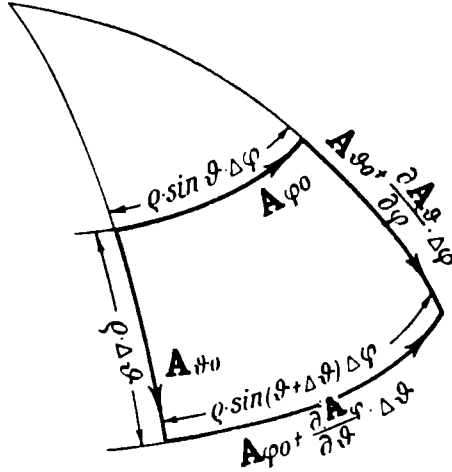


Fig. 28. Calculation of the curl in spherical coordinates

$$\begin{aligned}\int \mathbf{A}_\rho ds &= - \left(\mathbf{A}_\varphi + \frac{\partial \mathbf{A}_\varphi}{\partial \vartheta} \Delta \vartheta \right) \rho \sin (\vartheta + \Delta \vartheta) \Delta \varphi \\ &\quad + \left(\mathbf{A}_\vartheta + \frac{\partial \mathbf{A}_\vartheta}{\partial \varphi} \Delta \varphi \right) \rho \Delta \vartheta \\ &\quad + \mathbf{A}_\rho \rho \sin \vartheta \Delta \varphi - \mathbf{A}_\rho \rho \Delta \vartheta.\end{aligned}$$

Using the development $\sin(\vartheta + \Delta\vartheta) = \sin \vartheta + \Delta\vartheta \cos \vartheta$, dividing by the area $\rho^2 \sin \vartheta \Delta\vartheta \Delta\varphi$ and taking the limit for $\Delta\varphi \rightarrow 0$, $\Delta\vartheta \rightarrow 0$, we obtain the first of the formulae (3.18).

10. Divergence and curl of a gradient

Some combinations of our differential expressions are very important for physical applications. For example, we can investigate the special properties of the divergence and the curl of a vector field which is itself the gradient of a scalar or the curl of another vector.

Suppose we have $\mathbf{A} = \text{grad } \Phi$. Forming the divergence of this in rectangular coordinates, we obtain the following simple expression,

$$\begin{aligned} \text{div grad } \Phi &= \frac{\partial}{\partial x} \left(\frac{\partial \Phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \Phi}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial \Phi}{\partial z} \right) \\ &= \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}. \end{aligned} \quad (3.19)$$

This expression, which is named after Laplace, is most frequently encountered in the differential equations of physics. We designate it by $\Delta\Phi$ (or $\nabla^2\Phi$). Its representations in rectangular, cylindrical and spherical coordinates are

$$\Delta\Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \quad (3.20)$$

$$\Delta\Phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{\partial^2 \Phi}{\partial z^2} \quad (3.21)$$

$$\begin{aligned}\Delta\Phi = \frac{1}{\rho^2} \frac{\partial}{\partial\rho} \left(\rho^2 \frac{\partial\Phi}{\partial\rho} \right) + \frac{1}{\rho^2 \sin\vartheta} \frac{\partial}{\partial\vartheta} \left(\frac{\partial\Phi}{\partial\vartheta} \sin\vartheta \right) \\ + \frac{1}{\rho^2 \sin^2\vartheta} \frac{\partial^2\Phi}{\partial\varphi^2}.\end{aligned}\tag{3.22}$$

If we calculate the curl of grad Φ in rectangular coordinates we obtain

$$\begin{aligned}\text{curl}_x \text{grad } \Phi &= \frac{\partial}{\partial y} \left(\frac{\partial\Phi}{\partial z} \right) - \frac{\partial}{\partial z} \left(\frac{\partial\Phi}{\partial y} \right) \\ &= \frac{\partial^2\Phi}{\partial y \partial z} - \frac{\partial^2\Phi}{\partial z \partial y} = 0,\end{aligned}$$

and the same is true for the other components. It is quite clear that this result is independent of the coordinate system and so we have, in general,

$$\text{curl grad } \Phi = 0.\tag{3.23}$$

11. The divergence and the curl of a curl

If $\mathbf{B} = \text{curl } \mathbf{A}$ we see immediately that $\text{div } \mathbf{B} = 0$. For as we have seen earlier $\int \text{curl } \mathbf{A} \, df$ has the same value for two surfaces which are limited by the same curve (Fig. 29). These two surfaces enclose a region of space, and the vectors representing the curl on the surfaces are directed towards the interior for one surface and towards the exterior for the other. Since both integrals are equal, the outward flow is equal to the inward flow, and there is

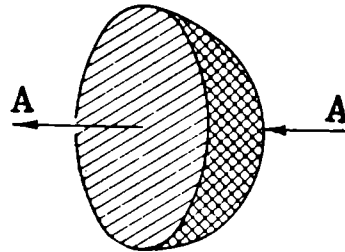


Fig. 29. Proof that
 $\text{curl } \mathbf{A} = 0$

thus no source or sink so that the divergence is equal to zero. This can easily be verified by calculation in any coordinate system. The general formula is

$$\operatorname{div} \operatorname{curl} \mathbf{A} = 0. \quad (3.24)$$

We shall finally compute the curl of a curl in rectangular coordinates:

$$\begin{aligned} \operatorname{curl}_x (\operatorname{curl} \mathbf{A}) &= \frac{\partial}{\partial y} (\operatorname{curl}_z \mathbf{A}) - \frac{\partial}{\partial z} (\operatorname{curl}_y \mathbf{A}) \\ &= \frac{\partial}{\partial y} \left(\frac{\partial \mathbf{A}_y}{\partial x} - \frac{\partial \mathbf{A}_x}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial \mathbf{A}_x}{\partial z} - \frac{\partial \mathbf{A}_z}{\partial x} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial \mathbf{A}_y}{\partial y} + \frac{\partial \mathbf{A}_z}{\partial z} \right) - \frac{\partial^2 \mathbf{A}_x}{\partial y^2} - \frac{\partial^2 \mathbf{A}_x}{\partial z^2}. \end{aligned}$$

By addition of

$$\frac{\partial^2 \mathbf{A}_x}{\partial x^2} - \frac{\partial^2 \mathbf{A}_x}{\partial x^2}$$

we obtain

$$\operatorname{curl}_x (\operatorname{curl} \mathbf{A}) = \frac{\partial}{\partial x} (\operatorname{div} \mathbf{A}) - \Delta \mathbf{A}_x$$

or

$$\operatorname{curl} \operatorname{curl} \mathbf{A} = \operatorname{grad} \operatorname{div} \mathbf{A} - \Delta \mathbf{A}. \quad (3.25)$$

It is remarkable that the operator Δ appears in this formula. This time it operates on a vector, i.e., the single components of the vector.

We can also express the operators in (3.25) in any coordinate system. Then, however, we must remember that in this equation \mathbf{A} stands for the Cartesian compo-

nents \mathbf{A}_r , \mathbf{A}_θ , \mathbf{A}_z . The computation is more complicated if we decompose \mathbf{A} itself into the components corresponding to another system, for example, if we use \mathbf{A}_r , \mathbf{A}_φ , \mathbf{A}_z in cylindrical coordinates.

12. Physical significance of the operator Δ

Since $\Delta\Phi$ is very important in the differential equations of physics we shall try to find a simple interpretation of this operator.

At a certain point 0 of the scalar field, Φ may have the value Φ_0 . We construct a cube around 0, the sides of which are of length a . For the average value $\bar{\Phi}$ of Φ in this cube we have

$$\bar{\Phi}a^3 = \iiint_{-a/2}^{+a/2} \Phi \, dx \, dy \, dz.$$

The Taylor representation gives for any point x, y, z

$$\begin{aligned} \Phi = & \Phi_0 + \left(\frac{\partial\Phi}{\partial x}\right)_0 x + \left(\frac{\partial\Phi}{\partial y}\right)_0 y + \left(\frac{\partial\Phi}{\partial z}\right)_0 z \\ & + \frac{1}{2} \left[\left(\frac{\partial^2\Phi}{\partial x^2}\right)_0 x^2 + \left(\frac{\partial^2\Phi}{\partial y^2}\right)_0 y^2 + \left(\frac{\partial^2\Phi}{\partial z^2}\right)_0 z^2 \right] \\ & + \left(\frac{\partial^2\Phi}{\partial x\partial y}\right)_0 xy + \left(\frac{\partial^2\Phi}{\partial y\partial z}\right)_0 yz + \left(\frac{\partial^2\Phi}{\partial z\partial x}\right)_0 zx + \dots \end{aligned}$$

If we integrate from $-a/2$ to $a/2$, the odd functions cancel out while from the other terms we obtain

$$\iiint_{-a/2}^{+a/2} x^2 \, dx \, dy \, dz = \frac{a^6}{12}, \text{ etc.}$$

We have finally

$$\overline{\Phi}a^3 = \Phi_0a^3 + \frac{a^5}{24} \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right)_0$$

or

$$\overline{\Phi} - \Phi_0 = \frac{a^2}{24} \Delta_0 \Phi.$$

The quantity $\Delta\Phi$ therefore is a measure of the difference between the value of the scalar Φ at a point (0) and the average value of Φ in an infinitesimal neighbourhood of this point.

CHAPTER IV

THE SIMPLEST PARTIAL DIFFERENTIAL EQUATIONS OF PHYSICS

1. The potential equation

In many important cases, the difference between the local value of a function and the average in the neighbourhood is essential for describing the space-time development of the field. Thus the operator Δ appears very frequently in differential equations.

a. There exist fields such that the value of Φ at any point is equal to the average value for the neighbourhood. Then we have the Laplace equation

$$\Delta\Phi = 0. \quad (4.1)$$

An illustration is given by the gravitational field in a region free of mass. The force \mathbf{K} which acts on a particle of unit mass in such a field is a function of the position of the particle only. In general, (positive or negative) work must be done to move the particle in the field. If the path along which the particle is moved is closed, however, the total work done is equal to zero. According to III 8, this can be expressed by

$$\text{curl } \mathbf{K} = 0. \quad (4.2)$$

This system of three equations (one for each component) can be reduced to one equation if we use (3.23). According to (3.23) equation (4.2) is satisfied by $\mathbf{K} = \text{grad } \Phi$. Now the gravitational field has the further

property that its sources appear only where there is mass. Thus in empty space there are no sources in the gravitational field and we have

$$\operatorname{div} \mathbf{K} = \operatorname{div} \operatorname{grad} \Phi = \Delta \Phi = 0. \quad (4.3)$$

We shall show later that this equation is identical with Newton's law of gravitation (VII, 1).

We shall find fields of the same kind in electrostatics, in magnetostatics, and in fluid dynamics. Wherever there is a vector field the curl of which is equal to zero, we can reduce the number of unknowns from three to one by representing the field as the gradient of a scalar. Φ is called the "potential" of the vector field. The difference of potential between two points 0 and 1 is equal to

$$\Phi_1 - \Phi_0 = \int_0^1 \mathbf{K}_s ds$$

where the path of integration is any arbitrary curve joining the points 0 and 1. If \mathbf{K} is a force, the integral represents the work to be done along the path, and in this case Φ is called the "potential energy" (of a unit mass in the case of the force of gravity).

b. There exist physical conditions such that the value of a scalar at a point differs from the average value in the neighbourhood. We then have the Poisson equation

$$\Delta \Phi = P, \quad (4.4)$$

where P is a given function of the space. When the force is due to gravity, the density of mass ρ is proportional to the magnitude of the divergence of the force \mathbf{K} . Then we have,

$$\Delta\Phi = K\rho. \quad (4.5)$$

(K = gravitational constant). The density of charge plays the same role in electrostatics, while there is no corresponding phenomenon in magnetostatics.

2. The equation of heat conduction

c. There are several physical processes in which the value of Φ at any point may be different from its average value in the neighbourhood at some time, but not for all values of time. In these cases this difference causes a tendency towards equalization with time. The smaller the local value than the average, the greater is its rate of increase with time. This is expressed mathematically by (k is factor of proportionality):

$$\Delta\Phi = k \frac{\partial\Phi}{\partial t}. \quad (4.6)$$

The conduction of heat is the classical example of such a tendency towards equalization. Equation (4.6) is thus usually called the equation of heat conduction. In this particular example Φ is the temperature T . The heat flow $\mathbf{q} = -\lambda \text{ grad } T$ (see 3.3) causes an amount of heat $dV \text{ div } \mathbf{q}$ to leave any volume element dV in unit time, and consequently the temperature decreases at the rate $dV \ c\rho(\partial T/\partial t)$, where c is the specific heat and ρ is the density of the material. It follows then that

$$\text{div } \mathbf{q} = -\lambda\Delta T = -c\rho \frac{\partial T}{\partial t}$$

or

$$\Delta T = \frac{c\rho}{\lambda} \frac{\partial T}{\partial t}. \quad (4.7)$$

$\lambda/c\rho$ is called "heat conductivity".

The laws of diffusion are expressed by the same differential equation. In this case Φ is the density of the diffusing material. This density changes at a rate proportional to the difference between its local value and the average value. The speed of diffusion is also proportional to the rate of change of density, as in the case of heat flow.

From the physical standpoint, friction in liquids and gases also is a similar phenomenon. This analogy is important in the kinetic theory of gases and supplies a relation between the constants of heat conduction and friction of the materials in question. These latter problems are rather difficult from the mathematical point of view, however, since these phenomena of friction cannot be described by a scalar or a vector, but require the concept of a tensor. Thus we cannot consider this problem here.

3. Wave equation

d. Finally, the deviation of a local value from the average in the neighbourhood can also have the same effect as a displacement from the position of equilibrium in the phenomena of oscillation (II 12). A force can appear which tends to give the variable the average value and so to restore the system to the equilibrium position. Thus we obtain a motion for which the dependence on time is expressed by a quantity analogous to the acceleration, namely the second derivative. This equation must be of the form

$$\Delta\Phi = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2}. \quad (4.8)$$

Here $1/c^2$ is written as a square since it is a positive factor. The value of Φ is "accelerated" and hence increases at those places where it is smaller than the average. The essential properties of this differential equation can be found by comparing it with mechanical oscillations (II 12). As a result of the acceleration, $\partial\Phi/\partial t$ is not equal to zero in the equilibrium position as it is in (4.6). The system "moves" through the position of equilibrium and the same process begins anew. The variable Φ oscillates about its value in the equilibrium position. (4.8) is the general wave equation. The constant c is the speed of propagation, as we shall see in (V 5).

4. Differential equations of perfect fluids

The hydrodynamical field can be described by the hydrostatic pressure p (which is a scalar) and the velocity \mathbf{v} (vector). We shall express both quantities as functions of position and time and not as functions of a fixed particle in the fluid. At a certain position in space there are different particles at different times, thus p and \mathbf{v} are functions of the time and the coordinates of a point which does not move with the fluid. We have one scalar and one vector differential equation according to the number of unknowns. The first equation expresses the theorem of the conservation of mass. The mass contained in a volume element is equal to ρdV , where ρ is the density. The mass which leaves the element in unit time is $dV \operatorname{div} (\rho \mathbf{v})$. This decreases the mass contained in the element which results in a decrease of the density of the remaining fluid in the volume element. In terms of the derivative of the density, the decrease

of mass in dV in unit time is equal to $-dV \partial\rho/\partial t$. Hence the equation of continuity of fluid dynamics has the form

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}) = 0. \quad (4.9)$$

If the fluid is incompressible ($\rho = \text{const}$), it has the simpler form

$$\operatorname{div} \mathbf{v} = 0. \quad (4.9a)$$

The vector equation of hydrodynamics contains the fundamental Newtonian law (2.3) for a single volume element dV . The mass is ρdV . Now in the Newton equation the acceleration is the derivative of the velocity of a definite particle and *not* the derivative of the velocity at a fixed point. This change of velocity is composed of two parts; first there is the change in velocity at the position of the particle, and second the change in velocity due to the change in position of the particle. The first change is expressed by the partial derivative $\partial\mathbf{v}/\partial t$, while the second is equal to

$$\frac{\partial\mathbf{v}}{\partial x} \frac{dx}{dt} + \frac{\partial\mathbf{v}}{\partial y} \frac{dy}{dt} + \frac{\partial\mathbf{v}}{\partial z} \frac{dz}{dt}. \quad (4.10)$$

In this equation, dx/dt , \dots denote the derivatives of the position of the particle with respect to the time. That means that they are identical with the components of the velocity \mathbf{v} at the point. If we denote these components by u , v , w , the three components of the expression (4.10) are

$$\begin{aligned}
& u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}, \\
& u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z}, \\
& u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}.
\end{aligned} \tag{4.10a}$$

As an abbreviation we shall write the vector symbol $(\mathbf{v} \text{ grad}) \mathbf{v}$ for the three expressions (4.10a). The force acting on the volume element dV consists of a volume force (for example the weight) $\mathbf{K} dV$ and the resultant of the pressure forces on the surface of the element, which is equal to $-dV \text{ grad } p$. We can derive this result by considering the element $dx dy dz$ in Fig. 30. The

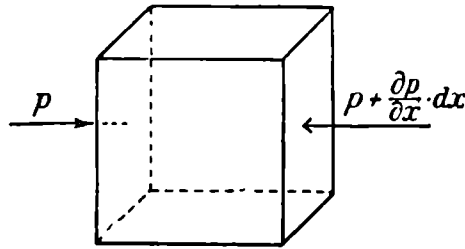


Fig. 30. Hydrostatic pressure

pressure is normal to the surface, so that the force $p dy dz$ acts in the direction of the positive x -axis and the force $-[p + (\partial p/\partial x) dx] dy dz$ in the negative direction. The resultant component in the x -direction is $-(\partial p/\partial x) dy dz$.

Thus we obtain the vector equation of fluid dynamics (Euler's equation):

$$\rho \left\{ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \text{ grad}) \mathbf{v} \right\} = \mathbf{K} - \text{grad } p. \quad (4.11)$$

If the fluid is incompressible (e.g., water), ρ is a constant, while if the fluid is compressible ρ is a given function of p for adiabatic processes, for instance. In the following we shall consider only the case $\rho = \text{const.}$

5. Vortices

Wherever possible we must reduce these four differential equations with four unknowns to one equation containing one unknown only for mathematical treatment. First we shall eliminate the pressure from (4.11) by calculating the curl of the right hand term (curl grad $p = 0$ according to (3.23)). If the volume force \mathbf{K} is derived from a potential Ω (see IV 1), it too drops out and we have

$$\frac{\partial}{\partial t} (\text{curl } \mathbf{v}) + \text{curl } \{(\mathbf{v} \text{ grad}) \mathbf{v}\} = 0. \quad (4.12)$$

We shall write this equation in Cartesian coordinates. The following calculations, however, are only made for the z -component. This equals

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left\{ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right\} \\ - \frac{\partial}{\partial x} \left\{ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right\} = 0 \end{aligned}$$

or

$$\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) + \frac{\partial u}{\partial y} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} +$$

$$\begin{aligned}
& + u \frac{\partial^2 u}{\partial x \partial y} + v \frac{\partial^2 u}{\partial y^2} + w \frac{\partial^2 u}{\partial y \partial z} \\
& - \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} - \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial w}{\partial x} \frac{\partial v}{\partial z} \\
& - u \frac{\partial^2 v}{\partial x^2} - v \frac{\partial^2 v}{\partial x \partial y} - w \frac{\partial^2 v}{\partial x \partial z} = 0
\end{aligned}$$

or, in a clearer form,

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \\
& + \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \frac{\partial v}{\partial z} = 0.
\end{aligned}$$

By use of

$$\operatorname{div} \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (4.9a)$$

and addition of

$$+ \frac{\partial w}{\partial y} \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y} \frac{\partial w}{\partial x} + \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \frac{\partial w}{\partial z} - \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \frac{\partial w}{\partial z},$$

we finally obtain the simpler form

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \\
& - \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \frac{\partial w}{\partial z} - \left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \right) \frac{\partial w}{\partial x} \\
& - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right) \frac{\partial w}{\partial y} = 0.
\end{aligned} \quad (4.13)$$

This is an equation for the three components of the curl. In addition we have two other equations. Now, the first term in (4.13) is equal to the derivative with respect to time of the curl at a fixed particle (according to page 70) and (4.13) expresses the fact that this derivative is equal to zero if the curl itself is equal to zero. The physical significance of the curl is the "vortex" (see III 8). Thus we see that a particle of the fluid which is not rotating at a certain instant will never rotate. This is true as long as our assumptions of perfect fluid and conservative volume forces hold. If a fluid particle moves in a curved path it does not necessarily mean that the particle itself rotates. If, as shown in Fig. 31, we produce

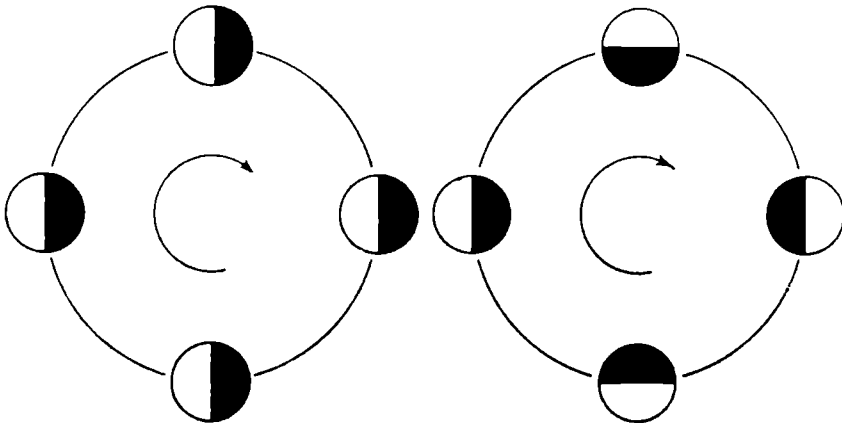


Fig. 31a. Circular motion without rotation (Ferris wheel)

Fig. 31b. Circular motion with rotation (moon circling the earth)

a vortex at the origin, the other particles which were not rotating before and which are moved only by the pressure, will remain without vorticity. They move as shown in Fig. 31a, not as in Fig. 31b.

6. Potential flow

While it is quite possible to treat vortex flows from (4.13), we shall only consider here flows without vorticity, since these represent a very clear example of the application of differential equations. If $\text{curl } \mathbf{v} = 0$ everywhere at a certain instant, the same will be true for any time. It follows then from (3.23) (and in analogy to (4.3)), that \mathbf{v} is the gradient of a scalar Φ which is called the "velocity potential"

$$\mathbf{v} = \text{grad } \Phi. \quad (4.14)$$

From the equation of continuity (4.9a) it follows that

$$\text{div grad } \Phi = \Delta \Phi = 0. \quad (4.15)$$

Furthermore we have the following boundary conditions: a. at a rigid wall, the fluid cannot penetrate into the wall and thus there is no velocity component normal to the wall; b. the pressure is constant at a free surface.

By a simple calculation we obtain

$$\begin{aligned} u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} &= \frac{\partial \Phi}{\partial x} \frac{\partial}{\partial x} \left(\frac{\partial \Phi}{\partial x} \right) \\ &+ \frac{\partial \Phi}{\partial y} \frac{\partial}{\partial x} \left(\frac{\partial \Phi}{\partial y} \right) + \frac{\partial \Phi}{\partial z} \frac{\partial}{\partial x} \left(\frac{\partial \Phi}{\partial z} \right) \\ &= \frac{1}{2} \frac{\partial}{\partial x} \left\{ \left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial y} \right)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right\} \end{aligned} \quad (4.15a)$$

or $(\mathbf{v} \text{ grad}) \mathbf{v} = \frac{1}{2} \text{grad } v^2$. If we replace \mathbf{v} by $\text{grad } \Phi$ in (4.11) and use (4.15a) we obtain

$$\begin{aligned} \rho \left[\text{grad } \frac{\partial \Phi}{\partial t} + \frac{1}{2} \text{grad } \left\{ \left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial y} \right)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right\} \right] \\ = - \text{grad } \Omega - \text{grad } p. \end{aligned}$$

We integrate this once, obtaining

$$\rho \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2} \left\{ \left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial y} \right)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right\} \right] + \Omega + p = \text{const.} \quad (4.16)$$

Hence we find p if we know Φ and if the constant is determinable from the given values of Φ and p at some point.

Flows without vorticity (potential flows) are only a special simplified case of real flows. However, a great number of practical problems can be solved by a potential flow approximation. Airfoil theory and related questions in the theory of airplanes, propellers, and turbines as well as surface waves and fluid jets are treated in just this way.

7. Differential equations of electrodynamics

The electromagnetic field in empty space is described by the two vectors of the electric and magnetic field (\mathbf{E} and \mathbf{H}). The differential equations of this field can be derived according to Maxwell from two empirical facts. The first is Faraday's law of induction: the rate of change $\partial \mathbf{H} / \partial t$ of the magnetic field is associated with an electric field \mathbf{E} such that

$$\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} = - \text{curl } \mathbf{E}. \quad (4.17)$$

The constant c is a factor of proportionality and has the dimension of a velocity, if \mathbf{E} and \mathbf{H} are defined by means of the forces in the electrostatic and magnetostatic field. This factor, which we shall again meet in (4.18),

is just equal to the velocity of light. This fact was one of the first which led to the electromagnetic theory of light.

The second empirical fact which was generalized into a fundamental law by Maxwell concerns the generation of a magnetic field by an electric current. The connection between both is an example of the concept of the curl. Originally the differential equation is

$$\frac{1}{c} \mathbf{J} = \text{curl } \mathbf{H}$$

where c is the same quantity as in (4.17) if the density \mathbf{J} of current is also defined in the electrostatic system. Maxwell, with reasoning based on certain properties of alternating currents and with the intention of finding an analogy between light and electromagnetic fields, completed this equation by introducing the concept of a "displacement current" $\partial \mathbf{E} / \partial t$. This current should be capable of generating a magnetic field in the same way as an ordinary current \mathbf{J} and analogous to $\partial \mathbf{H} / \partial t$ in (4.17).

The differential equation now becomes

$$\frac{1}{c} \left(\frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \right) = \text{curl } \mathbf{H}. \quad (4.18)$$

According to the electronic theory, one can write the current density \mathbf{J} as the product of the density of charge ρ by the velocity \mathbf{v} of the charge.

From (4.17) and (3.24) we obtain

$$\frac{\partial}{\partial t} (\text{div } \mathbf{H}) = 0$$

which means that $\text{div } \mathbf{H}$ is independent of the time. According to our experience no magnetic sources (charges) exist, thus we have

$$\text{div } \mathbf{H} = 0. \quad (4.19)$$

If we take the divergence of both sides of (4.18) we obtain the "continuity equation"

$$\frac{\partial}{\partial t} (\text{div } \mathbf{E}) + \text{div } \mathbf{J} = 0. \quad (4.20)$$

The density of charge ρ is defined by the following equation,

$$\text{div } \mathbf{E} = \rho. \quad (4.21)$$

With $\mathbf{J} = \rho \mathbf{v}$, (4.21) takes the form (4.9) of the continuity equation of hydrodynamics. This analogy is already expressed in the word "current".

8. The field equation in material bodies

Hitherto we have only talked about the field in empty space under the influence of electric currents and charges. From the standpoint of the electronic theory, this case covers all possibilities, for this theory only considers currents and charges and explains the influence which these bodies have upon the field by assuming distributions of charges and currents in the material bodies. The exact manner in which these currents and charges are distributed in the bodies is the object of special theories which lack of space does not allow us to consider here.

The older Maxwell theory avoids these difficulties by

assigning a generalized form to the fundamental equation whereby they contain certain constants of the material. These general equations containing the permeability μ , the dielectric constant ϵ and the conductivity σ , are

$$\begin{aligned} \frac{\mu}{c} \frac{\partial \mathbf{H}}{\partial t} &= - \operatorname{curl} \mathbf{E}, & \operatorname{div} (\mu \mathbf{H}) &= 0 \\ \frac{1}{c} \left[\epsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E} \right] &= \operatorname{curl} \mathbf{H}, & \operatorname{div} (\epsilon \mathbf{E}) &= 0. \end{aligned} \quad (4.22)$$

We shall now try to simplify the system of the Maxwell equations by the elimination of variables. For this purpose, however, we shall again consider the electromagnetic field in empty space.

9. Energy theorem

We can obtain a simple relation if we form the scalar product of (4.18) with \mathbf{E} and of (4.17) with \mathbf{H} and use the relation

$$\mathbf{E} \cdot \operatorname{curl} \mathbf{H} - \mathbf{H} \cdot \operatorname{curl} \mathbf{E} = - \operatorname{div} (\mathbf{E} \times \mathbf{H})$$

(This relation can be easily verified by using rectangular coordinates). We then obtain the following equation

$$\frac{1}{2c} \frac{\partial}{\partial t} (\mathbf{H}^2 + \mathbf{E}^2) + \frac{1}{c} \mathbf{E} \cdot \mathbf{J} + \operatorname{div} (\mathbf{E} \times \mathbf{H}) = 0. \quad (4.23)$$

This equation is an energy theorem: $\frac{1}{2} \mathbf{E}^2$ is the electric energy, $\frac{1}{2} \mathbf{H}^2$ the magnetic energy per unit volume, $\mathbf{E} \cdot \mathbf{J}$ is the Joule heat which is produced in a unit volume in unit time, $c \operatorname{div} (\mathbf{E} \times \mathbf{H})$ is the energy leaving the

volume in unit time and thus it follows that $c\mathbf{E} \times \mathbf{H}$ is the flow of energy through the surface in unit time. This vector is called the "Poynting vector".

10. Electromagnetic waves

Whenever we have $\rho = 0$ and $\mathbf{J} = 0$ the Maxwell equations can be reduced to a single equation containing only one unknown. If in

$$\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} = - \text{curl } \mathbf{E}$$

$$\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \text{curl } \mathbf{H}$$

we take the curl of the first equation and the derivative with respect to time of the second and then multiply the latter with $1/c$ and add, we obtain

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = - \text{curl curl } \mathbf{E}.$$

Using (3.25) and $\text{div } \mathbf{E} = 0$, we obtain the wave equation

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \Delta \mathbf{E}. \quad (4.24)$$

Each component of \mathbf{E} satisfies this equation. The same equation is valid for \mathbf{H} . This result is of great theoretical significance, since it proves the existence of electromagnetic waves and shows that their velocity of propagation is equal to the velocity of light. The propagation of light can therefore be treated by the Maxwell theory.

If we eliminate \mathbf{H} from (4.22) in the same way we obtain

$$\frac{\epsilon\mu}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{\sigma\mu}{c} \frac{\partial \mathbf{E}}{\partial t} = \Delta \mathbf{E}. \quad (4.25)$$

In this differential equation there is a term which corresponds to a wave having a velocity $c/(\epsilon\mu)^{1/2}$ and another term which corresponds to a damping of the oscillations. If we compare this term with (4.7) we see that the damping is greater the smaller the conductivity of the material.

11. Electromagnetic potentials

We can obtain another simplification of the Maxwell equations by introducing the concept of potentials. In this respect the static cases present nothing new, since if $\text{curl } \mathbf{E} = 0$ or $\text{curl } \mathbf{H} = 0$ the situation is similar to that previously considered for the gravitational field or for the (potential) flow of a fluid. Let us introduce a potential Φ such that $\mathbf{E} = -\text{grad } \Phi$. We then obtain from $\text{div } \mathbf{E} = 0$

$$\Delta \Phi = 0. \quad (4.26)$$

In the general case we can introduce a so called "vector potential" by

$$\mathbf{H} = \text{curl } \mathbf{A} \quad (4.27)$$

This is possible since $\text{div } \mathbf{H} = 0$ (see (3.24)). The introduction of \mathbf{A} does not have an immediate advantage, since \mathbf{A} is again a vector. Instead of the three original unknowns we have three others. However, if we introduce this expression of \mathbf{H} in (4.17) we obtain

$$\text{curl} \left(\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) = - \text{curl} \mathbf{E}.$$

From this we can not conclude that $\mathbf{E} = -\partial \mathbf{A}/c \partial t$. Since the curl of a gradient is always equal to zero [(3.23)], two quantities with equal curls may differ by the gradient of a scalar. Thus we have

$$\mathbf{E} = - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \text{grad} \Phi. \quad (4.28)$$

If we introduce these relations into the two other Maxwell equations (4.18) and (4.21), we obtain

$$\begin{aligned} - \frac{1}{c} \frac{\partial}{\partial t} (\text{div} \mathbf{A}) - \Delta \Phi &= \rho, \\ - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \frac{1}{c} \frac{\partial}{\partial t} \text{grad} \Phi + \frac{1}{c} \mathbf{J} \\ &= \text{grad} \text{div} \mathbf{A} - \Delta \mathbf{A}. \end{aligned} \quad (4.29)$$

Since the potentials have only to be chosen such that (4.27) and (4.28) are satisfied, we are free to prescribe another relation between the two potentials. We chose a relation that simplifies (4.29) essentially, namely

$$\frac{1}{c} \frac{\partial \Phi}{\partial t} + \text{div} \mathbf{A} = 0. \quad (4.30)$$

In this case we obtain a couple of equations from (4.29) which again are very similar to the wave equation

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta \mathbf{A} = \frac{1}{c} \mathbf{J}, \quad (4.31)$$

$$\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \Delta \Phi = \rho.$$

These four differential equations contain only four unknowns, while the original ones contained six. The importance of the symbol Δ should be noted, and also the importance of differential equations of the type we found in IV 1 to 3.

12. Boundary conditions

Let us consider a surface element f which contains a part of the boundary between two materials (Fig. 32) whose extension perpendicular to the boundary is infinitely small in comparison to its extension parallel to the boundary. If the difference of the components parallel to the boundary (E_{p1} and E_{p2} in Fig.

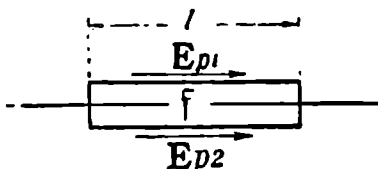


Fig. 32. The electric force at the boundary of two materials

32) were finite, the curl would be infinite. For the line integral $\oint \mathbf{E} \cdot d\mathbf{s}$ would be finite while the area f is infinitesimal. Since $\partial(\mu H)/\partial t$ cannot be infinite in (4.22), we have

$$E_{p1} = E_{p2} . \quad (4.32)$$

The same holds true for the magnetic components which are parallel to the boundary because the left hand term of the other equations (4.22) also cannot be infinite.

If we apply the two divergence equations to a prismatic region of base f (Fig. 32) and height h (normal to plane of Fig. 32), we obtain two boundary conditions for the normal components of \mathbf{E} and \mathbf{H} . Since $\text{div } \mathbf{H} = 0$, the two normal components cannot be different from each other, and thus we have

$$\mu_1 \mathbf{H}_{n1} = \mu_2 \mathbf{H}_{n2} . \quad (4.33)$$

On the other hand we have

$$\operatorname{div} (\epsilon \mathbf{E}) f h = (\epsilon_2 \mathbf{E}_{n2} - \epsilon_1 \mathbf{E}_{n1}) l h ,$$

and therefore

$$\epsilon_2 \mathbf{E}_{n2} - \epsilon_1 \mathbf{E}_{n1} = \lim_{f \rightarrow 0} \rho \frac{f}{l} = \omega . \quad (4.34)$$

ω can be different from zero, since at the boundary of the two materials there exist densities of charge so large that in the limit they may approach f/l . Charges of this kind (finite charge in an infinitesimal volume) are called “surface charges”.

CHAPTER V

SOLUTIONS BY EIGENFUNCTIONS

1. The product method

There may exist solutions of a partial differential equation which are the product of several functions, each of which depends only on one variable. This idea often allows us to reduce the solution of a partial differential equation to the solution of several ordinary ones. A solution of this kind is not general, of course, and the method can only be used if the boundary conditions satisfy certain conditions of symmetry and homogeneity. In many practical cases, however, the method is sufficiently general.

We demonstrate the method with the potential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad (5.1)$$

We try $u = XYZ$, where X is a function of x alone, Y of y alone and Z of z alone. If we denote differentiation by primes, we obtain from (5.1)

$$X''YZ + Y''XZ + Z''XY = 0 \quad (5.2)$$

After division by XYZ we obtain

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0.$$

Now X''/X is a function of x alone etc. These three functions of different variables can have the sum zero only if they are constants. Thus we have

$$\frac{X''}{X} = -\alpha, \quad \frac{Y''}{Y} = -\beta, \quad \frac{Z''}{Z} = -\gamma, \quad (5.3)$$

where the constants α, β, γ satisfy the condition

$$\alpha + \beta + \gamma = 0. \quad (5.4)$$

The negative signs in (5.3) were chosen for practical reasons.

Now the three ordinary differential equations (5.3) are to be solved; their solutions involve six constants of integration. The nature of the boundary conditions determines whether or not the constants can be determined and if they are sufficient to satisfy the boundary conditions. This also decides whether or not the method can be used at all. We shall now specialize our problem physically and formulate several boundary conditions.

2. Example: steady flow of heat

The equation $\Delta u = 0$ gives the temperature u in a body in which there is a steady distribution of heat, i.e. throughout which the temperature depends only on the position but not on time. We assume that the body is rectangular and that its edges have the lengths a, b, c . We chose a rectangular coordinate system which has its axes parallel to the edges of the body. Four sides of the body may be completely insulated so that there is no heat flowing through them. One side ($x = a$) may be subjected to a constant temperature (zero). On the remaining side ($x = 0$) the temperature may be given

by a function $u = f(y, z)$. What is the temperature in the interior of the body? We have the six boundary conditions

$$u = f(y, z) \text{ for } x = 0, u = 0 \text{ or } X = 0 \text{ for } x = a.$$

Furthermore, since the flow of heat is proportional to $\text{grad } u$ (according to (3.5)) we have

$$\frac{\partial u}{\partial y} = 0, \text{ or } Y' = 0 \text{ for } y = 0 \text{ and for } y = b \quad (5.5)$$

$$\frac{\partial u}{\partial z} = 0, \text{ or } Z' = 0 \text{ for } y = 0 \text{ and for } z = c.$$

These boundary conditions are homogeneous and linear except for the first. This means that they contain the first power of the unknown in each term.

We shall first make use of the second equation (5.3): $Y'' + \beta Y = 0$ with the boundary conditions $Y' = 0$ for $y = 0$ and $y = b$. The most general solution is

$$Y = A \sin (\beta^{\frac{1}{2}} y) + B \cos (\beta^{\frac{1}{2}} y).$$

In order to satisfy $Y' = 0$ for $y = 0$, we must have $A = 0$, and since

$$B\beta^{\frac{1}{2}} \sin (\beta^{\frac{1}{2}} b) = 0$$

(from the second condition), we have either $B = 0$ or $\sin (\beta^{\frac{1}{2}} b) = 0$. In the first case ($B = 0$) we have $Y = 0$ which is of no use. In the second case we have

$$\beta^{\frac{1}{2}} = n\pi/b \quad (n = \text{arbitrary integer}). \quad (5.6)$$

At the beginning of our computation we introduced β as an arbitrary constant. It has been determined to a

certain extent by the calculation. (5.6) gives an infinity of possible values for β . They are called the "eigenvalues". To each eigenvalue corresponds an eigenfunction $\cos(n\pi y/b)$, which satisfies the second equation in (5.3) and its boundary conditions. Moreover, each one of the infinitely many eigenfunctions multiplied by an arbitrary constant (B_n), is a solution. We now have the problem of deciding which of the eigenfunctions we can use in our case and how to determine the constants B_n .

First, however, we note that in a similar way we find that the eigenvalues

$$\gamma^{\frac{1}{2}} = m\pi/c \quad (m = \text{arbitrary integer}) \quad (5.7)$$

and the eigenfunctions $\cos(m\pi z/c)$ correspond to the equation

$$Z'' + \gamma Z = 0 \text{ with } Z' = 0 \text{ for } z = 0 \text{ and } z = c.$$

From (5.6), (5.7) and (5.4) it follows that

$$\alpha = -\left(\frac{n\pi}{b}\right)^2 - \left(\frac{m\pi}{c}\right)^2 \quad (5.8)$$

If we introduce the above into $X'' + \alpha X = 0$, we obtain

$$X'' - \left[\left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{c}\right)^2\right]X = 0$$

with the boundary condition $X = 0$ for $x = a$. With the abbreviation

$$\left(\left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{c}\right)^2\right)^{\frac{1}{2}} = \nu_{mn}$$

the solution is

$$X = A_{mn}[\cosh(\nu_{mn}x) - \coth(\nu_{mn}a) \sinh(\nu_{mn}x)]. \quad (5.9)$$

In this way we have found a double infinitude of solutions $u = XYZ$ of the differential equation with the homogeneous boundary conditions. Since the differential equation is linear the sum of several solutions is again a solution and now we must find a linear combination of solutions which also satisfies the inhomogeneous boundary condition for $x = 0$.

If $f(y, z)$ is of the form

$$f(y, z) = A_{mn} \cos \frac{n\pi}{b} y \cos \frac{m\pi}{c} z$$

the problem is, however, already solved by one of the particular solutions of the homogeneous problem.

3. Fourier series

Since the particular solutions of the differential equation can be additively combined to yield the general solution, we have the following result: if the function $f(y, z)$ can be represented in the form

$$f(y, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \cos \frac{n\pi y}{b} \cos \frac{m\pi z}{c}, \quad (5.10)$$

then the temperature field is

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} [\cosh \nu_{mn}x - \coth \nu_{mn}a \sinh \nu_{mn}x] \cos \frac{n\pi y}{b} \cos \frac{m\pi z}{c} \quad (5.11)$$

where

$$\nu_{mn} = \left(\left(\frac{n\pi}{b} \right)^2 + \left(\frac{m\pi}{c} \right)^2 \right)^{\frac{1}{2}}.$$

Thus our method is sufficient for all functions which can be expanded into a trigonometric series of the form (5.10).

For the sake of simplicity we now assume that in (5.10) all coefficients A_{mn} are equal to zero, when m is different from one. This means that $f(y, z)$ has the form $F(y) \cos (\pi z/c)$ and the problem is reduced to the expansion of $F(y)$ into a series

$$F(y) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi y}{b}. \quad (5.12)$$

Series of the type (5.12) are called Fourier series, after the mathematician who first used them and established their general properties.

The functions on the right-hand side of (5.12) are periodic. The greatest period is (if we omit $n = 0$) the one that corresponds to $n = 1$. Its magnitude is $2b$. The other periods are equal to $2b$ divided by some integer. The function on the right hand side of (5.12) is thus periodic with period $2b$. The function $F(y)$ is given for one half of the period only, namely between $y = 0$ and $y = b$. For our purposes it is sufficient if the series in (5.12) represents $F(y)$ in this interval. The values outside of this interval do not concern us. To extend $F(y)$ from the interval $(0, b)$ to an interval which is the whole period, i.e. $(-b, b)$, we may define $F(-y) = F(y)$ for any y in $(0, b)$. Indeed, any function represented by (5.12) is an even function, since only terms containing

cosines appear in the series. The function can be defined over the whole y -axis in a unique way, since it must be periodic.

First we obtain a simple representation for the A_n : multiplying equation (5.12) by $\cos (l\pi y/b)$ ($l = \text{integer}$) and integrating from $y = 0$ to $y = 2b$, we obtain

$$\begin{aligned} \int_0^{2b} F(y) \cos \frac{l\pi y}{b} dy \\ = \sum_{n=0}^{\infty} A_n \int_0^{2b} \cos \frac{n\pi y}{b} \cos \frac{l\pi y}{b} dy. \end{aligned} \quad (5.13)$$

Now we have

$$\begin{aligned} \int_0^{2b} \cos \frac{n\pi y}{b} \cos \frac{l\pi y}{b} dy \\ = \frac{1}{2} \int_0^{2b} \left[\cos (n+l) \frac{\pi y}{b} + \cos (n-l) \frac{\pi y}{b} \right] dy \end{aligned} \quad (5.14)$$

and all these integrals are equal to zero except in the case $n = l$. In this case the integral in (5.14) reduces to

$$\int_0^{2b} dy = 2b.$$

This operation makes all terms in (5.13), for which $n \neq l$ equal to zero, and the remaining term is $A_l b$.

This enables A_n to be calculated in a simple way:

$$A_n = \frac{1}{b} \int_0^{2b} F(y) \cos \frac{n\pi y}{b} dy. \quad (5.15)$$

Let us introduce $\eta = \pi y/b$ as a new variable, whose period is equal to 2π instead of $2b$. We then obtain the nicer mathematical form

$$A_n = \frac{1}{\pi} \int_0^{2\pi} F(\eta) \cos(n\eta) d\eta. \quad (5.16)$$

The case $n = 1$ is a special one. If in (5.12) we take the average value of all terms on the right side, we obtain zero as the result, since each cosine term is equally often positive as negative. For $n = 0$, $\cos 0 = 1$ and thus we have

$$A_0 = \frac{1}{2b} \int_0^{2b} F(y) dy = \frac{1}{2\pi} \int_0^{2\pi} F(\eta) d\eta. \quad (5.17)$$

We thus have a method of calculation for the coefficients A_n . No restrictions on $F(y)$ were necessary if only the integrals in (5.15) and (5.17) have a meaning, i.e., $F(y)$ must be integrable. However continuity and differentiability at every point are not necessary.

We must omit here the proof of the statement that the series with the calculated coefficients has $F(y)$ as limit. This method for the calculation of the coefficients is valid in much more general cases than our example, as we shall show later.

4. Example

We shall now carry through the computations for a specific example. The temperature may be a linearly dependent function of y for $x = 0$. Then we have $F(y) = C(b - y)$. This function and its continuation as an even periodic function is represented in Fig. 33.

According to this representation we have $F(y) = C(y - b)$ in $(b, 2b)$. From (5.17) follows that

$$A_0 = \frac{C}{2b} \left[\int_0^b (b - y) dy + \int_b^{2b} (y - b) dy \right] = C \frac{b}{2}.$$

From (5.15)

$$\begin{aligned} A_n &= \frac{C}{b} \left[\int_0^b (b - y) \cos \frac{n\pi y}{b} dy \right. \\ &\quad \left. + \int_b^{2b} (y - b) \cos \frac{n\pi y}{b} dy \right] \\ &= 2 \frac{Cb}{n^2 \pi^2} (1 - \cos n\pi) \end{aligned}$$

or

$$A_n = 4 \frac{Cb}{n^2 \pi^2} \quad \text{for odd } n,$$

$$A_n = 0 \quad \text{for even } n.$$

Thus we have the Fourier series

$$\begin{aligned} F(y) &= 4 \frac{Cb}{\pi^2} \left[\frac{\pi^2}{8} + \frac{1}{1^2} \cos \frac{\pi y}{b} + \frac{1}{3^2} \cos \frac{3\pi y}{b} \right. \\ &\quad \left. + \frac{1}{5^2} \cos \frac{5\pi y}{b} \dots \right] \end{aligned} \tag{5.18}$$

and the solution of our problem is

$$\begin{aligned}
 u = \frac{4Cb}{\pi^2} \cos \frac{\pi z}{c} & \left[\frac{\pi^2}{8} \left(\cosh \frac{\pi x}{c} - \coth \frac{\pi a}{c} \sinh \frac{\pi x}{c} \right) \right. \\
 & + \frac{1}{1^2} \left\{ \cosh \pi \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{\frac{1}{2}} x - \coth \pi \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{\frac{1}{2}} a \right. \\
 & \quad \left. \left. \sinh \pi \left(\frac{1}{b^2} + \frac{1}{c^2} \right)^{\frac{1}{2}} x \right\} \cos \frac{\pi y}{b} \right. \\
 & + \frac{1}{3^2} \left\{ \cosh \pi \left(\frac{9}{b^2} + \frac{1}{c^2} \right)^{\frac{1}{2}} x - \coth \pi \left(\frac{9}{b^2} + \frac{1}{c^2} \right)^{\frac{1}{2}} a \right. \\
 & \quad \left. \left. \sinh \pi \left(\frac{9}{b^2} + \frac{1}{c^2} \right)^{\frac{1}{2}} x \right\} \cos \frac{3\pi y}{b} + \dots \right] \quad (5.19)
 \end{aligned}$$

This gives us the temperature at any point x, y, z in the body. Fig. 33 shows how rapidly the series converges.

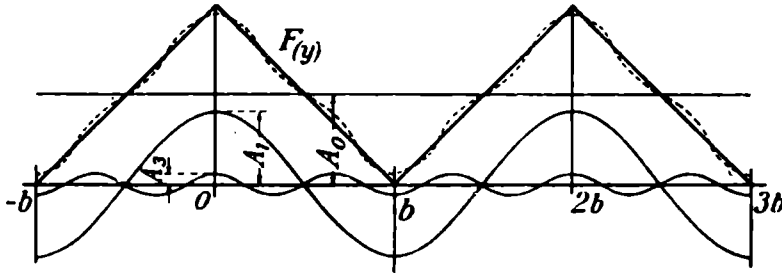


Fig. 33. Decomposition of a function $F(y)$ into a Fourier series
 — Sum of the first three terms of the Fourier expansion

5. Oscillations of a string

The mathematical method of Fourier series becomes very clear in the theory of acoustic oscillations. The

decomposition of a sound into fundamental and harmonic tones is an idea familiar to the reader. The importance of trigonometric functions in the calculation we just completed suggests taking an example from the theory of oscillations in which we have already encountered these functions (II 12). The equation (4.8) is valid for all instruments which generate tones. According to the particular case under consideration, this equation may be derived from the theory of elastic materials (strings, diaphragms) or the theory of compressible fluids (for pipes).

We shall consider the simple example of a string which is fixed at the points $x = 0$ and $x = l$ and is under the tension S (the force per unit cross section). We seek u , the perpendicular displacement from the equilibrium position as a function of the position x and the time t . The differential equation can be deduced by examining one element with mass $\rho \, dx$ (ρ = mass of per unit length) (Fig. 34). Its acceleration in the u -direction is $\partial^2 u / \partial t^2$. The force component in this direction is $S \sin \varphi_1 - S \sin \varphi_0$. For small angles, $\sin \varphi \sim \tan \varphi = \partial u / \partial x$ and we have

$$S \left\{ \left(\frac{\partial u}{\partial x} \right)_1 - \left(\frac{\partial u}{\partial x} \right)_0 \right\} = S \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) dx.$$

From Newton's law it follows that

$$\rho \frac{\partial^2 u}{\partial t^2} = S \frac{\partial^2 u}{\partial x^2}. \quad (5.20)$$

If we put $S/\rho = c^2$ we obtain the simplest form of (4.8):

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \quad (5.21)$$

If we put $u = X(x)T(t)$ here, we obtain the equations

$$\frac{X''}{X} = -\alpha \quad \text{and} \quad \frac{T''}{T} = -\alpha c^2, \quad (5.22)$$

where α is an undetermined constant and the negative sign was chosen in order to simplify the equations. To the first equation (5.22) there correspond the boundary conditions

$$u = 0 \quad \text{for } x = 0 \text{ and } x = l.$$

From $X = A \sin(\alpha^{\frac{1}{2}}x) + B \cos(\alpha^{\frac{1}{2}}x)$ we again obtain either the trivial solution $X = 0$ unless the number α equals one of the eigenvalues $n^2\pi^2/l^2$ ($n = \text{integer}$). The eigenfunctions are $\sin(n\pi x/l)$, where $n\pi/l$ is the space frequency of the oscillations. The wave length is $2l/n$, which is equal to twice the length of the string or an integral part of it (Fig. 35). This is the well known representation of fundamental and harmonic

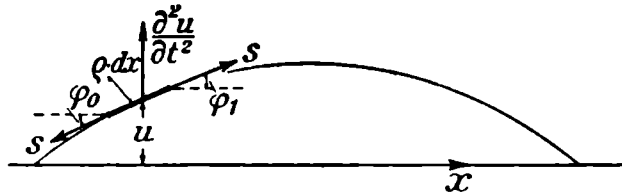


Fig. 34. Oscillating string

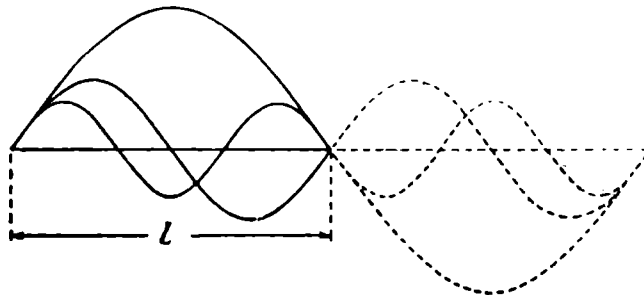


Fig. 35. Fundamental and harmonic oscillations

oscillations. The time frequencies follow immediately from the second equation (5.22), the solutions of which are

$$T = A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l}.$$

The frequencies are equal to $n\pi c/l$, and the period of oscillation is $2l/nc$. The velocity of propagation is generally equal to wave length/time of oscillation, which is equal to c (see IV 3).

We obtain the constants A_n and B_n here from the values corresponding to $t = 0$. The position u of the string and the speed $\partial u/\partial t$ must be given as functions of x at the beginning of the oscillation. In the case of plugging the initial position is given, in the case of striking the initial velocity. The precise manner of excitation differs widely for the various string instruments; it determines the constants A_n and B_n and hence the intensity of the harmonics and the quality of the sound.

If the initial conditions are $u = f(x)$ and $\partial u/\partial x = 0$, all B_n are equal to 0 and $f(x)$ must have the form of a sum

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}.$$

If the initial conditions are $u = 0$ and $\partial u/\partial x = g(x)$, the A_n are equal to zero and

$$g(x) = \sum_{n=1}^{\infty} B_n \frac{nc}{l} \sin \frac{n\pi x}{l}.$$

The coefficients of a sine series can be calculated in the same way as in the case of a cosine series. If the initial position and the initial velocity are both different from zero, the best thing to do is to represent the solution as a sum of two functions for one of which the initial velocity is zero and for the other of which the initial position is equal to zero. There are instruments which are able to decompose an oscillation into "partial oscillations" as they occur in the Fourier series. These are mostly based on the phenomenon of resonance. This is the principle of the ear. Other natural means are dispersion where the partial oscillations have different speeds, and finally damping where the harmonic oscillations are annihilated faster than the fundamental oscillation. We shall later consider an example of this latter phenomenon.

6. Generalization

We shall now show the extent of our method. Our previous considerations seem to have been dependent on the use of trigonometric functions which seem to satisfy the relation

$$\int_0^{2\pi} \cos n\eta \cos l\eta \, d\eta = 0 \quad \text{for } l \neq n,$$

by mere chance, thus resulting in a great simplification. It can easily be seen that the method is not restricted to pure cosine series, but that it can be extended to pure sine or even mixed series with the aid of

$$\int_0^{2\pi} \sin n\eta \sin l\eta \, d\eta = 0 \quad \text{for } l \neq n$$

and

$$\int_0^{2\pi} \cos n\eta \sin l\eta \, d\eta = 0 \quad \text{for all } l \text{ and } n.$$

Also the simplification from (5.10) to (5.12) was made only for convenience. In the general case we could determine the coefficients A_{mn} in the same way, since

$$\int_0^{2\pi} \int_0^{2\pi} \cos n\eta \cos m\zeta \cos l\eta \cos k\zeta \, d\eta \, d\zeta = 0,$$

unless $n = l$ and $m = k$.

This double integral is equal to π^2 if $n = l$ and $m = k$ and therefore we have in (5.10)

$$A_{mn} = \frac{1}{bc} \int_0^{2b} \int_0^{2c} f(y, z) \cos \frac{n\pi y}{b} \cos \frac{m\pi z}{c} \, dy \, dz. \quad (5.23)$$

In order to see the general content of these relations, we must ask how we happened to use the functions $\cos (n\pi y/b) \cos (m\pi z/c)$. We obtained them as eigenfunctions of a differential equation which was of the form

$$\Delta u = 0 \quad \text{or} \quad \Delta u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

with certain boundary conditions. These homogeneous linear differential equations from IV 1-3 (the method naturally does not work in the case of equation (4.5)) can always be transformed into the form

$$\Delta v = \lambda v, \quad (5.24)$$

by representing the unknown function as a product.

Here v depends on one, two, or three variables; Δv has one of the forms given in III 10 and the region under consideration is a curve, a surface, or a volume according to the number of variables. The constant λ corresponds to the constants α, β, γ in our examples and has different values which can be used for solutions. They are again called eigenvalues. We shall choose the function v such that it will satisfy homogeneous linear boundary conditions only. The most general condition of this kind is

$$hv + k \frac{\partial v}{\partial n} = 0. \quad (5.25)$$

Here n is the normal to the boundary of the region. In the example of the temperature field, the expression YZ in (5.2) corresponds to the present v . According to (5.2) and (5.3) we have the equation

$$\Delta v = Y''Z + Z''Y = -\frac{X''}{X} YZ = \alpha v.$$

The region of the function v is the y, z -plane between the limits 0 and b and 0 and c respectively. The boundary conditions are linear and homogeneous: $\partial v / \partial n = 0$.

However the functions Y and Z themselves satisfy the conditions (5.24) and (5.25), since we have

$$\Delta Y = Y'' = -\left(\frac{X''}{X} + \frac{Z''}{Z}\right)Y = (\alpha + \gamma)Y.$$

This time the region is a line, namely the y -axis between 0 and b , and the boundary conditions are $Y' = 0$ corresponding to (5.25) with $h = 0$. In the example of the

oscillating string X satisfies all conditions for v , since we have

$$\Delta X = X'' = -\alpha X;$$

The region is the segment from $x = 0$ to $x = l$ and the boundary conditions are $X = 0$, corresponding to $k = 0$ in (5.25).

7. Green's theorem and orthogonality

There exist certain theorems concerning functions of the type v which satisfy (5.24) and (5.25). We shall find them with the aid of Green's theorem which follows from the theorem of Gauss (3.12). In (3.12) we write

$$\mathbf{A} = \Phi \operatorname{grad} \Psi - \Psi \operatorname{grad} \Phi, \quad (5.26)$$

instead of \mathbf{A} , where Φ and Ψ are scalars. Since, for example, we have

$$\begin{aligned} \operatorname{div}\{\Phi \operatorname{grad} \Psi\} &= \frac{\partial}{\partial x} \left(\Phi \frac{\partial \Psi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\Phi \frac{\partial \Psi}{\partial y} \right) + \frac{\partial}{\partial z} \left(\Phi \frac{\partial \Psi}{\partial z} \right) \\ &= \Phi \Delta \Psi + \frac{\partial \Phi}{\partial x} \frac{\partial \Psi}{\partial x} + \frac{\partial \Phi}{\partial y} \frac{\partial \Psi}{\partial y} + \frac{\partial \Phi}{\partial z} \frac{\partial \Psi}{\partial z}, \end{aligned}$$

in rectangular coordinates, then

$$\operatorname{div} \mathbf{A} = \Phi \Delta \Psi - \Psi \Delta \Phi.$$

Furthermore

$$\mathbf{A}_n = \Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n}.$$

We thus obtain Green's theorem

$$\int (\Phi \Delta \Psi - \Psi \Delta \Phi) dV = \int \left(\Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n} \right) df. \quad (5.27)$$

Here V is any region (one, two, or three-dimensional) and f is the boundary of the region.

We shall now apply (5.27) for two eigenfunctions v_l and v_m which satisfy (5.24) and (5.25). According to (5.24) we introduce in

$$\int (v_l \Delta v_m - v_m \Delta v_l) dV = \int \left(v_l \frac{\partial v_m}{\partial n} - v_m \frac{\partial v_l}{\partial n} \right) df \quad (5.28)$$

the expressions

$$\Delta v_l = \lambda_l v_l \quad \text{and} \quad \Delta v_m = \lambda_m v_m$$

and notice that according to (5.25) the integrand on the right hand side is equal to zero. Thus we have

$$(\lambda_l - \lambda_m) \int v_l v_m dV = 0. \quad (5.29)$$

If now λ_l and λ_m are two different eigenvalues, we have

$$\int v_l v_m dV = 0, \quad (5.30)$$

and this is a generalization of a formula which we have already met and which enabled us to calculate the Fourier coefficients in a simple way. The property of the eigenfunctions which is expressed in (5.30) is called "orthogonality". If $m = l$, (5.30) is not true, of course, since $\int v_m^2 dV$ is positive.

8. Particular solutions in rectangular Cartesian coordinates

By the use of rectangular coordinates the solutions of the equations for X , Y , Z in our examples were expressed by trigonometric functions, and the same is true for T in the oscillation problem. Exponential functions appear in the solution of the equation for the conduction of heat. For if we write $u = vT(t)$ in the equation

$$\Delta u = \frac{c\rho}{\lambda} \frac{\partial u}{\partial t},$$

we obtain

$$\frac{T''}{T} = \text{const} = -\alpha,$$

the solution of which is given by

$$T = e^{-\alpha t}. \quad (5.31)$$

9. Particular solutions in cylindrical coordinates

If the problem has cylindrical symmetry, we shall express Δv in cylindrical coordinates according to (3.21). Equation (5.24) then takes the following form

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v}{\partial \varphi^2} + \frac{\partial^2 v}{\partial z^2} = \lambda v. \quad (5.32)$$

If we put $v = R(r)\Phi(\varphi)Z(z)$ in (5.32) and divide by $R\Phi Z$, we obtain

$$\frac{1}{rR} \frac{d}{dr} \left(r \frac{dR}{dr} \right) + \frac{1}{r^2 \Phi} \Phi'' + \frac{Z''}{Z} = \lambda. \quad (5.33)$$

Since Z''/Z and Φ''/Φ are independent of z and of φ respectively, the following relations follow from (5.33)

$$\frac{Z''}{Z} = -\alpha \quad \text{and} \quad \frac{\Phi''}{\Phi} = -\beta. \quad (5.34)$$

These relations are satisfied by trigonometric functions if α and β are positive, and by hyperbolic functions if they are negative. β can only be positive, however.

From (5.33) and (5.34) we obtain the equation

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) - \left\{ (\lambda + \alpha) + \frac{\beta}{r^2} \right\} R = 0$$

for R . If we put $-(\lambda + \alpha) = k^2$ and $\beta = p^2$ we have

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left(k^2 - \frac{p^2}{r^2} \right) R = 0. \quad (5.35)$$

This can be simplified by introducing the variable $kr = \rho$. Thus,

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(1 - \frac{p^2}{\rho^2} \right) R = 0. \quad (5.36)$$

This is Bessel's differential equation. Its solutions are called Bessel functions. If one wants to use these functions for calculations, one must use tables as for the trigonometric functions. p is called the order of the Bessel function. Since (5.36) is of order two, there must exist two independent Bessel functions of order p . If p is rational these two functions are called "of orders $+p$ and $-p$ " and are denoted by $J_p(\rho)$ and $J_{-p}(\rho)$. For integer p 's the tables usually contain a Bessel function $J_p(\rho)$ and a "Neumann function" $K_p(\rho)$ which is sometimes denoted by $N_p(\rho)$ or $Y_p(\rho)$.

There will be a short discussion in a later section on the most important methods in the theory of these functions (V 11 and 12).

The special case $\lambda + \alpha = 0$ can occur if, for example, we have the equation $\Delta v = 0$ and v is independent of z . The equation then has the form

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} - \frac{p^2}{r^2} R = 0 \quad (5.37)$$

and the two particular solutions are $R = r^p$ and $R = r^{-p}$.

10. Particular solutions in spherical coordinates

If the problem has spherical symmetry, we use equation (3.22):

$$\begin{aligned} \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial v}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial v}{\partial \vartheta} \right) \\ + \frac{1}{\rho^2 \sin^2 \vartheta} \frac{\partial^2 v}{\partial \varphi^2} = \lambda v. \end{aligned} \quad (5.38)$$

We put $v = R(r) \Theta(\vartheta) \Phi(\varphi)$ and divide by $R \Theta \Phi$. Thus,

$$\begin{aligned} \frac{1}{\rho^2 R} \frac{d}{d\rho} (\rho^2 R') + \frac{1}{\rho^2 \Theta \sin \vartheta} \frac{d}{d\vartheta} (\Theta' \sin \vartheta) \\ + \frac{1}{\rho^2 \Phi \sin^2 \vartheta} \Phi'' = \lambda. \end{aligned} \quad (5.39)$$

We now obtain that Φ''/Φ is independent of φ and consequently is a constant ($= -\alpha$). The left-hand side of (5.39) is thus the sum of a function of ρ and a function of ϑ ,

$$\left[\frac{1}{R} \frac{d}{d\rho} (\rho^2 R') - \lambda \rho^2 \right] + \left[\frac{1}{\Theta \sin \vartheta} \frac{d}{d\vartheta} (\Theta' \sin \vartheta) - \frac{\alpha}{\sin^2 \vartheta} \right] = 0. \quad (5.40)$$

Since, however, the sum is equal to zero, each term must be zero. The general theory of these problems is difficult and we shall consider only special cases.

First we shall simplify the equation

$$\frac{1}{R} \frac{d}{d\rho} (\rho^2 R') - \lambda \rho^2 = \text{const.} = \beta \quad (5.41)$$

by taking the special case $\lambda = 0$ which corresponds to the differential equation $\Delta v = 0$. It is easy to see that

$$\frac{d}{d\rho} (\rho^2 R) = \rho \frac{d^2}{d\rho^2} (\rho R).$$

The differential equation

$$\rho \frac{d^2}{d\rho^2} (\rho R) = \beta R \quad (5.42)$$

can be solved by setting $R = \rho^\nu$. We then obtain

$$\nu(\nu + 1) = \beta. \quad (5.43)$$

If we now write $\beta = m(m + 1)$, we have

$$\nu = m \quad \text{or} \quad \nu = -m - 1. \quad (5.44)$$

The other limiting case $\beta = 0$ corresponds to oscillation and heat problems which are independent of ϑ and φ . The equation is

$$\frac{d^2}{d\rho^2}(\rho R) = \rho R \lambda; \quad (5.45)$$

The solutions are $\rho R = \sin(\lambda^{\frac{1}{2}}\rho)$ and $\cos(\lambda^{\frac{1}{2}}\rho)$ or

$$R = \frac{\sin(\lambda^{\frac{1}{2}}\rho)}{\rho} \quad \text{and} \quad \frac{\cos(\lambda^{\frac{1}{2}}\rho)}{\rho}. \quad (5.46)$$

The second equation following from (5.40) is

$$\frac{1}{\sin \vartheta} \frac{d}{d\vartheta} (\Theta' \sin \vartheta) + \left(\gamma - \frac{\alpha}{\sin^2 \vartheta} \right) \Theta = 0. \quad (5.47)$$

Here we shall only consider the case $\alpha = 0$, that is, the case of independence of the "longitude".

In order to find the usual form we put $\gamma = m(m+1)$. This is somewhat arbitrary as long as we do not know anything about γ and m . However, if $\lambda = 0$ and $\alpha = 0$ for the problem under consideration, m has the same significance as in (5.44), since in this case we have $\gamma = \beta$. We obtain

$$\frac{1}{\sin \vartheta} \frac{d}{d\vartheta} \left(\frac{d\Theta}{d\vartheta} \sin \vartheta \right) + m(m+1)\Theta = 0 \quad (5.48)$$

or with $x = \cos \vartheta$

$$\frac{d}{dx} \left[(1-x^2) \frac{d\Theta}{dx} \right] + m(m+1)\Theta = 0. \quad (5.49)$$

The solutions of this differential equation are called spherical functions and are denoted by $P_m(x)$ and $Q_m(x)$, where m is the order of the function. Since $\cos \vartheta$ varies between -1 and $+1$ and is an even function, it suffices to know the spherical functions in the interval $0 \leq x \leq 1$ or $0 \leq \vartheta \leq \pi/2$. There also exist tables for these functions.

The solutions of (5.47) with arbitrary α can be expressed by spherical functions. They are called associated Legendre functions. Again lack of space does not allow us to consider this case here.

11. Solution of ordinary differential equations by expansion in series

We have twice encountered differential equations whose solutions are not elementary and known from previous courses. We found it necessary to use tables for practical calculations. We shall now use these functions as an example in order to demonstrate the procedure for finding the properties of functions which satisfy a given differential equation.

We assume that the solution of the differential equation (5.36)

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(1 - \frac{p^2}{\rho^2}\right) R = 0 \quad (5.50)$$

can be expanded in a power series about the point $\rho = 0$. The coefficients of this series are unknown and will be determined by introducing the series into the equation. Assume that the series is of the form

$$\begin{aligned} R = \rho^\nu + a_1 \rho^{\nu+1} + a_2 \rho^{\nu+2} + \cdots \\ + a_n \rho^{\nu+n} + \cdots \end{aligned} \quad (5.51)$$

The coefficients a_1, \cdots and the first exponent ν are unknown. From the series we find

$$\frac{dR}{d\rho} = \nu \rho^{\nu-1} + a_1(\nu+1)\rho^\nu + \dots + a_n(\nu+n)\rho^{\nu+n-1}$$

$$a_{n+1}(\nu+n+1)\rho^{\nu+n} + \dots$$

$$\frac{d^2R}{d\rho^2} = \nu(\nu-1)\rho^{\nu-2} + a_1(\nu+1)\nu\rho^{\nu-1} + \dots$$

$$a_{n+1}(\nu+n+1)(\nu+n)\rho^{\nu+n-1}$$

$$a_{n+2}(\nu+n+2)(\nu+n+1)\rho^{\nu+n} + \dots$$

By introducing these expressions in (5.50) and grouping the terms of equal powers of ρ we obtain

$$\begin{aligned} & \rho^{\nu-2}[\nu(\nu-1) + \nu - p^2] \\ & + \rho^{\nu-1}[a_1(\nu+1)\nu + a_1(\nu+1) - a_1p^2] \\ & + \rho^\nu[a_2(\nu+2)(\nu+1) \\ & + a_2(\nu+2) - a_2p^2 + 1] + \dots \\ & + \rho^{\nu+n}[a_{n+2}(\nu+n+2)(\nu+n+1) \\ & + a_{n+2}(\nu+n+2) - a_{n+2}p^2 + a_n] + \dots = 0. \end{aligned} \tag{5.52}$$

If this equation is to be satisfied for all values of ρ in a certain neighbourhood of $\rho = 0$, then the coefficient of each power of ρ must be equal to zero. The first coefficient contains only the first exponent ν as an unknown. Thus we find

$$\nu = \pm p. \tag{5.53}$$

We now have two solutions in the form of a series, as was to be expected. One begins with ρ^p and the other with ρ^{-p} . One is equal to zero for $\rho = 0$, the other has an infinite value for $\rho = 0$. From the second coefficient it follows that $a_1 = 0$. If in the third we put $\nu = \pm p$, we obtain

$$a_2\{(\pm p + 2)(\pm p + 2) - p^2\} + 1 = 0$$

or

$$a_2 = -\frac{1}{2(\pm 2p + 2)}.$$

Similarly we obtain the general recurrence formula

$$a_{n+2} = -\frac{a_n}{(n+2)(\pm 2p + n + 2)}. \quad (5.54)$$

Thus the two series representing the solutions of our differential equation are

$$\begin{aligned} R_1 = \rho^p & \left\{ 1 - \frac{\rho^2}{2(2p+2)} + \frac{\rho^4}{2 \cdot 4(2p+2)(2p+4)} \right. \\ & \left. - \frac{\rho^6}{2 \cdot 4 \cdot 6 \cdot (2p+2)(2p+4)(2p+6)} + \dots \right. \\ R_2 = \rho^{-p} & \left\{ 1 - \frac{\rho^2}{2(2-2p)} + \frac{\rho^4}{2 \cdot 4(2-2p)(2-4p)} \right. \\ & \left. + \frac{\rho^6}{2 \cdot 4 \cdot 6 \cdot (2-2p)(4-2p)(6-2p)} + \dots \right. \end{aligned} \quad (5.55)$$

In the tables of Bessel functions these functions are multiplied by a conventionally selected constant. The

series are convergent for each value of ρ . If ρ is large, however, they converge very slowly.

The series R_2 has no meaning if p is a positive integer, nor has R_1 if p is a negative integer. In these cases we can only use one series as a solution. Also for $p = 0$ we have only one solution. Here we can find the second solution in the following way; p shall not be equal to zero initially, but merely small. R_1 and R_2 do not differ much from each other. The difference is of the same order of magnitude as p . If, therefore, we take the limit as $p \rightarrow 0$

$$\lim_{p \rightarrow 0} \frac{R_1 - R_2}{2p},$$

we obtain a new solution of the equation which is identical to the one we previously called the Neumann function $K_0(\rho)$. The limit of the first term of the series is

$$\begin{aligned} \lim_{p \rightarrow 0} \frac{\rho^p - \rho^{-p}}{2p} &= \lim_{p \rightarrow 0} \frac{e^{p \log \rho} - e^{-p \log \rho}}{2p} \\ &= \lim_{p \rightarrow 0} \frac{(1 + p \log \rho + \cdots) - (1 - p \log \rho + \cdots)}{2p} \\ &= \log \rho, \end{aligned}$$

and the second term is

$$\begin{aligned} \lim_{p \rightarrow 0} \frac{1}{2p} \left[\frac{\rho^{p+2}}{2(2+2p)} - \frac{\rho^{-p+2}}{2(2-2p)} \right] \\ = \lim_{p \rightarrow 0} \frac{\rho^2}{8p} \left[\frac{1 + p \log \rho + \cdots}{1 + p} - \frac{1 - p \log \rho + \cdots}{1 - p} \right] \end{aligned} \quad (5.56)$$

$$\begin{aligned}
&= \lim_{p \rightarrow 0} \frac{\rho^2}{8p} \left[(1 + p \log \rho + \cdots)(1 - p + \cdots) \right. \\
&\quad \left. - (1 - p \log \rho + \cdots)(1 + p + \cdots) \right] \\
&= \frac{\rho^2}{4} (\log \rho - 1)
\end{aligned}$$

etc. We do not require the complete series, since we can already see the most important property. The series begins with the term $\log \rho$ and therefore for $\rho = 0$ becomes infinite as the logarithm and not like any power ρ^ν . This is the reason why we did not find this solution amongst the power series.

We shall now use this method of expansion into a series to determine the spherical functions which are defined by

$$\frac{d}{dx} \left[(1 - x^2) \frac{d\Theta}{dx} \right] + m(m + 1)\Theta = 0. \quad (5.57)$$

The series

$$\Theta = x^\nu + a_1 x^{\nu+1} + \cdots + a_n x^{\nu+n} + \cdots \quad (5.58)$$

gives us the relations

$$\nu(\nu - 1) = 0, \quad \text{i.e., } \nu_1 = 0 \quad \text{and} \quad \nu_2 = 1$$

$$a_1 = a_3 = \cdots = 0$$

$$a_2 = - \frac{m(m + 1) - \nu(\nu + 1)}{(\nu + 1)(\nu + 2)}$$

and

$$a_{n+2} = - \frac{m(m+1) - (\nu+n)(\nu+n+1)}{(\nu+n+1)(\nu+n+2)} a_n. \quad (5.59)$$

Thus we obtain the two solutions

$$\begin{aligned} \theta_1 = 1 &- \frac{m(m+1) - 0 \cdot 1}{1 \cdot 2} x^2 \\ &+ \frac{\{m(m+1) - 0 \cdot 1\} \{m(m+1) - 2 \cdot 3\}}{1 \cdot 2 \cdot 3 \cdot 4} x^4 \\ &- \dots \end{aligned} \quad (5.60)$$

$$\begin{aligned} \theta_2 = x &- \frac{m(m+1) - 1 \cdot 2}{2 \cdot 3} x^3 \\ &+ \frac{\{m(m+1) - 1 \cdot 2\} \{m(m+1) - 3 \cdot 4\}}{2 \cdot 3 \cdot 4 \cdot 5} x^5 \\ &- \dots \end{aligned}$$

A most important property of these functions is that one of the series has a finite number of terms whenever m is an integer. (This is always so in physical problems.) The coefficients of $x^{\nu+n+2}$ and all the higher powers are equal to zero if $\nu+n=m$. This behaviour is visible in θ_1 for even m , and in θ_2 for odd m . We have, for example, for $m=2$

$$\theta_1 = 1 - \frac{2 \cdot 3}{1 \cdot 2} x^2,$$

and for $m = 3$

$$\Theta_2 = x - \frac{3 \cdot 4 - 1 \cdot 2}{2 \cdot 3} x^3.$$

These finite series (after multiplication by a normalization factor) are called spherical functions of the first kind and are denoted by $P_m(x)$. The infinite series are called spherical functions of the second kind and are denoted by $Q_m(x)$. They have infinite values for $x = 1$.

12. Asymptotic expansions

The expansion which we have found is sufficient for the determination of the properties of spherical functions, since these functions are defined in the interval $0 < x < 1$ only. However, in the case of Bessel functions, the argument can have greater values and in some cases it may not be possible to use the series. In such cases the differential equation can be integrated "asymptotically" for very great values of the variable. We shall demonstrate this method with the Bessel functions. In equation (5.50),

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(1 - \frac{p^2}{\rho^2}\right) R = 0, \quad (5.61)$$

we shall first neglect the terms containing the large values ρ and ρ^2 in the denominator. We then have the equation.

$$\frac{d^2 R}{d\rho^2} + R = 0,$$

a solution of which is $R = e^{\pm i\rho}$. Now we shall try to find

a solution of (5.61) which can be represented as the product of this R by a series in decreasing powers of ρ

$$R = e^{i\rho}[\rho^\nu + a_1\rho^{\nu-1} + a_2\rho^{\nu-2} + \dots]. \quad (5.62)$$

(We do not continue to write $\pm i$, since the two corresponding solutions will simply be conjugate.) We obtain

$$\begin{aligned} \frac{dR}{d\rho} &= e^{i\rho}[\rho^\nu i + \rho^{\nu-1}(ia_1 + \nu) \\ &\quad + \rho^{\nu-2}(ia_2 + (\nu - 1)a_1) + \dots] \end{aligned}$$

$$\begin{aligned} \frac{d^2R}{d\rho^2} &= e^{i\rho}[\rho^\nu(-1) + \rho^{\nu-1}(-a_1 + 2i\nu) \\ &\quad + \rho^{\nu-2}\{-a_2 + i(\nu - 1)a_1 \\ &\quad + (\nu - 1)(ia_1 + \nu)\} + \dots] \end{aligned}$$

If we introduce these expressions in (5.61), we obtain $e^{i\rho}$ multiplied by a descending power series. The coefficient of each power of ρ in this series must be equal to zero (as in V 11). The coefficient of ρ^ν is identically zero. The next term is equal to $-a_1 + 2i\nu + i + a_1$. Thus we obtain $\nu = -1/2$. The coefficient of $\rho^{\nu-2}$ is

$$-a_2 + (\nu - 1)(2ia_1 + \nu) + ia_1 + \nu + a_2 - p^2$$

and it follows that

$$a_1 = \frac{i(\nu^2 - p^2)}{2\nu - 1} = i \frac{4p^2 - 1}{8}$$

Thus we obtain the asymptotic series

$$R = \frac{e^{i\rho}}{\rho^{\frac{1}{2}}} \left(1 + i \frac{4\rho^2 - 1}{8\rho} + \dots \right). \quad (5.63)$$

The following fact is specially interesting; for large values of ρ the Bessel functions behave like periodic functions whose amplitude is decreasing like $1/\rho^{\frac{1}{2}}$.

Asymptotic expansions are not convergent, they are "semiconvergent", i.e., the higher terms do not decrease but increase. Nevertheless they can be used for calculations if one stops before the terms start to increase.

13. Example: heat flow with cylindrical symmetry

Assume that we have an infinitely long cylinder composed of two half cylinders each of which has a semi-circular cross section. Let one of the cylinders have the temperature T , and let the other have zero temperature. Furthermore, let us assume the following law governing heat conduction through the surface to the exterior; the heat flow is proportional to the difference of temperature between the cylinder and exterior space, which is at zero temperature. We want to find the temperature u as a function of the time and of the position in the cylinder.

According to (4.7) and (3.21) the differential equation is

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} + \frac{1}{r^2} \frac{d^2 u}{d\varphi^2} = \frac{c\rho}{\lambda} \frac{\partial u}{\partial t}. \quad (5.64)$$

The direction of $\varphi = 0$ is taken along the middle of the

area with temperature T . Then u will certainly be an even function of φ . If we introduce $u = R\Phi T$, we have

$$\frac{T'}{T} = -\alpha \quad \text{and} \quad \frac{\Phi''}{\Phi} = -\beta$$

and hence

$$T = e^{-\alpha t} \quad \text{and} \quad \Phi = \cos(\beta^{\frac{1}{2}}\varphi).$$

We have not inserted any arbitrary factors here. Also, we have not added the particular solution $\sin(\beta^{\frac{1}{2}}\varphi)$, since it is an odd function of φ . The constant β is determined by the condition that Φ must have the period 2π . This corresponds to a boundary condition. The eigenvalues are

$$\beta^{\frac{1}{2}} = n \quad (= \text{integer}).$$

For R we now obtain the equation

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left(\frac{c\rho}{\lambda} \alpha - \frac{n^2}{r^2} \right) R = 0. \quad (5.65)$$

The solutions of this equation are Bessel functions of order n with the argument $c\rho\alpha r/\lambda$. Since n is integer these functions are the Bessel functions $J_n(c\rho\alpha r/\lambda)$ and the Neumann functions $K_n(c\rho\alpha r/\lambda)$, the values and fundamental properties of which are well known and can be found in tables. If $r = 0$, K_n is infinite; we thus cannot use this particular solution.

At the surface of the cylinder ($r = r_0$) we have in addition the condition that

$$\frac{dR}{dr} + bR = 0 \quad (b = \text{proportionality constant}).$$

From this and with $c\rho\alpha r_0/\lambda = \nu$ we obtain the transcendental equation for ν :

$$\frac{\nu}{r_0} J'_n(\nu) + bJ_n(\nu) = 0. \quad (5.66)$$

The infinitely many solutions of this equation are real. However, we are not able to prove this fact here. If we denote them by ν_m , we can write the result of our computations as

$$u = \sum_m \sum_n A_{mn} J_n\left(\nu_m \frac{r}{r_0}\right) \cos n\varphi \cdot \exp(-\nu_m \lambda t / c\rho r_0) \quad (5.67)$$

The A_{mn} can be determined by means of the initial condition which states that in the region from $r = 0$ to $r = r_0$ we have, for $t = 0$,

$$u = T \quad \text{for} \quad -\frac{\pi}{2} < \varphi < \frac{\pi}{2}$$

and

$$u = 0 \quad \text{for} \quad \frac{\pi}{2} < \varphi < \frac{3}{2}\pi.$$

The general theorem (5.30) is valid here and therefore we have for $m \neq l$ and $n \neq k$ the orthogonality condition

$$\int_0^{r_0} \int_{-\pi}^{+\pi} J_n\left(\nu_m \frac{r}{r_0}\right) \cos n\varphi J_k\left(\nu_l \frac{r}{r_0}\right) \cos k\varphi r dr d\varphi = 0.$$

We must keep in mind here that the volume element is

$r \, dr \, d\varphi$ (and not $dr \, d\varphi$). Therefore we now find the coefficients A_{mn} from the equation

$$\begin{aligned} \int_0^{r_0} \int_{-\pi/2}^{+\pi/2} T J_n\left(\nu_m \frac{r}{r_0}\right) \cos n\varphi \, r \, dr \, d\varphi \\ = A_{mn} \int_0^{r_0} \int_{-\pi}^{+\pi} \left\{ J_n\left(\nu_m \frac{r}{r_0}\right) \right\}^2 \cos^2 n\varphi \, r \, dr \, d\varphi. \end{aligned} \quad (5.68)$$

The evaluation of these double integrals is a mere calculation which we shall not perform here.

The calculations become much simpler if there is no dependence on φ at all. This is the case when the temperature in the cylinder is constant initially. We then have $n = 0$. We only need the function J_0 , for which there exist very extensive tables. The solutions of the transcendental equation can also be found in tables.

We can see in (5.67) (from the exponential function) that the terms are more highly damped the higher their order. After some time (which depends on λ/cpr_0), only the term with the smallest ν_m will remain.

14. Potential of a circular ring

The calculation of the potential u of a charged circular ring illustrates the use of spherical functions. (The ring may have an infinitesimal cross section.) Let the radius of the ring be a and the total charge, which is equally distributed, be M . We shall use spherical coordinates. This system may be chosen such that all points on the ring have the coordinate $\vartheta = \pi/2$. Then everything is independent of φ and the differential

equation (4.1) has the form (5.40) with $\lambda = \alpha = 0$. The particular solutions are

$$R = \rho^m \quad \text{and} \quad R = \frac{1}{\rho^{m+1}}$$

$$\Theta = P_m(\cos \vartheta) \quad \text{and} \quad \Theta = Q_m(\cos \vartheta).$$

Only integral values of m are considered in the solution, because R must be unique. According to V 11 the functions P_m are algebraic of degree m in $x = \cos \vartheta$. They are even for even m and odd for odd m . Each Q_m has an infinite value for $\cos \vartheta = 1$ or $\vartheta = 0$ and thus cannot be used for our problem. We therefore try an expression of the following form

$$u = \sum_{m=0}^{\infty} A_m \rho^m P_m(\cos \vartheta) + \sum_{m=0}^{\infty} \frac{B_m}{\rho^{m+1}} P_m(\cos \vartheta). \quad (5.69)$$

In order to determine the coefficients A_m and B_m we need an inhomogeneous boundary condition. We can obtain this condition from the values of the potential on the axis $\vartheta = 0$. Each point on this axis has the same distance from all points of the ring (Fig.

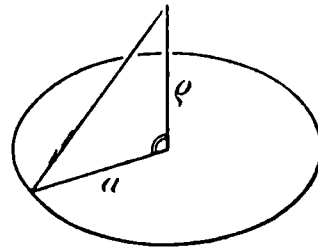


Fig. 36. Potential of a circular ring

36) namely $(a^2 + \rho^2)^{\frac{1}{2}}$. Each element of charge $M d\varphi/2\pi$ therefore produces the potential $M d\varphi/2\pi(a^2 + \rho^2)^{\frac{1}{2}}$. (See VII 1). The total potential for $\vartheta = 0$ is then

$$u_0 = \frac{M}{(a^2 + \rho^2)^{\frac{1}{2}}} \quad (5.70)$$

In order to compare this expression with (5.69), we must expand it in a power series in increasing powers of (ρ/a) if $\rho < a$,

$$u_0 = \frac{M}{a} \left(1 - \frac{1}{2} \frac{\rho^2}{a^2} + \frac{1 \cdot 3}{2 \cdot 4} \frac{\rho^4}{a^4} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{\rho^6}{a^6} + \dots \right) \quad (5.71a)$$

and in decreasing powers of (ρ/a) if $\rho > a$,

$$u_0 = M \left(\frac{1}{\rho} - \frac{1}{2} \frac{a^2}{\rho^3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{a^4}{\rho^5} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{a^6}{\rho^7} + \dots \right) \quad (5.71b)$$

For $\vartheta = 0$ all $P_m(1)$ are equal to one. By comparing (5.69) with (5.71) we find that for $\rho < a$

$$A_0 = \frac{M}{a}, \quad A_2 = \frac{M}{a} \left(-\frac{1}{2a^2} \right) \text{ etc.},$$

and consequently

$$\begin{aligned} &= \frac{M}{a} \left\{ P_0(\cos \vartheta) - \frac{1}{2} \frac{\rho^2}{a^2} P_2(\cos \vartheta) \right. \\ &\quad \left. + \frac{1 \cdot 3}{2 \cdot 4} \frac{\rho^4}{a^4} P_4(\cos \vartheta) - \dots \right\}. \quad (5.72a) \end{aligned}$$

For $\rho > a$ we obtain

$$u = \frac{M}{a} \left\{ \frac{a}{\rho} P_0(\cos \vartheta) - \frac{1}{2} \frac{a^3}{\rho^3} P_2(\cos \vartheta) + \frac{1 \cdot 3}{2 \cdot 4} \frac{a^5}{\rho^5} P_4(\cos \vartheta) - \dots \right\}. \quad (5.72b)$$

15. Example: charged hemispheres

In order to illustrate the expansion in spherical harmonics we consider two hemispheres with radius a and the potentials $+M$ and $-M$. They are so arranged as to form a complete sphere. For the potential we again have the equation (5.40) with $\alpha = \lambda = 0$. The boundary conditions are

$$u = 0 \quad \text{for} \quad \rho = 0 \quad \text{and for} \quad \rho = \infty$$

and for $\rho = a$ we have $u = +M$ for $0 < \vartheta < \pi/2$ and $u = -M$ for $\pi/2 < \vartheta < \pi$. The solutions are again

$$u = \sum_0^{\infty} A_m \rho^m P_m(\cos \vartheta)$$

or

$$u = \sum_0^{\infty} \frac{B_m}{\rho^{m+1}} P_m(\cos \vartheta)$$

according to whether $\rho < a$ or $\rho > a$. The coefficients are again found from the inhomogeneous condition for $\rho = a$. According to the condition of orthogonality we have

$$2\pi a^2 \int_0^{\pi} P_m(\cos \vartheta) P_l(\cos \vartheta) \sin \vartheta d\vartheta = 0 \quad (5.73)$$

$$\text{for} \quad m \neq l$$

and therefore

$$\begin{aligned} \int_0^{\pi/2} MP_m(\cos \vartheta) \sin \vartheta \, d\vartheta - \int_{\pi/2}^{\pi} MP_m(\cos \vartheta) \sin \vartheta \, d\vartheta \\ = A_m a^m \int_0^{\pi} (P_m(\cos \vartheta))^2 \sin \vartheta \, d\vartheta \end{aligned} \quad (5.74)$$

and a corresponding formula for B_m .

16. Example: propagation of waves

The properties of certain functions which are dependent on x , r , or ρ only become very clear if we compare the solutions of the equation

$$\Delta u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

for a given time frequency ω . If Δu depends on the Cartesian coordinate x only, we obtain the equation of a plane wave

$$u = e^{i\omega x/c} e^{i\omega t}, \quad (5.75)$$

if we again use complex exponential functions for the sake of simplicity. With cylindrical symmetry we have

$$u = J_0\left(\frac{\omega}{c} r\right) e^{i\omega t}, \quad (5.76)$$

and with spherical symmetry

$$u = \frac{e^{i\omega \rho/c}}{\rho} e^{i\omega t}. \quad (5.77)$$

The Bessel function of order zero represents a cylindrical wave. Bessel functions of higher order represent cylindrical waves which depend on φ . The expression

$$\frac{1}{\rho} e^{i\omega\rho/c}$$

decreases like $1/\rho$ for increasing ρ . The energy of each wave pulse emitted remains constant as the wave expands. Since, however, the surface over which the energy is distributed increases like ρ^2 , the amplitude must decrease like $1/\rho$ (since the energy is proportional to u^2). The energy of the cylindrical wave is distributed on a surface which increases like r . Thus the amplitude must decrease like $1/(r)^{\frac{1}{2}}$. The Bessel functions actually have this property in an asymptotic fashion, i.e., it is fulfilled more exactly the greater the value of r . This follows from (5.63). Also the subsequent zeros of the transcendent equation have asymptotically constant distances from each other like sine and cosine.

17. The Fourier integral

All of the problems which we have considered so far were concerned with a finite region. We can use the same methods for the whole space (propagation of oscillations or heat flows in the whole space). Here we must perform a limiting process which transforms the sum of the Fourier series into an integral.

For example, the function $f(x)$ may have the constant value 1 for x between $-a$ and $+a$ and the value 0 elsewhere. The process which we are considering shall not be restricted to the interval $-a, a$ as it was in the

case of the string. It may vary along the whole x -axis. It is not possible to work with a periodic function in this case. We must consider the function in the whole region from $-\infty$ to $+\infty$, which is the same as saying that we must give the value ∞ to the period. Since our method yields the eigenfunctions $\sin \alpha x$ and $\cos \alpha x$, we must construct the given non-periodic functions from these functions. An infinite limit for the period is equivalent to a zero limit for the frequency of the fundamental oscillation. The separated harmonic frequencies form a continuous sequence in the limit and the sum becomes an integral. The general Fourier series for a given period $-b, +b$ is

$$f(x) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{b} + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{b},$$

with

$$A_0 = \frac{1}{2b} \int_{-b}^{+b} f(\xi) d\xi, \quad A_n = \frac{1}{b} \int_{-b}^{+b} f(\xi) \cos \frac{n\pi\xi}{b} d\xi,$$

$$B_n = \frac{1}{b} \int_{-b}^{+b} f(\xi) \sin \frac{n\pi\xi}{b} d\xi.$$

Here we have written ξ for the variable of integration in order to have x at our further disposal. We now change the order of integration and summation and obtain

$$f(x) = \int_{-b}^{+b} f(\xi) \left[\frac{1}{2b} + \frac{1}{b} \sum_{n=1}^{\infty} \cos \frac{n\pi}{b} (x - \xi) \right] d\xi. \quad (5.78)$$

We put $n\pi/b = \alpha$ and keep in mind that only the large values of n have any influence if $b \rightarrow \infty$, while α goes

through all values from 0 to ∞ . The sum has a factor $1/b = \alpha/\pi n$ which becomes very small. In order to determine the limit, the term $1/2b$ can be neglected with respect to the sum. The sum, whose terms are infinitely small (order α/n), becomes equal to

$$\frac{1}{\pi} \int_0^{\infty} \cos \alpha(x - \xi) d\alpha,$$

if we write $d\alpha$ for α/n . We now have a representation of our arbitrary function by a Fourier integral:

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\xi f(\xi) \int_0^{\infty} \cos \alpha(x - \xi) d\alpha. \quad (5.79)$$

As an example we consider the conduction of heat in an infinitely large body whose temperature is equal to T in the strip $x = -a$ to $x = +a$ and equal to 0 at any other point at the time $t = 0$. The differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{c\rho}{\lambda} \frac{\partial u}{\partial t}$$

has the functions $\cos \alpha x$ and $\sin \alpha x$ (and hence also $\cos \alpha(x - \xi)$) multiplied by the time factor

$$e^{-\alpha^2 \lambda t / c\rho}$$

as particular solutions. This can easily be proven by our method, or verified directly. Thus the solution for the initial temperature field $u_0 = f(x)$ is

$$u = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\xi f(\xi) \int_0^{\infty} e^{-\alpha^2 \lambda t / c\rho} \cos \alpha(x - \xi) d\alpha. \quad (5.80)$$

In our example we have

$$f(x) = \frac{T}{\pi} \int_{-a}^{+a} d\xi \int_0^{\infty} \cos \alpha(x - \xi) d\alpha$$

and therefore

$$u = \frac{T}{\pi} \int_{-a}^{+a} d\xi \int_0^{\infty} e^{-\alpha^2 \lambda t / c \rho} \cos \alpha(x - \xi) d\alpha.$$

CHAPTER VI

SOLUTION BY CHANGE OF VARIABLES

For many differential equations we find general solutions if we use certain linear combinations of the space-time variables as new variables. This is a special case of the method of characteristics which is very important in the theory of partial differential equations. Since we shall only consider very simple cases we need not go into the general theory of characteristics.

1. Propagation of waves

The equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad (6.1)$$

is satisfied by any function of the variables $x \pm ct$ as can be easily verified. If we set

$$u = f(x \pm ct), \quad (6.2)$$

we have

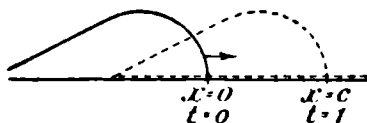
$$\frac{\partial^2 u}{\partial x^2} = f'' \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = c^2 f'',$$

and hence (6.1).

In our case the curves having the equations $x + ct = \text{const.}$ and $x - ct = \text{const.}$ are called the characteristics. The solution

$$u = f_1(x + ct) + f_2(x - ct) \quad (6.3)$$

contains two arbitrary functions which must be determined so that the solution satisfies the boundary conditions. The function $f_1(x + ct)$



represents a wave of constant shape given by u at the time $t = 0$ moving in the direction of the negative x -axis. (Propagation of a disturbance along a rope, or a tidal wave in a canal.) In the same way $f_2(x - ct)$ represents a wave moving in the positive direction of the x -axis. If the propagation occurs in a body of infinite extent, the boundary conditions are easy to satisfy. u and $\partial u / \partial t$ must be given initially for every x . Assume

$$t = 0 : u = F(x) \quad \frac{\partial u}{\partial t} = G(x). \quad (6.4)$$

Then for the determination of f_1 and f_2 we have the relations.

$$f_1(x) + f_2(x) = F(x)$$

$$cf'_1(x) - cf'_2(x) = G(x).$$

With $\int G(x) dx = H(x)$ we obtain from these equations

$$f_1(x) = \frac{1}{2} \left[F(x) + \frac{1}{c} H(x) \right],$$

$$f_2(x) = \frac{1}{2} \left[F(x) - \frac{1}{c} H(x) \right],$$

and the complete solution of the problem is

$$u = \frac{1}{2} \left[F(x + ct) + F(x - ct) + \frac{1}{c} H(x + ct) - \frac{1}{c} H(x - ct) \right]. \quad (6.5)$$

The method can also be used if the body is finite, e.g., an oscillating string. The string (Fig. 38) may be displaced

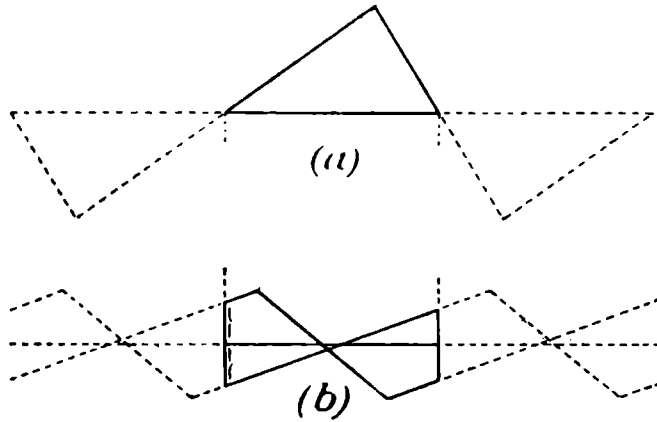


Fig. 38. Oscillation of a string as a superposition of two motions of opposite directions (b) from the initial state (a)

and then released with no acceleration (i.e., with $G(x) = 0$) at the time $t = 0$. Then we have $f_1 = f_2$, and the disturbance propagates in the same way in both directions. We can picture the oscillation of the string as due to the superpositions of two such waves travelling in opposite directions without change of shape. In order to satisfy the condition $u = 0$ at both ends we must continue the function (which is given physically for $0 \leq x \leq l$ only) in both directions so that it becomes antisymmetric and so that the disturbances propagating to the left and to the right continually cancel out one another at the points $x = 0$ and $x = l$. We say physically that the waves which arrive at those end points are reflected with a loss of half a wave length (inversion of the sign). The analogy between this extension of the functions and the Fourier series is evident.

2. Two-dimensional potential problems

The best known field of application of this method is in the solutions of the equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \quad (6.6)$$

by functions of the form

$$\Phi = f_1(x + iy) + f_2(x - iy) \quad (6.7)$$

Here i is the imaginary unit. Since Φ is real, f_1 and f_2 must be conjugate.

The solution can be simplified formally by addition of an imaginary solution $i\Psi$ which satisfies

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0. \quad (6.8)$$

Furthermore Ψ shall be chosen so that the sum

$$\Phi + i\Psi = f(x + iy) \quad (6.9)$$

is a function of the complex variable $x + iy$. We then have the Cauchy-Riemann equations relating the derivatives of Φ and Ψ :

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y}, \quad (6.10)$$

$$\frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x},$$

The two sides of the first of these equations represent the real part of f' , those of the second, i times the imaginary part of f' .

The physical significance of these equations becomes clear, if we consider the origin of the equation (6.6) (III 10 and 11). A vector field \mathbf{v} , which may represent the velocity in a potential flow or the force in a static field, satisfies the equations

$$\text{curl } \mathbf{v} = 0 \quad (6.11)$$

and

$$\text{div } \mathbf{v} = 0. \quad (6.12)$$

In addition we have the independence of the z -coordinate. We have previously introduced a scalar potential $\mathbf{v} = \text{grad } \Phi$ because of (6.11), and this potential function must satisfy $\Delta\Phi = 0$ (because of (6.12)).

In the same way we can introduce a vector potential according to (6.12), for which $\Delta\Psi = 0$ from (6.11). In this case of a plane problem, the vector potential only has one component and is therefore a scalar like Φ for the purposes of our computations.

3. Lines of constant potential and stream lines

The components u and v of the vector \mathbf{v} are

$$u = \frac{\partial\Phi}{\partial x} = \frac{\partial\Psi}{\partial y}, \quad v = \frac{\partial\Phi}{\partial y} = -\frac{\partial\Psi}{\partial x} \quad (6.13)$$

The curves $\Phi = \text{const.}$ (curves of constant potential) are given by

$$\frac{\partial\Phi}{\partial x} dx + \frac{\partial\Phi}{\partial y} dy = 0$$

or

$$\frac{dy}{dx} = -\frac{u}{v}. \quad (6.14)$$

Similarly we have for the curves $\Psi = \text{const.}$

$$\frac{dy}{dx} = \frac{v}{u}. \quad (6.15)$$

This means that the direction of these curves is the same as the direction of the vector \mathbf{v} . The curves have the direction of the field vector at every point in the field. For this reason these curves are called stream lines or lines of force (according to the nature of the problem). Equations (6.14) and (6.15) show that stream lines and the lines of constant potential are perpendicular to each other at every point.

By sketching this network of lines we may obtain a clear picture of the whole process. The direction of the field vector at any point is given by the direction of the stream line in this sketch. We can also find the length of the vector by the following consideration. Let us give to the constants in $\Phi = \text{const.}$ and $\Psi = \text{const.}$ a sequence of values having a constant difference ϵ which is sufficiently small. Now if we choose the coordinate system at any point such that the direction of the x -axis is the direction of the stream line and the direction of the y -axis that of the line of constant potential (Fig. 39), then the distance δx between the two equipotential lines whose difference of potential is ϵ satisfies

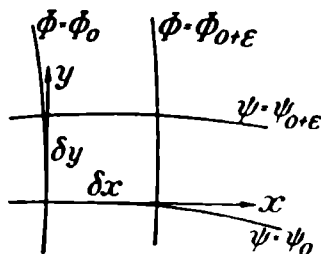


Fig. 39. Element of a net of potential and stream lines

$$\frac{\partial \Phi}{\partial x} \delta x = \epsilon.$$

The distance δy of two stream lines with Ψ -difference of ϵ satisfies

$$\frac{\partial \Psi}{\partial y} \delta y = \epsilon.$$

Thus according to (6.10) we have $\delta x = \delta y$ and the net is composed of small squares. Keeping ϵ fixed, the squares of this net are smaller the greater the absolute value of the field vector \mathbf{v} , since the latter is equal to $\partial \Phi / \partial x$. As an illustration we shall later show the representation of a potential flow and of the electric potential in a condenser.

In order to obtain such solutions we still must learn how to satisfy the boundary conditions. They can also be expressed in terms of Φ and Ψ in the simplest cases. In the case of the flow of a fluid, for example, the surface of any rigid body must be a stream line ($\Psi = \text{const.}$) since the direction of flow must be tangential to the surface. In electrostatic problems, the lines of force are always normal to a conducting surface, along which the potential is constant. ($\Phi = \text{const.}$). The two cases are equivalent from the mathematical standpoint. If

$$w = \Phi + i\Psi$$

and

$$z = x + iy,$$

we have the complex equation

$$w = f(z) \tag{6.16}$$

It is easy to find the function f in certain simple cases. For example, if the potential flow has a constant velocity U parallel to the x -axis we have

$$w = Uz, \quad (6.17)$$

and by differentiation we find

$$u = \frac{\partial \Phi}{\partial x} = U \quad \text{and} \quad v = \frac{\partial \Phi}{\partial y} = 0$$

The electrostatic field produced by a condenser which consists of two infinite planes at $y = 0$ and $y = a$ having the potentials 0 and Φ_0 , respectively, is

$$w = -iz\Phi_0/a \quad (6.18)$$

This means that $\Phi = \Phi_0 y/a$ and thus satisfies the boundary condition. All the lines of force $\Psi = -\Phi_0 x/a$ are normal to the plates.

4. Conformal mapping

Difficult cases, like the flow around a cylinder (Fig. 40), are treated by the following method. One introduces the new variable $\zeta = \xi + i\eta$ and tries to find a function $z = \varphi(\zeta)$ which establishes a correspondence between the boundary values ζ of the difficult problem with the boundary values z of a simple problem whose solution is known. For example, we may try to establish a correspondence between the central stream line of the simple parallel flow (6.17) and the stream line which curves around the cylinder.

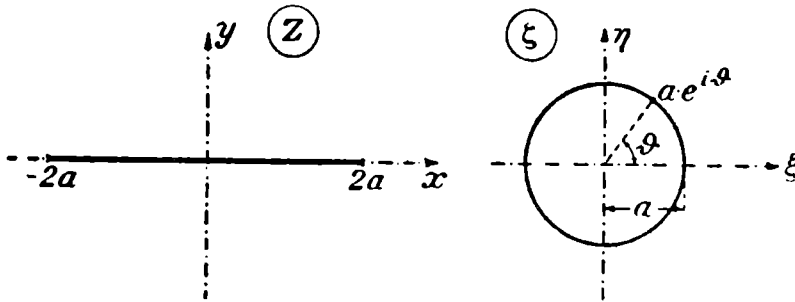


Fig. 40. Conformal mapping of a twice traversed segment onto a circle

If then $f(z)$ is the solution of the simple problem, the function

$$w = f[\varphi(\zeta)] = f_1(\zeta) \quad (6.19)$$

is the solution of the difficult problem. For every (analytic) function of a complex variable is a solution according to our fundamental idea. In addition, this solution satisfies the boundary condition and is therefore the solution we are looking for.

The variables x and y have been replaced by the variables ξ , η through the equation $z = \varphi(\zeta)$. This is usually expressed by saying that we have mapped the complex z -plane onto the complex ζ -plane. This mapping transforms the solution for the simple case of the variables x and y into the solution of the more difficult case as described by the variables ξ and η . For our example (parallel flow \rightarrow flow around cylinder) we can use the function

$$z = \zeta + \frac{a^2}{\zeta} \quad (6.20)$$

For if $\zeta = ae^{i\vartheta}$ (which is a point on the circle), the corresponding z is real and equal to $2a \cos \vartheta$. This is a

point on the x -axis between $-2a$ and $+2a$. All real values of ζ are also mapped into real values of z . Thus the x -axis of the z -plane is mapped onto the ξ -axis plus the circle of the ζ -plane. In both cases this is a stream line ($\Psi = 0$). Each stream line in the z -plane is transformed into a stream line of the ζ -plane by the same mapping, and the same is true for the lines of constant potential. This can be calculated by introducing (6.20) in (6.17)

$$w = U\left(\zeta + \frac{a^2}{\zeta}\right). \quad (6.21)$$

From this we obtain

$$\Phi + i\Psi = U\left(x + iy + \frac{a^2}{x + iy}\right)$$

or

$$\Phi = U\left(x + \frac{a^2x}{x^2 + y^2}\right) \text{ and } \Psi = U\left(y - \frac{a^2y}{x^2 + y^2}\right)$$

The result is represented in Fig. 42.

This method and the properties of conformal mapping are a subject of major interest in the theory of functions. One of the most important properties is the invariance of the angle between any two directions under a conformal mapping, except at singular points. This means, in particular, that the stream lines and lines of constant potential remain normal if a conformal mapping is applied to them.

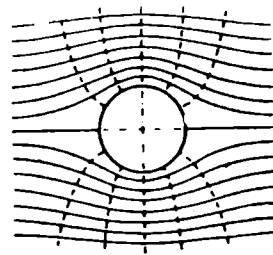


Fig. 41. Flow around a cylinder
— $\Psi = \text{const.}$,
... $\Phi = \text{const.}$

5. Examples

For many cases the mapping functions are completely known. There also exist general theorems concerning the mapping of a polygon onto a half plane or a circle, etc. However, we cannot consider special questions here. The example of the flow around a circle is of particular importance because it is possible to map the circle into the contour of an airfoil. As another illustration we shall use the example of a condenser by mapping from the trivial case (6.18). The mapping

$$z = \frac{a}{\pi} \log \zeta \quad (6.22)$$

maps every real z into a positive real ζ . Since

$$\log(-r) = i\pi + \log r,$$

there corresponds a negative real ζ to every point on the line $z = x + ia$ which represents one of the condenser plates in (6.18). Thus (6.22) maps the two plates into the two parts of the ξ -axis and the equation

$$w = -i \frac{\Phi_0}{\pi} \log \zeta \quad (6.23)$$

describes the electrostatic field produced by two parallel plates at different potentials. By separation of the real and the imaginary part in (6.23) we find

$$\begin{aligned} \Phi + i\Psi &= -i \frac{\Phi_0}{\pi} \log(re^{i\vartheta}) = -i \frac{\Phi_0}{\pi} (\log r + i\vartheta) \\ &= -i \frac{\Phi_0}{\pi} \log r + \frac{\Phi_0}{\pi} \vartheta. \end{aligned}$$

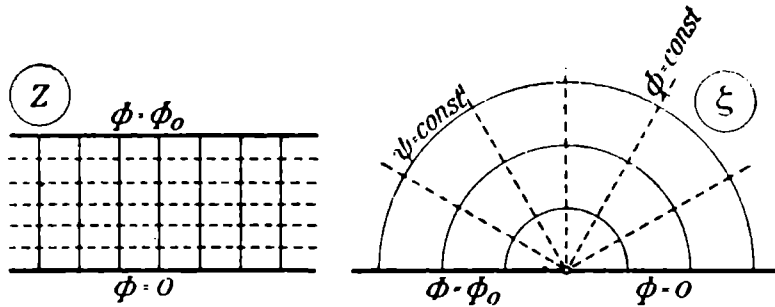


Fig. 42. Mapping of a strip into a half-plane

The lines of constant potential are the lines through the origin ($\vartheta = \text{const.}$) and the lines of force are the circles with the origin as center ($r = \text{const.}$). We shall now apply another mapping to the ξ -plane transforming it into the Z -plane by the function

$$Z = \arcsin \xi \quad \text{or} \quad \xi = \sin Z. \quad (6.24)$$

The real axis of the ξ -plane is then transformed into the contour of the Z -plane which is shown in Fig. 43. To

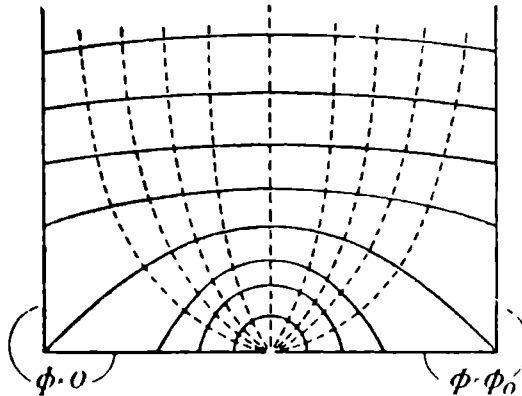


Fig. 43. Example of a conformal mapping
 $- \Psi = \text{const.}, \dots \Phi = \text{const.}$

the values of ζ between -1 and $+1$ there correspond real values of Z between $-\pi/2$ and $+\pi/2$. To the larger and smaller values of ζ there correspond Z -values whose real part is $-\pi/2$ or $+\pi/2$.

The field quantities are contained in the equation

$$w = -i \frac{\Phi_0}{\pi} \log \sin Z \quad (6.25)$$

and numerical calculation gives the curves of Fig. 43.

CHAPTER VII

SOLUTION BY THE USE OF SINGULARITIES

It is often easy to find the solution of a differential equation which is defined in all space and is equal to zero at points at infinity, but which has a singularity at one point, i.e., becomes infinite at this point. Whether or not this function has a physical meaning depends on the nature of the singularity.

1. Source

The clearest example of the above is the potential flow produced by a source. The differential equation

$$\Delta\Phi = 0 \tag{7.1}$$

must be valid in the whole space, but at the origin where there is a source. The equation cannot be valid there, for according to (4.15) it expresses the fact that there is no source in the region of its validity. The solution evidently has spherical symmetry with the point of origin as center and is therefore a function of the radius only. From the form (3.22) of the differential equation

$$\frac{d}{d\rho} \left(\rho^2 \frac{d\Phi}{d\rho} \right) = 0, \tag{7.2}$$

we obtain the two particular solutions $\Phi = \text{const.}/\rho$ and $\Phi = \text{const.}$ The latter has no physical meaning. The former is equal to zero at infinity and has a singularity at the origin. It is therefore the representation of

the source flow we seek. The constant depends on the magnitude of the source. If we write $\Phi = -C/\rho$, the vector \mathbf{v} which is free of sources in the whole space except at the origin has the magnitude

$$v = \frac{d\Phi}{d\rho} = + \frac{C}{\rho^2}.$$

For the integral of the divergence of \mathbf{v} over any sphere which encloses the origin we obtain, with the use of the theorem of Gauss (3.12),

$$Q = \int \operatorname{div} \mathbf{v} dV = \int \frac{C}{\rho_0^2} \rho_0^2 d\omega, \quad (7.3)$$

where $d\omega$ is the element of the solid angle and ρ_0 is the radius of the sphere. We find

$$Q = 4\pi C. \quad (7.4)$$

The quantity of fluid leaving the sphere in unit time must be independent of the radius of the sphere, since otherwise the region between two spheres would not be free of sources. The solution of (7.2) is

$$\Phi = - \frac{Q}{4\pi\rho}. \quad (7.5)$$

This is the mathematical expression for a point source. The result is the same for the potential in the gravitation field of a single mass point, since the differential equation is the same (4.3). To the constant Q there corresponds the number $-4\pi fM$, if f is the constant of gravitation and M the mass of the point. By differentiation with respect to ρ we obtain Newton's law that the force per unit mass is $-fM/r^2$.

2. Superposition of positive and negative sources

By summation of such fundamental solutions we can find solutions of more complicated cases, since the equation is linear and homogeneous. In this way we find the potential of a continuous distribution of sources whose strength per unit volume is a function μ of position,

$$\begin{aligned}\Phi &= - \int \frac{\mu dV}{4\pi\rho} \\ &= - \int \frac{\mu(\xi, \eta, \zeta)}{4\pi\rho(\xi, \eta, \zeta; x, y, z)} d\xi d\eta d\zeta.\end{aligned}\tag{7.6}$$

This is a solution of the differential equation

$$\Delta\Phi = \mu,\tag{7.7}$$

since $\Delta\Phi = \text{div } \mathbf{v}$ is the strength per unit volume. In order to calculate the integral (7.6) we must remember that the origin of ρ depends on the point here. If the potential at the point x, y, z is to be calculated, for example, we have in the integrand

$$\rho = [(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2]^{\frac{1}{2}}.\tag{7.8}$$

μ can also have negative values. A particularly important case is the one where a positive and a negative source (for example electric charges) of equal magnitude are very close together. This is called a dipole (Fig. 44). If the distance ρ from the dipole is large compared with the length ϵ of the dipole, the potential can be written in the following way

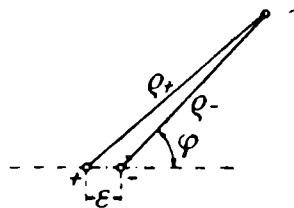


Fig. 44. Dipole

$$\Phi = \frac{Q}{4\pi} \left(\frac{1}{\rho_+} - \frac{1}{\rho_-} \right) = + \frac{Q\epsilon \cos \varphi}{4\pi\rho^2}. \quad (7.9)$$

$Q\epsilon$ is called moment of the dipole.

3. Satisfying the boundary conditions

Simple boundary conditions can also be satisfied with our fundamental solutions if we use the method of reflection. If, for example, we have a source and a wall on which the normal component must be zero (potential flow), then we introduce an ideal source of the same magnitude whose position is symmetrical to the first source with respect to the wall. If on the other hand the wall has a constant potential (electric potential), the second source must be assumed to have the same magnitude but opposite sign (Fig. 45).

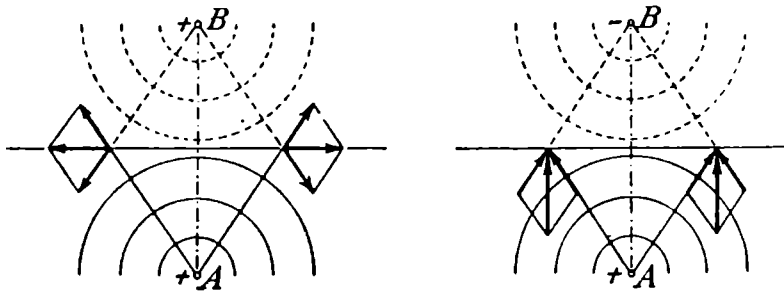


Fig. 45. Reflections

In the first case the field is

$$\Phi = \frac{Q}{4\pi\rho_A} + \frac{Q}{4\pi\rho_B}, \quad (7.10)$$

and in the second case it is

$$\Phi = \frac{Q}{4\pi\rho_A} - \frac{Q}{4\pi\rho_B}. \quad (7.11)$$

The boundary conditions can be satisfied for other than plane walls by superposition of negative and positive sources and dipoles. By superposition of a parallel flow along the x -axis and a dipole at the origin we obtain, for example,

$$\Phi = Ux + \frac{Q\epsilon \cos \varphi}{4\pi\rho^2} = \left(U\rho + \frac{Q\epsilon}{4\pi\rho^2} \right) \cos \varphi \quad (7.12)$$

If we calculate the component $\partial\Phi/\partial\rho$ (the radial component of the velocity) on the sphere with radius $\rho_0 = (Q\epsilon/2\pi U)^{\frac{1}{2}}$, we obtain zero. Thus (7.12) represents a potential flow around a sphere. By point sources or a continuous distribution of sources along the x -axis, we obtain bodies of revolution (other than the sphere) for which the potential flow can be easily calculated.

If the flow is plane, we have cylindrical symmetry. In this case the potential of the source can be calculated from the differential equation

$$\frac{d}{dr} \left(r \frac{d\Phi}{dr} \right) = 0$$

and we obtain

$$\Phi = \frac{Q}{2\pi} \log r. \quad (7.13)$$

The corresponding dipole potential is

$$\Phi = \frac{Q\epsilon \cos \varphi}{2\pi r}.$$

We obtain the flow around a circular cylinder by superposition of this dipole with a parallel flow

$$\Phi = Ux + \frac{Q}{2\pi} \frac{\cos \varphi}{r}, \quad (7.13a)$$

With $Q/2\pi a^2 = U$ we again obtain formula (6.21).

4. Solutions with the aid of the theorem of Green

From Green's theorem (5.27) we obtain a general and physically clear solution of $\Delta\Phi = \mu$, if we introduce $\Psi = 1/\rho$. As origin for the coordinate ρ we take the point at which we want to know the potential. Since $\Delta(1/\rho) = 0$, we obtain

$$\int \Delta\Phi \frac{1}{\rho} dV = \int \left(\frac{1}{\rho} \frac{\partial\Phi}{\partial n} - \Phi \frac{\partial}{\partial n} \left(\frac{1}{\rho} \right) \right) df. \quad (7.14)$$

We shall apply this equation to a region whose exterior is arbitrarily limited or unlimited, but from which we have cut out a small sphere of radius ρ_0 whose center is at $\rho = 0$. On this sphere we have

$$\frac{\partial}{\partial n} \left(\frac{1}{\rho} \right) = + \frac{1}{\rho^2}$$

(n is the *outward* normal, i.e., in this case the direction toward the point of origin!). Furthermore we have $df = \rho^2 d\omega$ ($d\omega =$ solid angle). If we now take the limit $\rho_0 \rightarrow 0$, the first term of the integral on the right side of (7.14) cancels out and the second becomes equal to $-4\pi\Phi_0$, if Φ_0 is the potential at the origin which we are looking for. Thus we have

$$\begin{aligned} 4\pi\Phi_0 = & - \int \Delta\Phi \frac{1}{\rho} dV + \int \frac{1}{\rho} \frac{\partial\Phi}{\partial n} df \\ & - \int \Phi \frac{\partial}{\partial n} \left(\frac{1}{\rho} \right) df, \end{aligned} \quad (7.15)$$

where we must now integrate only over the exterior surface of the region.

The three terms on the right hand side of (7.15) have a special physical meaning. The space integral, which is the same as (7.6), represents the potential of the charges or masses. The surface integrals show the influence of the boundary values. $\partial\Phi/\partial n$ is the normal component of the vector grad Φ at the boundary and thus is equal to the intensity of the field vector which enters the region V through the surface f . If the region is limited by a metallic cover $\partial\Phi/\partial n$ can be considered as "surface charge" according to (4.34).

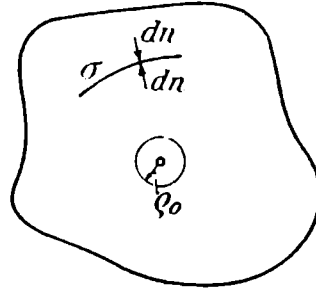


Fig. 46. Charge on a surface and on a point

For an interpretation of the last integral in (7.15) we shall consider a region of infinitesimal thickness, which in the limit is "bounded by the two sides of a surface". On both sides of this region, $\partial(1/\rho)/\partial n$ has the same magnitude but opposite signs. If, therefore, Φ is the same on both sides, the integral is equal to zero. It is finite if the potential is not continuous through the surface. Such a discontinuity is caused by a distribution of dipoles (opposite charges on both sides of the surface).

5. Heat source

This method is also useful in the theory of the conduction of heat. We shall consider a one-dimensional region in which the differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \frac{\partial u}{\partial t} \quad (7.16)$$

is valid. Let us assume that at the point $x = 0$ there is a "heat source". This means that at the instant $t = 0$ the temperature at this point becomes infinite, but in such a way that the total quantity of heat remains finite. The solution of (7.17), which describes the flow of heat from the source is then

$$u = \frac{1}{(4\pi kt)^{\frac{1}{2}}} e^{-x^2/4kt}, \quad (7.17)$$

as is easily verified. In Fig. 47 we have drawn this function for various times. By the use of reflections it is also possible to satisfy the most important boundary

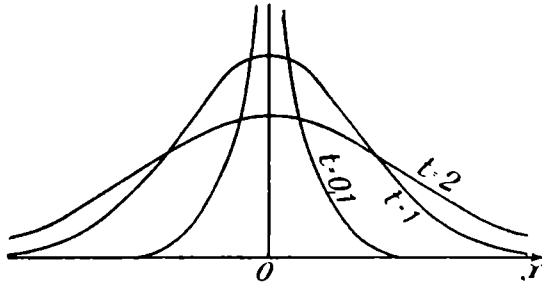


Fig. 47. The flow of heat from a heat source

conditions in this case. An arbitrary initial temperature $f(x)$ can be considered as the sum of heat sources of strength $f(\xi) d\xi$ at the points $x = \xi$ and produces a solution in the form of an integral

$$u = \frac{1}{(4\pi kt)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} f(\xi) e^{-(x-\xi)^2/4kt} d\xi. \quad (7.18)$$

This formula is equivalent to the formula (5.80) which we obtained from the Fourier integral, since

$$\int_0^{\infty} e^{-\alpha^2 kt} \cos \alpha(x - \xi) d\alpha = \left(\frac{\pi}{4kt} \right)^{\frac{1}{2}} e^{-(x-\xi)^2/4kt}.$$

This formula, however, cannot be proved here. In the two-dimensional case, (7.17) is replaced by

$$u = \frac{1}{4\pi kt} e^{-r^2/4kt} \quad (7.19)$$

and in the three-dimensional by

$$u = \frac{1}{(4\pi kt)^{3/2}} e^{-\rho^2/4kt}. \quad (7.20)$$

6. Flash of light

We can also find a solution containing a singularity for the wave equation. A light flash which is emitted from the origin at the instant $t = 0$ obeys the differential equation

$$\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial u}{\partial \rho} \right) = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad (7.21)$$

in spherical coordinates. A solution of this equation is

$$u = \frac{C}{\rho} F(\rho - ct) \quad (7.22)$$

as can be easily verified.

Now let $F(x)$ be the "delta" function which is equal to zero everywhere except at $x = 0$. At this point the function is infinite in such a way that

$$\int_{-\infty}^{+\infty} F(x) dx = 1. \quad (7.23)$$

(For a more exact theory, we should consider a sequence of functions which have the formulated properties in the limit). The constant C is determined by the magnitude Q of the flash. We obtain the relation by calculating the flow from a region with the small radius ρ_0 and the point of origin as center (df = surface element)

$$Q = \int_0^\infty dt \int \frac{\partial u}{\partial \rho} df = - \int_0^\infty 4\pi \rho_0^2 \frac{C}{\rho_0^2} F(\rho_0 - ct) dt,$$

since

$$\frac{\partial u}{\partial \rho} = - \frac{C}{\rho^2} F + \frac{C}{\rho} F'$$

and since the second term with $\rho_0 \rightarrow 0$ becomes equal to zero on integration. According to (7.23) we now have

$$Q = \frac{4\pi C}{c}. \quad (7.24)$$

This representation is of special importance in electronic theory. There we have the equation (7.21) for the scalar potential and the three components of the vector potential in empty space, according to (4.31). At the distance ρ from the point where we seek the scalar potential Φ , there may now be a volume element dV which at the time $t = \tau$ contains the charge μdV during the time $d\tau$. According to (7.22) and (7.24) we have

$$d\Phi = \frac{c}{4\pi} \mu dV \frac{1}{\rho} F[\rho - c(t - \tau)] d\tau. \quad (7.25)$$

We obtain the total potential by integrating over the whole space and all time elements from $-\infty$ to $+\infty$.

$$4\pi\Phi = \int \frac{dV}{\rho} \mu^*. \quad (7.26)$$

Here μ^* is the charge density in the volume element dV at the time $t - \tau = \rho/c$. This instant is earlier than the time t at which we want to know the potential. The difference of time is ρ/c , which is just the time required by the impulse to travel the distance ρ . That is why the integrand in (7.26) and the corresponding expressions for **A** are called the retarded potentials.

7. Transition to integral equations

The solutions which we obtained by the method of singularities are expressed by an integral, the domain of which is the whole domain of an auxiliary variable. The typical form is clearly illustrated in the example (7.18), which contains only one variable

$$u(x, t) = \int_{-\infty}^{+\infty} f(\xi)K(x - \xi, t) d\xi. \quad (7.27)$$

The function K is a solution of the equation which has a singularity at $x = \xi$. The dependence on t is of no importance in evaluating the integral. The elementary process with the singularity at $x = \xi$ is described by the function K . Here we need no longer bother about the deduction of K from the differential equation.

It may be that the function $f(x)$ is not given initially. The function $u(x)$ might be given instead (for a certain t), and then the problem is to determine $f(x)$. In this case $f(x)$ is the unknown in the equation (7.27) which is an *integral equation of the first type*. K is called the kernel of the equation.

The problem of finding the initial heat sources from the heat function at a certain instant seldom occurs in practice. However, problems of this kind occur frequently in other fields. If, for example, one has to determine the flow around an airship of a certain shape, one uses an integral equation similar to (7.6) in order to calculate the sources which can generate the flow according to VII 12 from the given form of the stream lines.

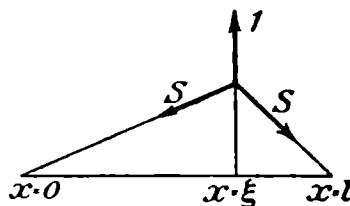


Fig. 48. String, stretched by a single force

The representation by an integral equation becomes even more important if u itself is unknown, while there exists a known relation between u and $f(\xi)$. We have such a case in the example of the oscillating string. The oscillation is caused by the equilibrium between the forces of inertia and the tension. In order to find a fundamental solution we consider a string which is displaced by a single unit force (see Fig. 48). If u is the displacement and S the tension, then we have the equations

$$K(x, \xi) = \frac{1}{Sl} x(l - \xi) \quad \text{for} \quad x < \xi$$

(7.28)

$$K(x, \xi) = \frac{1}{Sl} \xi(1 - x) \quad \text{for} \quad x > \xi.$$

This function $K(x, \xi)$ is the kernel of the integral equation. The force of inertia at the point ξ corresponds to the function $f(\xi)$. If the oscillation is

$$u = y(\xi) \sin \nu t, \quad (7.29)$$

this force is $\rho \nu^2 y(\xi) d\xi \sin \nu t$. Thus we obtain the integral equation

$$y(x) = \rho \nu^2 \int_0^l y(\xi) K(x, \xi) d\xi, \quad (7.30)$$

which is a so-called *homogeneous integral equation of the second type*. If, on the left hand side, we have an additional known function of x besides $y(x)$, the equation is inhomogeneous and of the second kind. The parameter ν has here the same properties as an eigenvalue in the case of differential equations. In our examples, the same problem can be expressed by a differential equation with boundary conditions or by an integral equation. The boundary conditions are automatically satisfied by the form of the kernel in the latter case.

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