Estimation of parameters of the Gumbel type-II distribution under AT-II PHCS with an application of Covid-19 data

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Abstract

In this paper, we investigate the classical and Bayesian estimation of unknown parameters of the Gumbel type-II distribution based on adaptive type-II progressive hybrid censored sample (AT-II PHCS). The maximum likelihood estimates (MLEs) and maximum product spacing estimates (MPSEs) are developed and computed numerically using Newton-Raphson method. Bayesian approaches are employed to estimate parameters under symmetric and asymmetric loss functions. Bayesian estimates are not in explicit forms. Thus, Bayesian estimates are obtained by using Markov chain Monte Carlo (MCMC) method along with the Metropolis-Hastings (MH) algorithm. Based on the normality property of MLEs the asymptotic confidence intervals are constructed. Also, bootstrap intervals and highest posterior density (HPD) credible intervals are constructed. Further a Monte Carlo simulation study is carried out. Finally, the data set based on the death rate due to Covid-19 in India is analyzed for illustration of the purpose.
Keywords : Adaptive type-II progressive hybrid censoring; Maximum product spacing estimation; Markov chain Monte Carlo; Highest posterior density credible interval; Coverage probability; Covid-19 data set.

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1 Introduction

In recent times, life testing experiments and reliability studies have achieved more acceptance. In such experiments, various situations appear where experimental units are eliminated before the time of failure. In these cases, the investigator may not have full information about the failure times of the whole experimental unit. Such kind of data collected from all these experiments are called censored data. It is worth mentioning that the censoring was introduced in practice to save time and reduce the number of failed units in a life-testing experiment. Censoring can be done with respect to a prefixed time or prefixed number of

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failures or sometimes a combination of both prefixed time and number of failures. Depending upon these criteria, there are different types of censoring schemes. Let us consider a life testing experiment with n number of experimental units and $X_{i:n}$ denotes the time of i-th failure of the experimental units, where $i = 1, \ldots, n$. Let T be a pre-specified time. If all the experiments fail before that pre-specified time T, it is known as a type-I censoring scheme. In the case of the type-II censoring scheme, the experiment will terminate after m-th number of failures observed, where $m \leq n$. Both type-I and type-II censoring schemes have some disadvantages. In the case of type-I censoring, the number of failures may be zero, and in the case of type-II censoring, experimental time may be very large. For further information about these schemes, we refer to Lawless (2011), Sirvanci and Yang (1984) and Balakrishnan and Chan (1992).

Epstein (1954) introduced a censoring scheme which is a mixture of type-I and type-II censoring schemes, known as the type-I hybrid censoring scheme. In this scheme, the experiment is terminated at time $T^* = min\{T, X_{m:n}\}$, where $X_{m:n}$ denotes the m-th failure and T is pre-specified time. Childs et al. (2003) proposed type-II hybrid censoring scheme where experiment will be terminated at time $T^* = max\{X_{m:n}, T\}$. For further studies on these hybrid censoring schemes, one may read Balakrishnan and Kundu (2013), Kundu (2007), Kundu and Pradhan (2009) and Dey and Pradhan (2014). In practical life testing experiments, there are many cases when the experimental units are removed from experiments before failure for other reasons. To overcome such scenarios, a type-II right censoring scheme is introduced, known as a progressive type-II censoring scheme. It has more flexibility in allowing the removal of units at time points other than the terminal point of the experiments. For progressive type-II censoring scheme, let us consider that n experimental units are placed on a life testing experiment. Let $X_{i:m:n}$ be the failure time of the i-th experimental unit, and m be the number of failures that is fixed before the experiment starts. After first failure at $X_{1:m:n}$, R_1 number of units are removed randomly from $n-1$ surviving units. After second failure at $X_{2:m:n}$, R_2 number of units are removed randomly from $n - R_1 - 2$ surviving units. The test continues until m-th failure occurs and all remaining $R_m = n - \sum_{i=1}^{m} R_i - m$ surviving units are removed. In the progressive type-II censoring scheme, the values of R_i's are pre-fixed. Many researchers have studied statistical inferences of different distributions under this censoring scheme. For example, see Balakrishnan et al. (2003), Rastogi and Tripathi (2012) and Maiti and Kayal (2019).

In recent years, progressive type-II censoring scheme gained more popularity in life testing experiments and reliability studies. But there is a major drawback of this censoring scheme that it may take a large time to complete an experiment. To overcome this drawback, Kundu and Joarder (2006) introduced progressive hybrid censoring scheme. Basically, there are two types of progressive hybrid censoring schemes. Let us consider progressively censored ordered statistics $X_{1:m:n}, \ldots, X_{m:m:n}$ from a life testing experiments of n units. Further, $(R_1, \ldots, R_m)$ are assumed to be the corresponding removals. If the experiment terminates at a time $T^* = min\{T, X_{m:m:n}\}$, where $T \in (0, \infty)$ is prefixed, then it is called type-I progressive hybrid censoring scheme (PHCS). In case of type-I PHCS, at the time of first failure $X_{1:m:n}$, R_1 surviving units are removed randomly from $(n-1)$ units. At time of second failure $X_{2:m:n}$, R_2 number of surviving units are removed randomly from $(n - R_1 - 2)$ units. In similar way if m-th failure occurs before time T, i.e. $X_{m:m:n} < T$, then the observed failures are $X_{1:m:n}, \ldots, X_{m:m:n}$ and all remaining units $R_m = n - m - \sum_{i=1}^{m-1} R_i$ are removed at termination point $X_{m:m:n}$. If $X_{m:m:n} > T$, then the observed failures are $X_{1:m:n}, \ldots, X_{j:m:n}$, where $X_{j:m:n} < T < X_{j+1:m:n}$ and all remaining units $R_j^* = n - j - \sum_{i=1}^{j} R_i$ at the time of termination T. In recent years, statistical inference under type-I PHC has been studied by several authors. One may refer to Lin and Huang (2012), Tomer and Panwar (2015) and Kayal et al. (2017). There is a drawback of type-I progressive hybrid censoring scheme, since the effective sample size may turn out to a small number. Thus, it has lower efficiency in computing statistical inferences for some problems.

To increase such efficiency, Ng et al. (2009) proposed adaptive type-II progressive hybrid censoring scheme (AT-II PHCS) where the effective sample size m and time T are pre-specified. This censoring scheme is similar to the type-I progressive hybrid censoring scheme. It is different when $T < X_{m:m:n}$. If $X_{j:m:n} < T < X_{j+1:m:n}$, where $j + 1 < m$, all remaining units will be terminated by setting $R_{j+1} = \ldots = R_{m-1} = 0$ and $R_m = n - m - \sum_{i=1}^{j} R_j$. Thus the experiment will be terminated after getting the expected sample size. In recent years many authors studied inferences of various lifetime models under AT-II PHCS. Lin et al. (2009) obtained classical estimates on the Weibull lifetime model based on AT-II PHCS. Hemmati and Khorram (2013) obtained maximum likelihood and approximated maximum likelihood estimates of model parameters of two parameters log-normal distribution based on AT-II PHCS. Nassar and Abo-Kasem (2016) discussed maximum likelihood estimates and Bayesian estimates of parameters of Burr XII distribution model based on AT-II PHCS. Panahi and Moradi (2020) discussed classical and Bayesian estimation of parameters of inverted exponential Rayleigh distribution. Panahi and Asadi (2021) discussed the maximum likelihood estimates and Bayesian estimates of parameters of Burr III distribution based on AT-II PHCS.

Gumbel (2004) introduced a two-parameter distribution, known as Gumbel type-II distribution, which is very useful to model meteorological phenomena such as floods, earthquakes, and natural disasters. Also it can used in life expectancy tables, hydrology and rainfall. The cumulative distribution function (CDF) of Gumbel type-II distribution is

$$F(x) = e^{-\beta x^{-\alpha}}, \quad x > 0, \quad \alpha, \beta > 0, \tag{1.1}$$

where α is the shape parameter, β is the scale parameter. The corresponding probability density function (PDF) is

$$f(x) = \alpha\beta x^{-(\alpha+1)} e^{-\beta x^{-\alpha}}, \quad x > 0, \quad \alpha, \beta > 0. \tag{1.2}$$

The hazard rate function of Gumbel type-II distribution is given by

$$h(x) = \frac{\alpha\beta x^{-(\alpha+1)}}{e^{\beta x^{-\alpha}} - 1}, \quad x > 0, \quad \alpha, \beta > 0. \tag{1.3}$$

Figure 1 represents graphs of the PDFs and hazard rate functions of the Gumbel type-II distribution based on different values of the parameters α and β. It is noticed that the shape of the hazard rate function of Gumbel type-II distribution is decreasing and upside-down bathtub (UTB). Due to these shapes of the hazard rate function, the Gumbel type-II

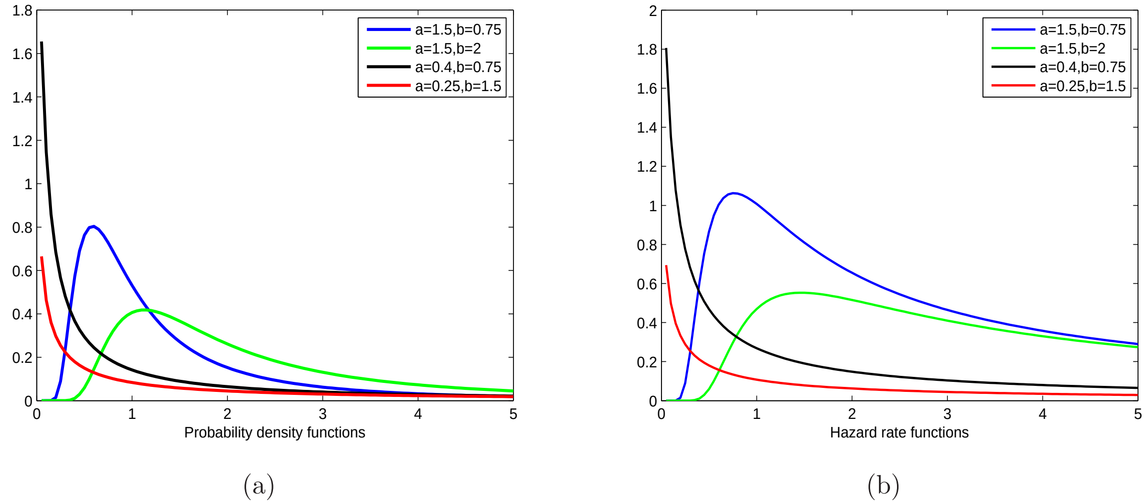


Figure 1: (a) PDF and (b) Hazard rate functions of Gumbel type-II distribution for different values of α and β.

distribution is very flexible to use in different research areas such as clinical, reliability and survival studies. Recently, many authors have studied statistical properties of the estimators of the model parameters of Gumbel type-II distribution. As example, Abbas et al. (2013) discussed Bayesian estimation of the model parameters. Reyad and Ahmed (2015) studied E-Bayesian estimation of the unknown model shape parameter. Sindhu et al. (2016) obtained Bayes estimates and corresponding risk of the unknown model parameters based on left-censored data. Abbas et al. (2020) proposed Bayesian estimation of the model parameters under type-II censored sample with some medical applications.

Due to importance of the Gumbel type-II distribution and the AT-II PHCS, in this paper, we have considered the problem of estimation of the parameters of Gumbel type-II distribution under AT-II PHCS. To the best of our information, this problem has not been studied yet. The rest of the article is organized as follows. In Section 2, MLEs and MPSEs are computed by using the Newton-Raphson method. The Bayesian estimates are obtained under symmetric and asymmetric loss functions using the Markov chain Monte Carlo (MCMC) technique in Section 3. In Section 4, the asymptotic confidence intervals using the normality property of the MLEs and bootstrap confidence intervals are constructed. Also, MCMC samples are used to build HPD credible intervals. A Monte Carlo simulation study is carried out to compare the performance of different estimates in Section 5. Section 6 deals with a real data set based on the death rates due to Covid-19 in India. Finally, conclusions of this paper have been drawn in Section 7.

2 Classical estimation

In this section, classical estimation of unknown model parameters of Gumbel type-II distribution is obtained by using two methods: (a) maximum likelihood estimation and (b)

maximum product spacing.

2.1 Maximum likelihood estimation

Let $x_{1:m:n} < \ldots < x_{j:m:n} < T < x_{j+1:m:n} < \ldots < x_{m:m:n}$ be an adaptive type-II progressive censored ordered sample from (1.2) along with a censoring scheme $(R_1, \ldots, R_j, 0, \ldots, 0, R_j^*)$, where $R_j^* = n - m - \sum_{i=1}^{j} R_i$ and T is pre-specified. Then, the likelihood function can be written as

$$L(\alpha, \beta|\underline{x}) \propto \prod_{i=1}^{m} f(x_{i:m:n}) \prod_{i=1}^{j} [1 - F(x_{i:m:n})]^{R_i} [1 - F(x_{m:m:n})]^{R_j^*}. \tag{2.1}$$

Whenever $x_{m:m:n} < T$, then $R_j^* = 0$ and the likelihood function becomes

$$L(\alpha, \beta|\underline{x}) \propto \prod_{i=1}^{m} f(x_{i:m:n}) [1 - F(x_{i:m:n})]^{R_i}. \tag{2.2}$$

Replacing $F(x)$ and $f(x)$ from Equations (1.1) and (1.2) in Equation (2.1) we get,

$$L \propto \alpha^m \beta^m \prod_{i=1}^{m} \left[{x_i}^{-(\alpha+1)} e^{-\beta x_i^{-\alpha}} \right] \prod_{i=1}^{j} \left[1 - e^{-\beta x_i^{-\alpha}} \right]^{R_i} \left[1 - e^{-\beta x_m^{-\alpha}} \right]^{R_j^*}, \tag{2.3}$$

where x_i represents the i-th failure time $x_{i:m:n}$. The log-likelihood function is given as

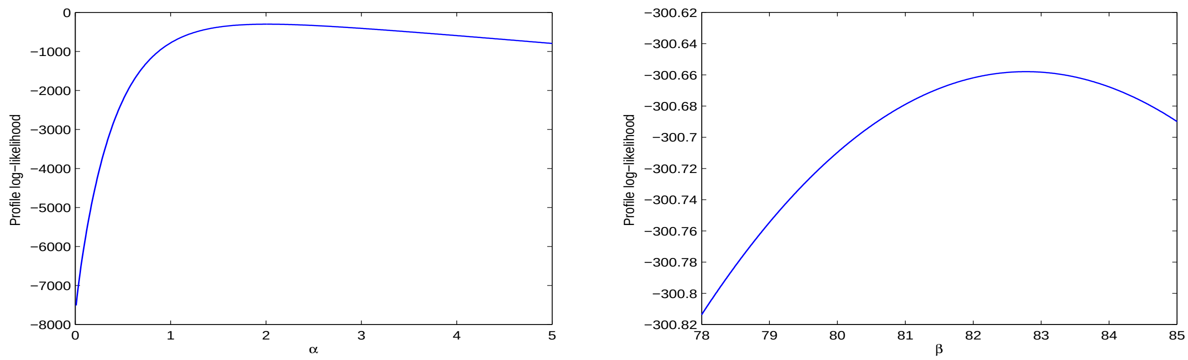
$$\begin{aligned} \log L \propto \;& m \log \alpha + m \log \beta - (\alpha + 1) \sum_{i=1}^{m} \log x_i - \beta \sum_{i=1}^{m} {x_i}^{-\alpha} + \sum_{i=1}^{j} R_i \log(1 - e^{-\beta x_i^{-\alpha}}) \\ & + R_j^* \log(1 - e^{-\beta x_m^{-\alpha}}). \end{aligned} \tag{2.4}$$

After differentiating (2.4) partially with respect to α and β respectively, and equating to zero we get

$$\begin{aligned} \frac{\partial l}{\partial \alpha} =& \frac{m}{\alpha} - \sum_{i=1}^{m} \log x_i + \beta \sum_{i=1}^{m} x_i^{-\alpha} \log x_i - \beta \sum_{i=1}^{j} \frac{R_i x_i^{-\alpha} e^{-\beta x_i^{-\alpha}} \log x_i}{1 - e^{-\beta x_i^{-\alpha}}} \\ & - \beta \frac{R_j^* x_m^{-\alpha} e^{-\beta x_m^{-\alpha}} \log x_m}{1 - e^{-\beta x_m^{-\alpha}}} = 0, \end{aligned} \tag{2.5}$$

$$\frac{\partial l}{\partial \beta} = \frac{m}{\beta} - \sum_{i=1}^{m} x_i^{-\alpha} + \sum_{i=1}^{j} \frac{R_i x_i^{-\alpha} e^{-\beta x_i^{-\alpha}}}{1 - e^{-\beta x_i^{-\alpha}}} + \frac{R_j^* x_m^{-\alpha} e^{-\beta x_m^{-\alpha}}}{1 - e^{-\beta x_m^{-\alpha}}} = 0. \tag{2.6}$$

The maximum likelihood estimates of unknown parameters α and β can be obtained from Equations (2.5) and (2.6), which are not in closed form, thus need to be solved numerically. The maximum likelihood estimates of the unknown parameters are denoted by $\widehat{\alpha}$ and $\widehat{\beta}$. Further, when computing MLEs numerically, it is always of interest to study the existence and uniqueness of the MLEs of the parameters. In order to achieve this, one requires to show two conditions proposed by Mäkeläinen et al. (1981). These are difficult to establish due to the complicated nature of the expressions of the second order partial derivatives of the log-likelihood function. To have a rough idea of that, the profile log-likelihood functions for the parameters α and β are presented in Figure 2. These figures represent that the MLEs may exist uniquely.



(a) (b)

Figure 2: Profile log-likelihood functions of α and β for the real data set.

2.2 Maximum product spacing estimation

Cheng and Amin (1983) proposed maximum product spacing estimation (MPSE) method as an alternative to the MLE. Almetwally et al. (2019) discussed MPSE method under AT-II PHCS for generalized Rayleigh distribution. The product spacing under AT-II PHCS can be written as

$$M = \prod_{i=1}^{m+1} D_i \prod_{i=1}^{j} \Big[1 - F(x_i; \alpha, \beta)\Big]^{R_i} \Big[1 - F(x_m; \alpha, \beta)\Big]^{R_j^*}, \tag{2.7}$$

where

$$D_i = \begin{cases} F(x_1), & i = 1 \\ F(x_i) - F(x_{i-1}), & i = 2, 3, \ldots, m \\ 1 - F(x_m), & i = m + 1, \end{cases} \tag{2.8}$$

such that $\sum D_i = 1$. Then, the product spacing function under AT-II PHCS based on Gumbel type-II model can be written as

$$M = e^{-\beta x_1{}^{-\alpha}} \Big[1 - e^{-\beta x_m{}^{-\alpha}}\Big] \prod_{i=2}^{m} \Big[e^{-\beta x_i{}^{-\alpha}} - e^{-\beta x_{i-1}{}^{-\alpha}}\Big] \Big[1 - e^{-\beta x_m{}^{-\alpha}}\Big]^{R_j^*}. \tag{2.9}$$

Now, the logarithm of space function can be written as

$$\begin{aligned} \log M =& -\beta x_1{}^{-\alpha} + \log(1 - e^{-\beta x_m{}^{-\alpha}}) + \sum_{i=2}^{m} \log\left(e^{-\beta x_i{}^{-\alpha}} - e^{-\beta x_{i-1}{}^{-\alpha}}\right) \\ &+ \sum_{i-1}^{j} R_i \log(1 - e^{-\beta x_i{}^{-\alpha}}) + R_j^* \log(1 - e^{-\beta x_m{}^{-\alpha}}). \end{aligned} \tag{2.10}$$

To obtain the normal equations from the above Equation (2.10), differentiate partially with respect to unknown parameters and equating to zero, we get

$$\frac{\partial \log M}{\partial \alpha} = -\frac{\beta x_m{}^{-\alpha} e^{-\beta x_m{}^{-\alpha}}}{(1-e^{-\beta x_m{}^{-\alpha}})} + \beta \sum_{i=2}^{m} \left[\frac{x_i{}^{-\alpha} e^{-\beta x_i{}^{-\alpha}} \log x_i - x_{i-1}{}^{-\alpha} e^{-\beta x_{i-1}{}^{-\alpha}} \log x_{i-1}}{e^{-\beta x_i{}^{-\alpha}} - e^{-\beta x_{i-1}{}^{-\alpha}}} \right]$$
$$- \beta \sum_{i=1}^{j} \frac{R_i x_i{}^{-\alpha} e^{-\beta x_i{}^{-\alpha}} \log x_i}{(1-e^{-\beta x_i{}^{-\alpha}})} - \frac{R_j^* \beta x_m{}^{-\alpha} e^{-\beta x_m{}^{-\alpha}} \log x_m}{(1-e^{-\beta x_m{}^{-\alpha}})} + \beta x_1{}^{-\alpha} \log x_1 = 0. \tag{2.11}$$

$$\frac{\partial \log M}{\partial \beta} = -x_1{}^{-\alpha} + \sum_{i=2}^{m} \frac{x_{i-1}{}^{-\alpha} e^{-\beta x_{i-1}{}^{-\alpha}} - x_i{}^{-\alpha} e^{-\beta x_i{}^{-\alpha}}}{e^{-\beta x_i{}^{-\alpha}} - e^{-\beta x_{i-1}{}^{-\alpha}}} + \sum_{i=1}^{j} \frac{R_i x_i{}^{-\alpha} e^{-\beta x_i{}^{-\alpha}}}{(1-e^{-\beta x_i{}^{-\alpha}})}$$
$$+ \frac{R_j^* x_m{}^{-\alpha} e^{-\beta x_m{}^{-\alpha}}}{(1-e^{-\beta x_m{}^{-\alpha}})} + \frac{x_m{}^{-\alpha} e^{-\beta x_m{}^{-\alpha}}}{(1-e^{-\beta x_m{}^{-\alpha}})} = 0. \tag{2.12}$$

Above these two equations can not be solved analytically, so numerical method will be applied to obtain the MPSEs of the unknown parameters α and β, as $\widehat{\alpha}_{MPS}$ and $\widehat{\beta}_{MPS}$, respectively.

3 Bayesian estimation

In this section, Bayes estimates of unknown parameters of Gumbel type-II distribution are obtained with respect to symmetric and asymmetric loss functions under AT-II PHCS. To obtain the Bayes estimates of α and β, squared error loss function (SELF), which is symmetric and general entropy loss function (GELF) and LINEX loss function (LLF), which are asymmetric, are considered. Let $\tilde{\theta}$ be an estimator of θ. Then, SELF, LLF and GELF are respectively given by

$$L_{SE}(\tilde{\theta}, \theta) = (\tilde{\theta} - \theta)^2 \tag{3.1}$$

$$L_{LI}(\tilde{\theta}, \theta) = e^{p(\tilde{\theta}-\theta)} - p(\tilde{\theta} - \theta) - 1, \ p \neq 0 \tag{3.2}$$

and

$$L_{GE}(\tilde{\theta}, \theta) = \left(\frac{\tilde{\theta}}{\theta}\right)^q - q \log\left(\frac{\tilde{\theta}}{\theta}\right) - 1, \ q \neq 0. \tag{3.3}$$

Under the loss functions given by (3.1), (3.2) and (3.3), the Bayes estimates of θ can be respectively written as

$$\widehat{\theta}_{SE} = E_\theta(\theta|\underline{x}), \tag{3.4}$$

$$\widehat{\theta}_{LI} = -p^{-1} \log[E_\theta(e^{-p\theta}|\underline{x})], \tag{3.5}$$

and

$$\widehat{\theta}_{GE} = [E_\theta(\theta^{-q}|\underline{x})], \ q \neq 0. \tag{3.6}$$

In Bayesian estimation, choosing priors for the unknown model parameters is an important as well as challenging problem. There is no clear methodology to choose best priors for Bayesian estimation problem. Here, we assume the prior distributions for the parameters as $\alpha \sim Gamma(a,b)$ and $\beta \sim Gamma(c,d)$, where $Gamma(a,b)$ denotes gamma distribution with shape parameter a and scale parameter b. The joint prior density of α and β can be written as

$$\pi^*(\alpha,\beta) = \alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}, \quad \alpha,\beta > 0,\ a,b,c,d > 0. \tag{3.7}$$

Based on the likelihood function (2.3) and the joint prior density function (3.7), the posterior density function of α and β can be written as

$$\begin{aligned} \pi(\alpha,\beta|\underline{x}) &= k^{-1}\alpha^{m+a-1}e^{-\alpha(b+\sum_{i=1}^{m}\log x_i)}\beta^{m+c-1}e^{-\beta(d+\sum_{i=1}^{m}x_i{}^{-\alpha})}\left(1-e^{-\beta x_m{}^{-\alpha}}\right)^{R_j^*} \\ &\quad \times \prod_{i=1}^{j}\left(1-e^{-\beta x_i{}^{-\alpha}}\right)^{R_i}, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} k &= \int_0^\infty\int_0^\infty k^{-1}\alpha^{m+a-1}e^{-\alpha(b+\sum_{i=1}^{m}\log x_i)}\beta^{m+c-1}e^{-\beta(d+\sum_{i=1}^{m}x_i{}^{-\alpha})}\left(1-e^{-\beta x_m{}^{-\alpha}}\right)^{R_j^*} \\ &\quad \times \prod_{i=1}^{j}\left(1-e^{-\beta x_i{}^{-\alpha}}\right)^{R_i}\, d\alpha d\beta \end{aligned} \tag{3.9}$$

and all the hyper-parameters a,b,c,d are non-negative and known. Now, consider a function of parameters α and β, say $\psi(\alpha,\beta)$. Then, from (3.4), (3.5) and (3.6), the Bayes estimates of $\psi(\alpha,\beta)$ with respect to SELF, LLF and GELF are given by

$$\widehat{\psi}_{SE} = \int_0^\infty\int_0^\infty \psi(\alpha,\beta)\pi(\alpha,\beta|\underline{x})\, d\alpha d\beta, \tag{3.10}$$

$$\widehat{\psi}_{LI} = -\left(\frac{1}{p}\right)\log\left[\int_0^\infty\int_0^\infty e^{-p\psi(\alpha,\beta)}\pi(\alpha,\beta|\underline{x})\, d\alpha d\beta\right], \tag{3.11}$$

and

$$\widehat{\psi}_{GE} = \left[\int_0^\infty\int_0^\infty (\psi(\alpha,\beta))^{-q}\pi(\alpha,\beta|\underline{x})\, d\alpha d\beta\right]^{-\frac{1}{q}}. \tag{3.12}$$

respectively. To derive Bayes estimates of α and β in respect of loss functions given by (3.1), (3.2) and (3.3), $\psi(\alpha,\beta)$ is replaced by α and β respectively in Equations (3.10), (3.11) and

(3.12). Then, the Bayes estimates of α are given by

$$\begin{aligned}
\widehat{\alpha}_{SE} &= k_1^{-1}\int_0^\infty\int_0^\infty \alpha^{m+a}e^{-\alpha(b+\sum_{i=1}^m \log x_i)}\beta^{m+c-1}e^{-\beta(d+\sum_{i=1}^m x_i^{-\alpha})} \\
&\quad \times\left(1-e^{-\beta x_m^{-\alpha}}\right)^{R_j^*}\prod_{i=1}^{j}\left(1-e^{-\beta x_i^{-\alpha}}\right)^{R_i}\, d\alpha d\lambda, \qquad (3.13)\\
\widehat{\alpha}_{LI} &= -\left(\frac{1}{p}\right)\log\left[k_1^{-1}\int_0^\infty\int_0^\infty \alpha^{m+a-1}e^{-\alpha(b+p+\sum_{i=1}^m \log x_i)}\beta^{m+c-1}e^{-\beta(d+\sum_{i=1}^m x_i^{-\alpha})}\right. \\
&\quad \left.\times\left(1-e^{-\beta x_m^{-\alpha}}\right)^{R_j^*}\prod_{i=1}^{j}\left(1-e^{-\beta x_i^{-\alpha}}\right)^{R_i}\, d\alpha d\lambda\right], \qquad (3.14)
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\alpha}_{GE} &= \left[k_1^{-1}\int_0^\infty\int_0^\infty \alpha^{m+a-q-1}e^{-\alpha(b+\sum_{i=1}^m \log x_i)}\beta^{m+c-1}e^{-\beta(d+\sum_{i=1}^m x_i^{-\alpha})}\right. \\
&\quad \left.\times\left(1-e^{-\beta x_m^{-\alpha}}\right)^{R_j^*}\prod_{i=1}^{j}\left(1-e^{-\beta x_i^{-\alpha}}\right)^{R_i}\, d\alpha d\lambda\right]^{-\frac{1}{q}}. \qquad (3.15)
\end{aligned}$$

In a similar way, by replacing $\psi(\alpha,\beta)$ as β in Equations (3.10), (3.11) and (3.12) Bayes estimates $\widehat{\beta}_{SE}$, $\widehat{\beta}_{LI}$ and $\widehat{\beta}_{GE}$ can be obtained. There are ratios of two integrals given in Equations (3.13), (3.14) and (3.15) which can not be obtained in a closed form. To overcome such situations, MCMC technique will be used to obtain desired Bayes estimates.

3.1 MCMC method

In this section, the Bayes estimates of the unknown parameters α and β are computed. To generate samples from (3.8), MCMC method has been used. From posterior probability density function given by (3.8), the conditional posterior densities of the parameters can be written as

$$\begin{aligned}
\pi_1(\alpha|\beta,\underline{x}) &\propto \alpha^{m+a-1}e^{-\alpha(b+\sum_{i=1}^m \log x_i)}e^{-\beta(d+\sum_{i=1}^m x_i^{-\alpha})}\prod_{i=1}^{D}(1-e^{-\beta x_i^{-\alpha}})^{R_i} \\
&\quad \times(1-e^{-\beta x_m^{-\alpha}})^{R_j^*} \qquad (3.16)
\end{aligned}$$

and

$$\pi_2(\beta|\alpha,\underline{x}) \propto \beta^{m+c-1}e^{-\beta(d+\sum_{i=1}^m x_i^{-\alpha})}\prod_{i=1}^{D}(1-e^{-\beta x_i^{-\alpha}})^{R_i}(1-e^{-\beta x_m^{-\alpha}})^{R_j^*}. \qquad (3.17)$$

Note that the density functions $\pi_1(\alpha|\beta,\underline{x})$ and $\pi_2(\beta|\alpha,\underline{x})$ can not be brought into some well known classes of distributions. Thus, the MCMC samples for α and β can not be generated directly. In this context, the Metropolis-Hastings algorithm (see Chen et al. (2012)) is used to generate samples from (3.16) and (3.17). This method has been discussed as follows:

Step 1 Set $i=1$ and choose initial guesses as $\alpha^{(1)}=\widehat{\alpha}$ and $\beta^{(1)}=\widehat{\beta}$.
Step 2 Generate new samples and $\beta^{(i)}$ with proposal distribution $\alpha^{(i)}\sim N(\alpha^{(i-1)}, var(\widehat{\alpha}))$

and $\beta^{(i)} \sim N(\beta^{(i-1)}, var(\widehat{\beta}))$.
Step 3 Compute $h = min\{1, \frac{\pi(\alpha^{(i)},\beta^{(i)}|\underline{x})}{\pi(\alpha^{(i-1)},\beta^{(i-1)}|\underline{x})}\}$.
Step 4 Generate a sample u from $U(0,1)$.
Step 5 Set $(\alpha,\beta) = (\alpha^{(i)},\beta^{(i)})$, if $u \leq h$; otherwise $(\alpha,\beta) = (\alpha^{(i-1)},\beta^{(i-1)})$.
Step 6 Set $i = i + 1$.
Step 7 Repeat Steps (2-6) N number of times to get $\alpha^{(1)},\ldots,\alpha^{(N)}$ and $\beta^{(1)},\ldots,\beta^{(N)}$.

Then, the Bayes estimates of the parameters under SELF are given by

$$\widehat{\alpha}_{SE} = \frac{1}{N}\sum_{i=1}^{N}\alpha^{(i)} \quad \text{and} \quad \widehat{\beta}_{SE} = \frac{1}{N}\sum_{i=1}^{N}\beta^{(i)}. \tag{3.18}$$

The Bayes estimates of the parameters under LLF are obtained as

$$\widehat{\alpha}_{LI} = -\left(\frac{1}{p}\right)\log\left(\frac{1}{N}\sum_{i=1}^{N}e^{-p\alpha^{(i)}}\right) \quad \text{and} \quad \widehat{\beta}_{LI} = -\left(\frac{1}{p}\right)\log\left(\frac{1}{N}\sum_{i=1}^{N}e^{-p\beta^{(i)}}\right), \quad p \neq 0. \tag{3.19}$$

Further, the Bayes estimates of the parameters under GELF are proposed as

$$\widehat{\alpha}_{GE} = \left[\frac{1}{N}\sum_{i=1}^{N}(\alpha^{(i)})^{-q}\right]^{-q} \quad \text{and} \quad \widehat{\beta}_{GE} = \left[\frac{1}{N}\sum_{i=1}^{N}(\beta^{(i)})^{-q}\right]^{-q}, \quad q \neq 0. \tag{3.20}$$

4 Confidence intervals

In this section, three types of confidence intervals for α and β, namely asymptotic confidence intervals, bootstrap confidence intervals and highest posterior density intervals are constructed.

4.1 Asymptotic confidence interval

The $100(1-\gamma)\%$ asymptotic confidence intervals for α and β can be constructed by using asymptotic normality property of the MLEs $\widehat{\alpha}$ and $\widehat{\beta}$. In doing so, variance of $\widehat{\alpha}$ and $\widehat{\beta}$ are required. These can be obtained from main diagonal elements of the inverse of the observed Fisher information matrix $\widehat{I}^{-1}(\widehat{\alpha},\widehat{\beta})$, where

$$\widehat{I}(\widehat{\alpha},\widehat{\beta}) = \begin{bmatrix} -l_{20} & -l_{11} \\ -l_{11} & -l_{02} \end{bmatrix}_{(\alpha,\beta)=(\widehat{\alpha},\widehat{\beta})} \tag{4.1}$$

and $l_{ij} = \frac{\partial^2 logL}{\partial\theta_i\partial\theta_j}$, $\Theta = (\theta_1,\theta_2) = (\alpha,\beta)$. Therefore, the $100(1-\gamma)\%$ approximate confidence intervals for α and β are respectively given by

$$\left(\widehat{\alpha} \pm z_{\frac{\gamma}{2}}\sqrt{Var(\widehat{\alpha})}\right) \quad \text{and} \quad \left(\widehat{\beta} \pm z_{\frac{\gamma}{2}}\sqrt{Var(\widehat{\beta})}\right),$$

where $z_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma}{2}$-th percentile point of a standard normal distribution.

4.2 Bootstrap confidence interval

The approximate confidence intervals are acceptable when the effective sample size m is large. When m is small, efficiency of construction of the asymptotic confidence intervals may not work properly. Bootstrap re-sampling may produce approximate confidence intervals with more accuracy. Here, two types of parametric bootstrap confidence intervals are constructed.

4.2.1 Percentile bootstrap (Boot-p) confidence interval

Step 1 Compute $\widehat{\alpha}$ and $\widehat{\beta}$.
Step 2 Use these $\widehat{\alpha}$, $\widehat{\beta}$ and same T, R_i's and m to generate a bootstrap re-sample.
Step 3 Obtain bootstrap estimates $\widehat{\alpha}^B$ and $\widehat{\beta}^B$ from these bootstrap sample.
Step 4 Repeat Steps 2-3 up to N times to get $\widehat{\alpha}^{B[1]}, \ldots, \widehat{\alpha}^{B[N]}$ and $\widehat{\beta}^{B[1]}, \ldots, \widehat{\beta}^{B[N]}$.
Step 5 Rearrange these bootstrap estimates in ascending order as $\widehat{\alpha}^{B(1)}, \ldots, \widehat{\alpha}^{B(N)}$ and $\widehat{\beta}^{B(1)}, \ldots, \widehat{\beta}^{B(N)}$.

The $100(1-\gamma)\%$ percentile bootstrap confidence intervals for α and β are respectively given by

$$\left(\widehat{\alpha}^{B(N\gamma/2)}, \widehat{\alpha}^{B(N(1-\gamma/2)}\right) \text{ and } \left(\widehat{\beta}^{B(N\gamma/2)}, \widehat{\beta}^{B(N(1-\gamma/2)}\right).$$

4.2.2 Bootstrap-t (Boot-t) confidence interval

Steps 1-2 are similar to above discussed Boot-p method. Thus, we only state Steps 3 and 4 below:
Step 3 Set t-statistics as $T_1 = \frac{\widehat{\alpha}^B - \alpha}{\sqrt{Var(\widehat{\alpha}^B)}}$ and $T_2 = \frac{\widehat{\beta}^B - \beta}{\sqrt{Var(\widehat{\beta}^B)}}$ and then compute.
Step 4 Repeat Steps 2-3 up to N times to get ${T_1}^{[1]}, \ldots, {T_1}^{[N]}$ and ${T_2}^{[1]}, \ldots, {T_2}^{[N]}$ and then rearranging these to get ${T_1}^{(1)}, \ldots, {T_1}^{(N)}$ and ${T_2}^{(1)}, \ldots, {T_2}^{(N)}$.

Then, $100(1-\gamma)\%$ bootstrap-t confidence intervals are obtained as

$$\left({T_1}^{(N\gamma/2)}, {T_1}^{(N(1-\gamma/2))}\right) \text{ and } \left({T_2}^{(N\gamma/2)}, {T_2}^{(N(1-\gamma/2))}\right),$$

respectively.

4.3 HPD credible intervals

In this section, using MCMC samples $\alpha^{(1)}, \ldots, \alpha^{(N)}$ and $\beta^{(1)}, \ldots, \beta^{(N)}$ and the method given by Chen et al. (2012), HPD credible intervals for model parameters are obtained. After ordering the MCMC samples in increasing way, these samples can be written as $\alpha_{(1)}, \ldots, \alpha_{(N)}$ and $\beta_{(1)}, \ldots, \beta_{(N)}$. The $100(1-\gamma/2)\%$ credible intervals are obtained as

$$(\alpha_{(k)}, \alpha_{(k+(1-\gamma)N)}) \text{ and } (\beta_{(k)}, \beta_{(k+(1-\gamma)N)})$$

where $k = 1, \ldots, [\gamma N]$, where $[\cdot]$ represent the greatest integer function and the corresponding interval lengths are $l_{1k} = \alpha_{(k+(1-\gamma)N)} - \alpha_{(k)}$ and $l_{2k} = \beta_{(k+(1-\gamma)N)} - \beta_{(k)}$. Then, put away these intervals for which l_{1k} and l_{2k} become the smallest to obtain the HPD credible intervals.

5 Simulation study

In this section, a Monte Carlo simulation study is carried out to compare the performance of different estimates of parameters for Gumbel type-II distribution based on adaptive type-II progressive hybrid censoring schemes. The performance of all estimates has been compared in terms of their absolute bias (AB) and mean squared errors (MSEs). For this purpose, 10000 AT-II progressive hybrid censored samples are generated by using different values of n, m, T along with three following censoring schemes:

Scheme 1 : $R_1 = \ldots = R_{m-1} = 0, R_m = n - m.$
Scheme 2 : $R_1 = n - 2m + 1, R_2 = \ldots = R_m = 1.$
Scheme 3 : $R_1 = n - m - 5, R_2 = \ldots = R_{m-5} = 0, R_{m-4} = \ldots = R_m = 1.$

The average bias and MSEs of the MLEs, MPSEs and Bayesian estimates are computed for $\alpha = 1.5$ and $\beta = 0.75$. These are presented in Table 1, Table 2 , Table 4 and Table 5. The Bayesian estimates are computed by using MCMC method along with 5000 MCMC samples. The hyper parameters in gamma prior are considered as $a = 3, b = 2, c = 3,$ and $d = 4$. Further, the 95% confidence interval lengths and coverage probabilities for asymptotic confidence intervals, bootstrap intervals and HPD credible intervals are computed and tabulated in Table 3 and Table 6. From the tables, the following conclusions are made :

(i) For fixed values of n and T, when value of m increases, then the values of absolute bias and MSEs of MLEs, MPSEs and the Bayes estimates decrease.

(ii) For different values of n and T, when $m = 10$, then Scheme 2 performes better than other two schemes; but when $m = 15$, then Scheme 3 performes better than other two schemes.

(iii) In most of the cases, for fixed values of n and m, when T deceases the values of MSEs of the estimates increase.

(iv) For fixed values of n and T, when value of m increases the values of the average lengths increase.

(v) It have been noticed that the Bayes estimates perform better than the classical estimates in terms of the absolute bias and MSEs. In classical estimates, MPSEs perform better than MLEs in terms of the absolute bias and MSEs.

(vi) Bayes estimates with respect to the LLF when $p = 0.25$ provide superior performance than other Bayes estimates.

(vii) The performance of the HPD credible intervals is better than other confidence intervals in the sense of the average lengths and coverage probabilities.

6 Real data

In this section, a real data set is considered to illustrate the established estimates. Gumbel type-II distribution can be used as an alternative to some well known two parameter distri-

Table 1: Absolute bias and MSE of estimates of α when $T = 1.5$.

(n,m)	scheme	$\widehat{\alpha}$	$\widehat{\alpha}_{MPS}$	$\widehat{\alpha}_{LI}$		$\widehat{\alpha}_{GE}$		$\widehat{\alpha}_{SE}$
				$p=-0.25$	$p=0.25$	$q=-0.25$	$q=0.25$	
(30,10)	I	0.5098	0.3317	0.1786	0.1772	0.1778	0.1780	0.1778
		0.3336	0.1986	0.0501	0.0489	0.0492	0.0490	0.0495
	II	0.3658	0.2987	0.1803	0.1789	0.1795	0.1796	0.1795
		0.2691	0.1588	0.0508	0.0497	0.0499	0.0498	0.0503
	III	0.3563	0.2919	0.1765	0.1754	0.1761	0.1765	0.1759
		0.2490	0.1479	0.0492	0.0482	0.0485	0.0484	0.0486
(30,15)	I	0.3191	0.2568	0.1701	0.1687	0.1692	0.1693	0.1694
		0.3577	0.1085	0.0458	0.0446	0.0448	0.0446	0.0452
	II	0.3053	0.2747	0.1732	0.1723	0.1731	0.1735	0.1727
		0.2005	0.1168	0.0473	0.0465	0.0468	0.0469	0.0469
	III	0.2913	0.2779	0.1756	0.1751	0.1761	0.1767	0.1753
		0.1617	0.1174	0.0484	0.0478	0.0482	0.0485	0.0481
(40,10)	I	0.4630	0.3317	0.1688	0.1675	0.1679	0.1680	0.1681
		0.9015	0.1990	0.0451	0.0439	0.0441	0.0439	0.0445
	II	0.3432	0.2854	0.1710	0.1696	0.1700	0.1699	0.1703
		0.2281	0.1437	0.0458	0.0447	0.0448	0.0447	0.0453
	III	0.3344	0.2797	0.1708	0.1696	0.1702	0.1703	0.1702
		0.2124	0.1367	0.0455	0.0455	0.0447	0.0446	0.0450
(40,15)	I	0.3484	0.2588	0.1641	0.1628	0.1631	0.1630	0.1634
		0.5966	0.1110	0.0427	0.0416	0.0417	0.0415	0.0422
	II	0.2876	0.2627	0.1647	0.1637	0.1643	0.1645	0.1642
		0.1702	0.1066	0.0429	0.0421	0.0423	0.0423	0.0425
	III	0.2745	0.2666	0.1668	0.1661	0.1669	0.1667	0.1664
		0.1353	0.1081	0.0437	0.0431	0.0434	0.0435	0.0434

Table 2: Absolute bias and MSE of estimates of β when $T = 1.5$.

(n,m)	scheme	$\widehat{\beta}$	$\widehat{\beta}_{MPS}$	$\widehat{\beta}_{LI}$		$\widehat{\beta}_{GE}$		$\widehat{\beta}_{SE}$
				$p=-0.25$	$p=0.25$	$q=-0.25$	$q=0.25$	
(30,10)	I	0.9689	0.1753	0.0827	0.0826	0.0828	0.0831	0.0826
		0.5365	0.0479	0.0108	0.0107	0.0107	0.0108	0.0107
	II	0.2237	0.1899	0.0859	0.0858	0.0862	0.0865	0.0859
		0.1384	0.0575	0.0115	0.0114	0.0115	0.0116	0.0115
	III	0.2251	0.1919	0.0852	0.0850	0.0854	0.0857	0.0851
		0.1251	0.0582	0.0114	0.0113	0.0114	0.0115	0.0114
(30,15)	I	0.2694	0.1414	0.0818	0.0816	0.0819	0.0821	0.0817
		0.2713	0.0319	0.0106	0.0105	0.0105	0.0106	0.0105
	II	0.1815	0.1586	0.0835	0.0834	0.0836	0.0838	0.0835
		0.0855	0.0404	0.0110	0.0109	0.0109	0.0110	0.0109
	III	0.1879	0.1690	0.0848	0.0846	0.0849	0.0850	0.0847
		0.0669	0.0459	0.0113	0.0112	0.0113	0.0114	0.0113
(40,10)	I	1.2630	0.1804	0.0826	0.0825	0.0828	0.0831	0.0826
		0.7557	0.0505	0.0107	0.0106	0.0106	0.0107	0.0106
	II	0.2193	0.1923	0.0859	0.0858	0.0861	0.0863	0.0858
		0.1234	0.0589	0.0114	0.0113	0.0115	0.0116	0.0114
	III	0.2218	0.1965	0.0863	0.0862	0.0866	0.0869	0.0862
		0.1054	0.0608	0.0116	0.0115	0.0116	0.0117	0.0115
(40,15)	I	0.7871	0.1412	0.0805	0.0804	0.0806	0.0807	0.0805
		0.5675	0.0313	0.0102	0.0101	0.0101	0.0102	0.0102
	II	0.1807	0.1601	0.0833	0.0831	0.0834	0.0835	0.0832
		0.0809	0.0412	0.0109	0.0108	0.0109	0.0109	0.0108
	III	0.1844	0.1709	0.0846	0.0845	0.0847	0.0849	0.0846
		0.0591	0.0470	0.0113	0.0112	0.0112	0.0113	0.0113

Table 3: Average length and coverage probability of 95% C.I. for α and β when $T = 1.5$.

		α				β			
(n,m)	scheme	ACI	boot-p	boot-t	HPD	ACI	boot-p	boot-t	HPD
(30,10)	I	1.6866	2.4919	2.6417	0.8570	0.8230	1.2655	1.6682	0.4053
		0.9283	0.8129	0.8295	0.9480	0.8701	0.8653	0.9459	0.9492
	II	1.5017	2.0124	2.2382	0.8601	0.8981	1.0208	1.5391	0.4114
		0.9257	0.8677	0.8739	0.9487	0.8871	0.9006	0.9095	0.9498
	III	1.4682	1.9503	1.9832	0.8584	0.9140	1.0408	1.3678	0.4171
		0.9210	0.8779	0.8934	0.9473	0.8892	0.9059	0.9190	0.9496
(30,15)	I	1.2722	1.6068	1.8069	0.8157	0.7057	1.1223	1.3871	0.3992
		0.9426	0.8615	0.8725	0.9515	0.9081	0.9201	0.9234	0.9401
	II	1.2325	1.5764	1.7768	0.8344	0.7489	0.8629	1.2013	0.4049
		0.9194	0.8924	0.9227	0.9483	0.9033	0.9199	0.9305	0.9492
	III	1.1782	1.4872	1.5273	0.8353	0.7883	0.8960	1.0720	0.4113
		0.9064	0.9018	0.9186	0.9476	0.9055	0.9178	0.9621	0.9490
(40,10)	I	1.7059	2.4172	2.6725	0.8122	0.8322	1.2006	1.9418	0.3977
		0.9360	0.7936	0.8493	0.9481	0.8685	0.8387	0.9353	0.9496
	II	1.4316	1.8620	1.9062	0.8151	0.8971	0.9866	1.2241	0.4120
		0.9258	0.8656	0.9065	0.9475	0.8842	0.8930	0.9019	0.9492
	III	1.3989	1.8115	1.8417	0.8209	0.9155	1.0130	1.2131	0.4154
		0.9238	0.8702	0.8926	0.9485	0.8876	0.8987	0.9104	0.9492
(40,15)	I	1.2829	1.6239	1.8337	0.7890	0.6861	0.8595	0.8488	0.3908
		0.9388	0.8591	0.8994	0.9501	0.9026	0.9117	0.9477	0.9501
	II	1.1784	1.4744	1.6841	0.7951	0.7468	0.8402	1.0552	0.4038
		0.9185	0.8963	0.9163	0.9488	0.9028	0.9159	0.9350	0.9498
	III	1.1314	1.4043	1.6104	0.7951	0.7833	0.8786	1.0143	0.4103
		0.9077	0.9041	0.9183	0.9483	0.9033	0.9138	0.9368	0.9598

Table 4: Absolute bias and MSE of estimates of α when $T = 0.75$.

(n,m)	scheme	$\widehat{\alpha}$	$\widehat{\alpha}_{MPS}$	$\widehat{\alpha}_{LI}$		$\widehat{\alpha}_{GE}$		$\widehat{\alpha}_{SE}$
				$p=-0.25$	$p=0.25$	$q=-0.25$	$q=0.25$	
(30,10)	I	0.4871	0.3250	0.1777	0.1764	0.1770	0.1772	0.1771
		1.2142	0.1950	0.0495	0.0483	0.0485	0.0484	0.0488
	II	0.3655	0.2971	0.2001	0.2016	0.2039	0.2062	0.2008
		0.2699	0.1567	0.0607	0.0615	0.0615	0.0645	0.0611
	III	0.3437	0.2863	0.1959	0.1972	0.1994	0.2015	0.1965
		0.2349	0.1476	0.0581	0.0588	0.0601	0.0613	0.0584
(30,15)	I	0.3245	0.2566	0.1707	0.1691	0.1696	0.1695	0.1699
		0.3725	0.1069	0.0456	0.0443	0.0445	0.0442	0.0449
	II	0.3572	0.4163	0.2278	0.2314	0.2349	0.2386	0.2296
		0.2207	0.2267	0.0768	0.0791	0.0816	0.0842	0.0779
	III	0.2671	0.3230	0.1907	0.1924	0.1945	0.1966	0.1915
		0.1248	0.1439	0.0543	0.0550	0.0562	0.0573	0.0546
(40,10)	I	0.4769	0.3398	0.1676	0.1661	0.1664	0.1663	0.1668
		0.9496	0.2050	0.0448	0.0435	0.0436	0.0433	0.0441
	II	0.3451	0.2880	0.1874	0.1885	0.1904	0.1921	0.1879
		0.2222	0.1429	0.0535	0.0539	0.0550	0.0560	0.0537
	III	0.3308	0.2794	0.1823	0.1832	0.1849	0.1865	0.1827
		0.2135	0.1412	0.0501	0.0504	0.0119	0.0118	0.0120
(40,15)	I	0.3420	0.2571	0.1639	0.1625	0.1629	0.1628	0.1632
		0.5346	0.1084	0.0428	0.0416	0.0417	0.0414	0.0422
	II	0.3347	0.3945	0.2135	0.2164	0.2192	0.2222	0.2149
		0.1797	0.2064	0.0672	0.0689	0.0708	0.0727	0.0681
	III	0.2593	0.3116	0.1785	0.1798	0.1815	0.1833	0.1791
		0.1121	0.1331	0.0479	0.0485	0.0494	0.0502	0.0482

Table 5: Absolute bias and MSE of estimates of β when $T = 0.75$.

(n,m)	scheme	$\widehat{\beta}$	$\widehat{\beta}_{MPS}$	$\widehat{\beta}_{LI}$		$\widehat{\beta}_{GE}$		$\widehat{\beta}_{SE}$
				$p=-0.25$	$p=0.25$	$q=-0.25$	$q=0.25$	
(30,10)	I	0.7260	0.1730	0.0828	0.0827	0.0831	0.0834	.0828
		0.5283	0.0466	0.0107	0.0106	0.0107	0.0108	0.0107
	II	0.2716	0.2289	0.0881	0.0877	0.0877	0.0875	0.0879
		0.1582	0.0803	0.0123	0.0122	0.0122	0.0121	0.0123
	III	0.2475	0.2145	0.0865	0.0862	0.0862	0.0861	0.0864
		0.1259	0.0716	0.0117	0.0116	0.0116	0.0115	0.0117
(30,15)	I	0.2638	0.1406	0.0819	0.0818	0.0822	0.0824	0.0819
		0.3930	0.0316	0.0105	0.0104	0.0105	0.0106	0.0105
	II	0.2615	0.2453	0.0873	0.0867	0.0863	0.0859	0.0870
		0.1399	0.0910	0.0121	0.0119	0.0118	0.0116	0.0120
	III	0.1886	0.1797	0.0831	0.0827	0.0827	0.0825	0.0829
		0.0649	0.0516	0.0109	0.0108	0.0108	0.0107	0.0109
(40,10)	I	0.3957	0.1840	0.0817	0.0816	0.0819	0.0821	0.0817
		0.1709	0.0520	0.0104	0.0103	0.0104	0.0105	0.0104
	II	0.2683	0.2361	0.0885	0.0881	0.0880	0.0878	0.0883
		0.1397	0.0859	0.0122	0.0121	0.0121	0.0120	0.0122
	III	0.2532	0.2250	0.0870	0.0867	0.0867	0.0866	0.0869
		0.1260	0.0783	0.0120	0.0119	0.0119	0.0118	0.0120
(40,15)	I	0.6908	0.1411	0.0798	0.0797	0.0800	0.0802	0.0797
		0.3047	0.0312	0.0100	0.0099	0.0100	0.0101	0.0100
	II	0.2547	0.2507	0.0893	0.0887	0.0882	0.0877	0.0890
		0.1159	0.0963	0.0126	0.0123	0.0122	0.0120	0.0124
	III	0.1914	0.1879	0.0829	0.0825	0.0825	0.0823	0.0827
		0.0669	0.0557	0.0108	0.0107	0.0107	0.0106	0.0108

Table 6: Average length and coverage probability of 95% C.I. for α and β when $T = 0.75$.

		α				β			
(n,m)	scheme	ACI	boot-p	boot-t	HPD	ACI	boot-p	boot-t	HPD
(30,10)	I	1.6775	2.0947	2.3369	0.8511	0.8240	1.3510	1.5282	0.4020
		0.9348	0.8235	0.8749	0.9479	0.8799	0.8763	0.9409	0.9491
	II	1.3180	1.7148	1.9864	0.8880	0.9686	1.2647	1.5943	0.4233
		0.8184	0.8494	0.8962	0.9483	0.8686	0.8495	0.9414	0.9490
	III	1.2966	1.6360	1.8635	0.8649	0.9847	1.1818	1.4015	0.4133
		0.8392	0.8899	0.9162	0.9492	0.8970	0.8661	0.9239	0.9484
(35,15)	I	1.2752	1.6161	1.8547	0.8144	0.7048	1.1695	1.3268	0.3988
		0.9392	0.8621	0.9034	0.9501	0.9087	0.9198	0.9255	0.9535
	II	1.0247	1.2023	1.3168	0.8723	0.8260	1.0679	1.2178	0.4121
		0.9053	0.8429	0.8936	0.9493	0.8475	0.8310	0.8752	0.9494
	III	1.0743	1.1932	1.3591	0.8290	0.8296	0.9052	1.0084	0.4019
		0.8648	0.8968	0.9106	0.9481	0.9464	0.8820	0.9293	0.9495
(40,10)	I	1.4135	1.7339	1.8531	0.8191	0.8299	1.2996	1.5237	0.3939
		0.9331	0.8825	0.9146	0.9485	0.8541	0.8267	0.9317	0.9499
	II	1.2681	1.6423	1.7529	0.8374	0.9665	1.2478	1.3757	0.4231
		0.8226	0.8490	0.9205	0.9490	0.8555	0.8464	0.9344	0.9488
	III	1.2541	1.5694	1.7325	0.8137	0.9796	1.1670	1.3149	0.4213
		0.8426	0.8839	0.9314	0.9478	0.8831	0.8698	0.9208	0.9494
(40,15)	I	1.2852	1.6241	1.7638	0.7915	0.6868	1.0593	1.2304	0.3885
		0.9434	0.8604	0.9126	0.9501	0.9056	0.9154	0.9415	0.9571
	II	0.9953	1.1725	1.2621	0.8439	0.8193	1.0496	1.0636	0.4144
		0.8567	0.8665	0.9021	0.9495	0.8337	0.9137	0.9311	0.9575
	III	1.0347	1.1417	1.2136	0.7788	0.8264	0.9005	0.9658	0.4015
		0.8656	0.8887	0.9054	0.9475	0.9426	0.8857	0.9653	0.9495

butions such as Burr III , Nandrajah Haghighi (NH) and Inverted Kumaraswamy (Ikum) distributions. For the purpose of goodness of fit test, negative log-likelihood criterion, Alkaikes-information criterion (AIC), Bayesian information criterion (BIC), Cramer-von Mises (C^*) measure and Anderson-Darling (A^*) measure along with p-values are computed and tabulated in Table 7. If we get value of C^* and A^* smaller but p-values greater, then the distribution fits better than other one.

Data : Death rates due to Covid-19 in India.

This following data set is taken from https://www.worldometers.info/coronavirus/country/india/, which represents the death rates due to Covid-19 in India from March 16 to May 13, 2020. The data set is provided in the following:

13.33, 17.65, 17.65, 16.67, 17.86, 17.86, 22.58, 22.73, 20, 21.82, 30.77, 21.51, 22.22, 22.13, 23.88, 22.15, 28.16, 27.38, 30.94, 30.18, 26.46, 26.16, 25.48, 26.02, 26.33, 24.34, 22.91, 23.46, 23.26, 22.43, 21.87, 20.22, 19.23, 17.46, 16.38, 15.32, 13.96, 13.48, 12.58, 12.43, 12.20, 11.90, 11.63, 11.51, 11.34, 11.29, 10.89, 10.90, 10.57, 10.87, 10.69, 10.43, 10.12, 9.99, 9.82, 9.54, 9.23, 9, 8.81, 8.65, 8.34, 7.74, 7.60, 7.45, 7.24, 7.03, 6.87, 6.71, 6.64, 6.52, 6.43, 6.33, 6.27,6.23,5.68,5.63,5.56, 5.53, 5.49, 5.53, 5.54, 5.55, 5.53, 5.50, 5.47, 5.44, 5.44, 5.47, 5.45, 5.36.

For the purpose of goodness of fit test, we consider different plots in Figure 3, Figure 4 and Figure 5. In Figure 3, the theoretical CDFs of the distributions are compared with the empirical CDF of the given real data set. In Figure 4, the QQ-plots are used to compare the fitted theoretical models. In Figure 5(a), the box plot of the real data set is displayed, which represents that the distribution is right skewed. The TTT plot for given real data set is shown by Figure 5(b). From Table 7, it is concluded that Gumbel type-II distribution is fitted to that data better than NH, Burr III and IKum distributions. From the real data set, different AT-II progressive hybrid censored samples are considered by using different values of n, m and T. The computed values of the MLEs, MPSEs and Bayes estimates based on the real data set are tabulated in Table 8. Further, the confidence intervals are constructed and tabulated in Table 9. From Table 8, it is observed that the Bayes estimates perform better than the other estimates. From Table 9, it is seen that the HPD credible interval performs better than asymptotic confidence intervals.

Table 7: The values of MLEs and statistics of different distributions along with goodness-of-fit measures for Covid-19 data set.

Model	$\widehat{\alpha}$	$\widehat{\beta}$	-log L	AIC	BIC	C^*	A^*	p-value
GT-II	2.0130	82.7737	300.6597	605.3194	610.3190	0.1694	1.2297	0.9674
NH	138.7024	0.0003	311.8292	627.6584	632.6580	0.2528	1.4784	0.8398
Burr III	2.0256	85.8196	300.7166	605.4332	610.4328	0.3204	1.8480	0.6588
IKum	2.2073	163.2839	300.6774	605.3548	610.3544	0.2219	1.3392	0.8927

Table 8: Average values of the estimates of parameters for real data set.

(n,m)	T	scheme	θ	MLE	$MPSE$	LLF		$GELF$		SELF
						$p=-0.25$	$p=0.25$	$q=-0.25$	$q=0.25$	
(90,40)	10	(0*39,50)	α	1.8701	1.6657	2.0298	2.0291	2.0290	2.0286	2.0294
			β	63.4539	52.9911	85.9635	85.9626	85.9631	85.9630	85.9631
	15		α	1.9209	1.7090	2.0285	2.0280	2.0279	2.0277	2.0282
			β	68.4234	56.2260	86.1488	86.1461	86.1474	86.1475	86.1475
	10	(0*35,10*5)	α	1.9380	1.7362	2.0310	2.0298	2.0295	2.0289	2.0304
			β	71.7506	60.2997	86.0327	86.0322	86.0325	86.0326	86.0325
	15		α	2.0281	1.8208	2.0266	2.0257	2.0255	2.0251	2.0262
			β	83.0597	69.0278	85.9584	85.9576	85.9580	85.9580	85.9581
	10	(0*30,5*10)	α	2.1209	1.9269	2.0449	2.0443	2.0442	2.0439	2.0446
			β	99.6236	85.2831	86.0510	86.0502	86.0507	86.0506	86.0507
	15		α	2.2134	2.0142	2.0692	2.0685	2.0683	2.0680	2.0688
			β	115.7839	98.1035	86.1114	86.1075	86.1094	86.1093	86.1095
(90,50)	10	(0*49,40)	α	1.7442	1.5885	1.9779	1.9773	1.9772	1.9769	1.9776
			β	53.1119	48.2612	86.0341	86.0333	86.0337	86.0337	86.0337
	15		α	1.9620	1.7843	2.0283	2.0277	2.0276	2.0273	2.0281
			β	73.7502	63.7147	85.9503	85.9502	85.9503	85.9501	85.9503
	10	(0*45,5*8)	α	1.7442	1.5885	1.9682	1.9678	1.9677	1.9674	1.9680
			β	53.1119	48.2612	85.9022	85.8993	85.9008	85.9007	85.9008
	15		α	2.0428	1.8740	2.0410	2.0403	2.0402	2.0399	2.0407
			β	85.5752	75.4799	85.8843	85.8827	85.8835	85.8834	85.8835
	10	(0*40,4*10)	α	1.7442	1.5885	1.9857	1.9851	1.9850	1.9847	1.9854
			β	53.1119	48.2612	85.8008	85.7989	85.7998	85.7998	85.7999
	15		α	2.1158	1.9539	2.0516	2.0510	2.0508	2.0506	2.0513
			β	97.8510	87.7249	86.2835	86.2772	86.2803	86.2802	86.2804

Table 9: Confidence intervals of parameters for real data set when $T = 15$.

			α		β	
(n,m)	T	scheme	ACI	HPD	ACI	HPD
(90,40)	10	(0*39,50)	(1.40,2.33)	(1.93,2.13)	(4.71,122.18)	(85.83,86.06)
	15		(1.44,2.40)	(1.96,2.07)	(3.62,133.22)	(85.75,86.42)
	10	(0*35,10*5)	(1.45,2.42)	(1.91,2.18)	(3.37,140.12)	(85.96,86.12)
	15		(1.52,2.53)	(1.90,2.12)	(0.78,165.33)	(85.87,86.06)
	10	(0*30,5*10)	(1.59,2.64)	(1.93,2.12)	(0,201.80)	(85.94,86.16)
	15		(1.66,2.76)	(1.96,2.16)	(0,238.78)	(85.94,86.36)
(90,50)	10	(0*49,40)	(1.36,2.12)	(1.88,2.07)	(11.36,94.85)	(85.94,86.16)
	15		(1.53,2.39)	(1.93,2.10)	(9.94,137.55)	(85.89,86.01)
	10	(0*45,8*5)	(1.36,2.12)	(1.88,2.04)	(11.36,94.85)	(85.76,86.11)
	15		(1.59,2.48)	(1.92,2.13)	(9.22,161.92)	(85.74,86.01)
	10	(0*40,4*10)	(1.36,2.12)	(1.89,2.07)	(11.36,94.85)	(85.68,86.15)
	15		(1.65,2.57)	(1.95,2.13)	(7.96,187.74)	(85.91,86.49)

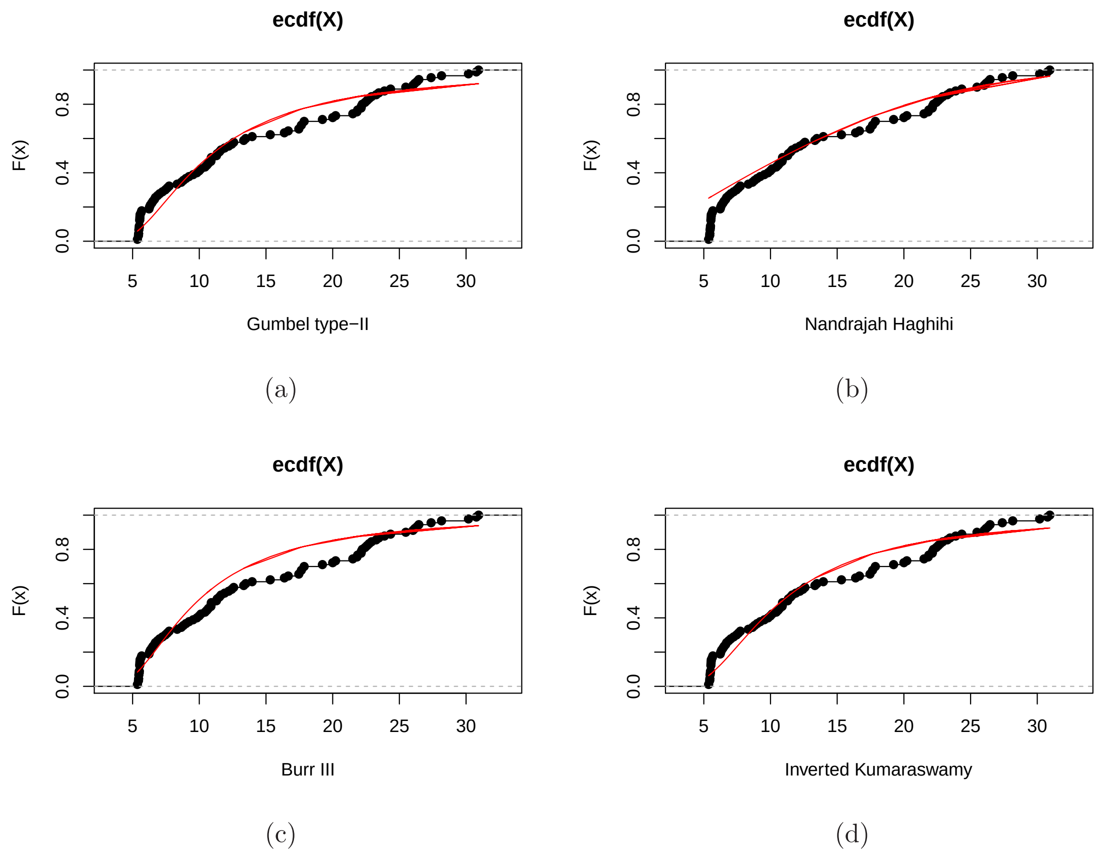


Figure 3: ECDF and CDF comparison for different distributions fitted to given real data set.

7 Conclusion

In this article, different estimates of the unknown model parameters of the Gumbel type-II distribution based on AT-II PHCS have been developed. Both classical and Bayesian estimation methods are used to obtain the estimates. It is observed that the MLEs and MPSEs can not be obtained explicitly. So, Newton Raphson iterative method is employed to compute these estimates. Bayes estimates are obtained based on the symmetric and asymmetric loss functions under the assumption of independent gamma priors using MCMC method. Three types of confidence intervals for unknown parameters are constructed. Then, Monte carlo simulation study is performed to compare the performance of the estimates in terms of the absolute bias and MSEs. It is observed that the Bayes estimates under LINEX loss function perform better than the other estimates. Further, a real data set is considered for illustrative purposes.

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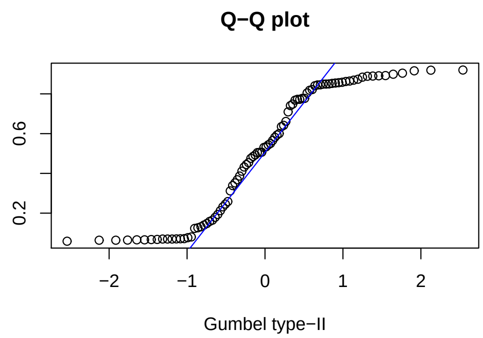
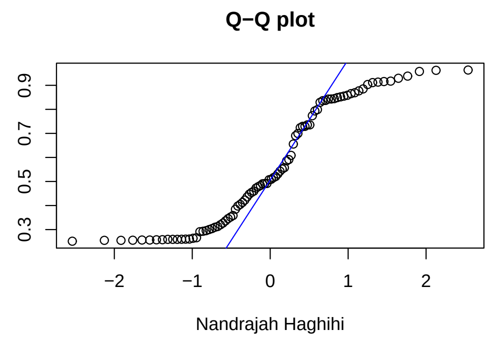
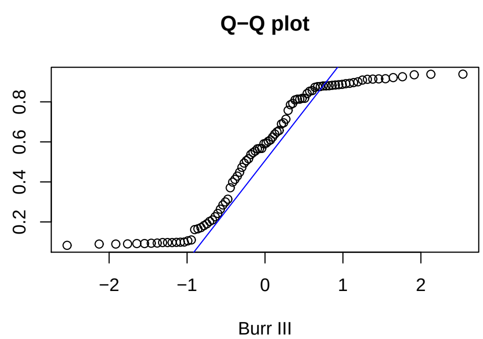


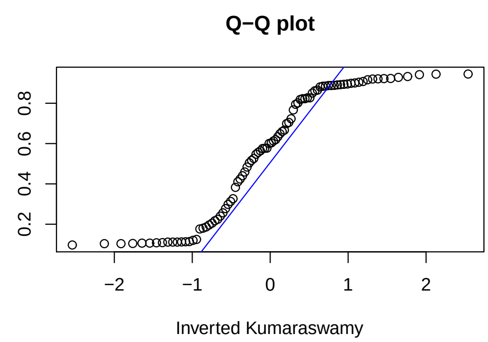
(a)



(b)

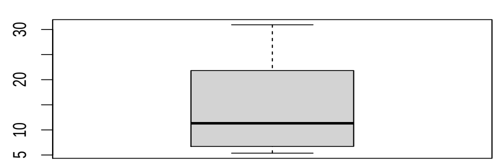


(c)

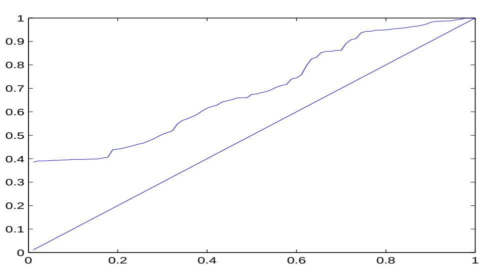


(d)

Figure 4: QQ-plot comparison for different distributions fitted to given real data set.



(a)



(b)

Figure 5: (a) Boxplot and (b) TTT plot for given real data set.

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