

# Data-driven deep learning algorithms for time-varying infection rates of COVID-19 and mitigation measures

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## Abstract

Epidemiology models with constant parameters may not capture the infection patterns in the presence of pharmaceutical and non-pharmaceutical mitigation measures during a pandemic, since infectiousness is a function of time. In this paper, Epidemiology Informed Neural Network (EINN) algorithms are introduced to discover time-varying infection rates for the COVID-19 pandemic. Since there are asymptomatic infectives, mostly unreported, EINN learns the probability that an infective individual is asymptomatic. Using cumulative and daily reported cases of infectives, we simulate the impact of non-pharmaceutical mitigation measures such as early detection of infectives, contact tracing, and social distancing on the basic reproduction number. We demonstrate the effectiveness of vaccination, a pharmaceutical mitigation measure, together with non-pharmaceutical mitigation measures on the daily reported infectives. The EINN algorithms discover time-varying infection and recovery rates. The Mean Squared Error is used to demonstrate the accuracy of the proposed EINN algorithms. Simulations are presented for Italy, South Korea, United Kingdom, and the United States.

*Keywords:* data-driven algorithms, deep learning, Epidemiology Informed Neural Network, time-varying, COVID-19, asymptomatic infectives

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## 1. Introduction

COVID-19 data from different countries reflects various mitigation measures [1, 2], such as social distancing, early detection of infectives, contact tracing, and vaccination [3, 4, 5]. Time-varying infection rates have been suggested to efficiently model the spread of an infectious disease such as COVID-19. For example, nonlinear time-dependent infection rate functions were introduced in [6, 7]. Given that it may not be possible to know the most accurate form of a time-varying infection rate, we propose a data-driven approach that learns time-varying infection and recovery rates.

A widely used threshold parameter for the spread or extinction of an infectious disease in an epidemiology model is the basic reproduction number [8]. It is defined as the average number of persons an infected person can infect. When the basic reproduction number is less than one, the infectious disease vanishes. In the SIR model [9], the basic reproduction number is computed as the ratio of the infection rate to the recovery rate. In this paper, we adopt the basic reproduction number for the asymptomatic-SIR model introduced in [10]. We denote the basic reproduction number as BRN when the model parameters are constant and when one or more of the model parameters are time-varying, we also have a time-varying reproduction rate, which is denoted as  $R_t$ . This time-varying reproduction rate,  $R_t$ , demonstrates the spread pattern of COVID-19 throughout the duration of the pandemic.

There is an asymptomatic period for every infective individual, some papers have reported a range of 7 to 14 days [11]. There are also asymptomatic infectives that never show symptoms but are infectious [10]. Early study of the spread of COVID-19 shows that some of the infectives are asymptomatic infectives [12, 13] and they are mostly unreported in the publicly available data [10]. An asymptomatic-SIR model

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is a modification to the traditional SIR infectious disease model [9], that includes asymptomatic infectives and asymptomatic recovered compartments, such as introduced in [10]. We introduce modifications to the asymptomatic-SIR model by including a vaccination compartment determined by a vaccination rate, time-varying infection and recovery rates, and a probability term that determines the portion of the total infectives that are asymptomatic infectives.

In March and April 2020, many countries instituted widespread lockdown [14]. A model describing lockdown and lockdown relaxation was presented [15]. The purpose of a lockdown is to limit human contacts and potentially lower infection rates. Unfortunately, the impact of a comprehensive lockdown on the economy is severe and thus other measures are more desirable.

Neural networks are universal approximators of continuous functions [16, 17]. In [18, 19], a feedforward neural network (FNN) is developed to learn approximate solutions of differential equations by constraining the residual. This FNN is called Physics Informed Neural Network (PINN). The authors in [20] developed a deep learning solution of a COVID-19 model using an FNN. PINN has been used successfully to solve ordinary differential equations [21] and partial differential equations [18]. In [22], a PINN with a modified loss function is used to learn the diffusivity in a nonlinear diffusion equation. A Biologically Informed Neural Network (BINN) is proposed in [23] to learn the nonlinear diffusion and reaction terms in a 1-D spatial dimensional reaction-diffusion model. BINN is an extension to PINN for the discovery of underlying dynamics of biological systems from sparse experimental data.

The Epidemiology Informed Neural Network (EINN) algorithms presented in this paper are extensions to PINN for epidemiology models. The loss function is extended to include time-varying rates using some known epidemiology facts about the infectious disease. Interpolation is used to generate sufficient training data. The proposed EINN algorithms are able to capture the dynamics of the spread of the disease and the influence of the mitigation measure. The asymptomatic infectives population is usually unreported in the publicly available data [24]. EINN algorithms can learn the asymptomatic infectives population by training on symptomatic infectives data that are available in the reported public data.

The paper is organized as follows. In Section 2, we introduce and discuss the asymptomatic-SIR model. The definition of the basic reproduction number (BRN) and the time-varying reproduction rate  $R_t$  are also presented. In Section 3, we describe the FNN. The architecture and loss function of EINN is presented. In Section 4, the EINN algorithm for constant infection and recovery rates is presented and applied to data from Italy, South Korea, and the USA. In Section 5, using the EINN algorithm introduced in Section 4, we simulate the impact of non-pharmaceutical mitigation measures including social distancing, contact tracing, and early detection of infectives on the BRN and the spread of COVID-19. The effectiveness of vaccination in reducing the daily reported infective cases, a pharmaceutical mitigation measure, is discussed in Section 6. In Section 7 and 8, EINN algorithms are presented for the discovery of time-varying infection and recovery rates subject to non-pharmaceutical mitigation measures. Finally, a summary of the results in this paper is presented in Section 9.

## 2. Asymptomatic-SIR Model

The asymptomatic-SIR model introduced in [10] assumes that some of the infectives are asymptomatic infectives. This group is infectious despite not showing symptoms of COVID-19, probably do not get tested, and are usually unreported in the various publicly available data. In Figure 1, the interactions between the six different compartments of the asymptomatic-SIR model are shown. The dynamics of these interactions can be represented as a system of ordinary differential equations with time-varying parameters in eq. (1).

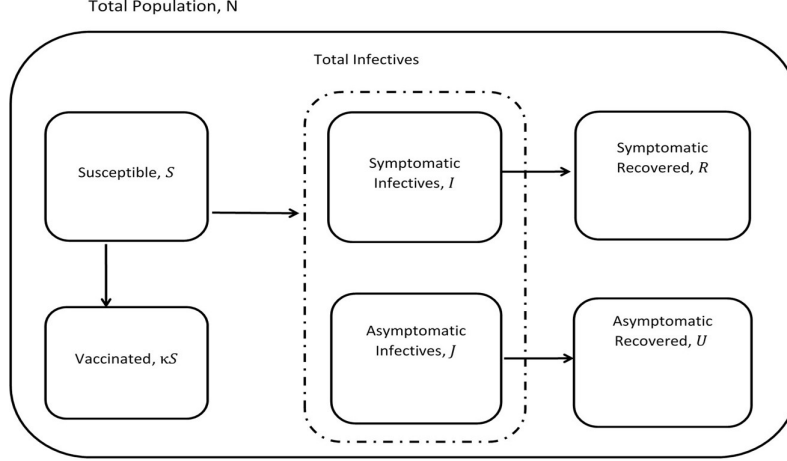


Figure (1) Compartments in an Asymptomatic-SIR model with vaccination

$$\begin{aligned}
 \frac{dS(t)}{dt} &= -\frac{1}{N}\beta(t)\left(I(t) + J(t)\right)S(t) - \kappa S(t) \\
 \frac{dI(t)}{dt} &= \frac{1}{N}\beta(t)\xi\left(I(t) + J(t)\right)S(t) - \gamma(t)I(t) \\
 \frac{dJ(t)}{dt} &= \frac{1}{N}\beta(t)\left(1 - \xi\right)\left(I(t) + J(t)\right)S(t) - \mu(t)J(t) \\
 \frac{dR(t)}{dt} &= \gamma(t)I(t) + \kappa\xi S(t) \\
 \frac{dU(t)}{dt} &= \mu(t)J(t) + \kappa(1 - \xi)S(t).
 \end{aligned} \tag{1}$$

The continuity equation is given by

$$N = S + I + J + R + U. \tag{2}$$

The initial conditions are given by  $S(t_0) = S_0$ ,  $I(t_0) = I_0$ ,  $J(t_0) = J_0$ ,  $R(t_0) = R_0$ , and  $U(t_0) = U_0$ .  $t \geq t_0$  represent time in days and  $t_0$  is the start date of the pandemic in the model.  $\beta(t)$ ,  $\gamma(t)$ , and  $\mu(t)$  are time-varying parameters.  $\gamma(t)$  represents the time-varying recovery rate for the symptomatic infectives.  $N$  represents the total population eq. (2). It is assumed that  $N$  does not change throughout the pandemic and that infective individuals are immediately infectious. We show the interactions between the compartments of the model in Figure 1. The time-varying recovery rate of the asymptomatic infectives is denoted by  $\mu(t)$ .  $\beta(t)$  is the time-varying infection rate, it usually depends on the infection vector. In the COVID-19 pandemic,  $\beta(t)$  depends also on contacts between individuals.

The asymptomatic-SIR model, eq. (1) considers six population compartments: the Susceptible  $S$ , the symptomatic Infectives  $I$  which correspond to the reported infectives in the publicly available data, the asymptomatic Infectives  $J$  which correspond to the unreported infectives. The total infectives are  $I + J$ . The rest of the compartments are the symptomatic Recovered  $R$ , the asymptomatic Recovered  $U$ , and the vaccinated compartment  $\kappa S$ , which is a loss from the susceptible group.  $\kappa$  is the average percentage of individuals that are vaccinated daily.  $\xi(I + J)$  corresponds to the portion of the total infectives that are symptomatic and reported. That is,  $\xi$  represents the probability that an infective individual is reported. On the other hand,  $(1 - \xi)(I + J)$  represents the asymptomatic infectives. And so,  $(1 - \xi)$  is the probability that an infective is an asymptomatic infective.

**Definition 1.** [10] *The basic reproduction number (BRN) for the asymptomatic-SIR model eq. (1) with*

constant parameters is given by

$$BRN = \frac{\beta}{\xi\gamma + (1 - \xi)\mu}, \quad \xi \in (0, 1). \quad (3)$$

If  $\xi = 0$ ,  $BRN = \beta/\mu$ , when all the infective population are asymptomatic.

If  $\xi = 1$ ,  $BRN = \beta/\gamma$ , when all the infective population are symptomatic.

It is assumed that during the early period of the pandemic,  $\xi = \frac{I(0)}{I(0)+J(0)}$ .

**Definition 2.** The time-varying reproduction rate  $R_t$  for the asymptomatic-SIR model eq. (1) with time-varying infection rate is given by

$$R_t = \frac{\gamma\beta(t)}{\left[\gamma\left(\frac{I(t)}{I(t)+J(t)}\right) + \mu\left(1 - \frac{I(t)}{I(t)+J(t)}\right)\right]^2}. \quad (4)$$

Where  $\gamma$  and  $\mu$  may be constant or time-varying.

### 3. Neural network structure

#### 3.1. Feedforward Neural Network (FNN)

We write an FNN as a function of  $L$  layers,  $t$  input and an output  $\mathcal{N}$

$$\mathcal{N}(t; \theta) = \sigma(W_L \sigma(\dots \sigma(W_2 \sigma(W_1 t + b_1) + b_2) \dots) + b_L), \quad (5)$$

where  $\theta := (W_1, \dots, W_L, b_1, \dots, b_L)$ .  $W_k$ ,  $k = 1, \dots, L$  is the set of the neural network weight matrices while  $b_k$ ,  $k = 1, \dots, L$  is the set of the bias vectors.  $\sigma$  is the activation function. Given a collection of sample pairs  $(t_j, u_j)$   $j = 1, \dots, M$ , where  $u$  is some target function. The goal is to find  $\theta^*$  by solving the optimization problem

$$\theta^* = \arg \min_{\theta} \frac{1}{M} \sum_{j=1}^M \|\mathcal{N}(t_j; \theta) - u_j\|_2^2. \quad (6)$$

The function  $\frac{1}{M} \sum_{j=1}^M \|\mathcal{N}(t_j; \theta) - u_j\|_2^2$  on the right hand side of (6) is called the mean squared error (MSE) loss function. A major task in training a network is to determine the suitable number of layers and the number of neurons per layer needed, the choice of activation function, and an appropriate optimizer for the loss function [25].

#### 3.2. Epidemiology Informed Neural Network (EINN)

EINN is an FNN that includes the known epidemiology dynamics in its loss function. EINN MSE loss function is a modification of the PINN MSE loss function in order to learn time-varying model parameters. In this paper, EINN is adapted for the asymptomatic-SIR model eq. (1), where the EINN MSE loss function includes the known epidemiology dynamics such as a lockdown. We observe that training data is not available for all the compartments in the asymptomatic-SIR model, however, EINN is able to capture the epidemiology interactions between the compartments because the epidemiology model

residual is included in the EINN MSE loss function. The EINN MSE loss function is given by

$$\begin{aligned}
MSE &= \frac{1}{M} \sum_{j=1}^M \|I(t_j; \theta) - \tilde{I}(t_j)\|_2^2 + \frac{1}{M} \sum_{j=1}^M \|R(t_j; \theta) - \tilde{R}(t_j)\|_2^2 \\
&+ \frac{1}{M_\beta} \sum_{j=1}^{M_\beta} \|\beta(t_j; \theta) - \tilde{\beta}(t_j)\|_2^2 \\
&+ \frac{1}{M_\gamma} \sum_{j=1}^{M_\gamma} \|\gamma(t_j; \theta) - \tilde{\gamma}(t_j)\|_2^2 \\
&+ \frac{1}{M_\kappa} \sum_{j=1}^{M_\kappa} \|\kappa S(t_j; \theta) - V(t_j)\|_2^2 \\
&+ \|J(0; \theta) - \tilde{J}(0)\|_2^2 + \|U(0; \theta) - \tilde{U}(0)\|_2^2 \\
&+ \frac{1}{M} \sum_{i=1}^6 \sum_{j=1}^M \|L_i(t_j; \theta)\|_2^2,
\end{aligned} \tag{7}$$

where the residual  $L_i$ ,  $i = 1, \dots, 6$  is as follows

$$\begin{aligned}
L_1(t_j; \theta) &= \frac{dS(t_j; \theta)}{dt_j} + \frac{1}{N} \beta(t_j; \theta) (I(t_j; \theta) + J(t_j; \theta)) S(t_j; \theta) + \kappa S(t_j; \theta) \\
L_2(t_j; \theta) &= \frac{dI(t_j; \theta)}{dt_j} - \frac{1}{N} \beta(t_j; \theta) \xi (I(t_j; \theta) + J(t_j; \theta)) S(t_j; \theta) + \gamma(t_j; \theta) I(t_j; \theta) \\
L_3(t_j; \theta) &= \frac{dJ(t_j; \theta)}{dt_j} - \frac{1}{N} \beta(t_j; \theta) (1 - \xi) (I(t_j; \theta) + J(t_j; \theta)) S(t_j; \theta) + \mu(t_j; \theta) J(t_j; \theta) \\
L_4(t_j; \theta) &= \frac{dR(t_j; \theta)}{dt_j} - \gamma(t_j; \theta) I(t_j; \theta) - \kappa \xi S(t_j; \theta) \\
L_5(t_j; \theta) &= \frac{dU(t_j; \theta)}{dt_j} - \mu(t_j; \theta) J(t_j; \theta) - \kappa (1 - \xi) S(t_j; \theta) \\
L_6(t_j; \theta) &= N - (S(t_j; \theta) + I(t_j; \theta) + J(t_j; \theta) + R(t_j; \theta) + U(t_j; \theta)).
\end{aligned} \tag{8}$$

In Figure 2, EINN includes the time-varying infection and recovery rates as an output of the neural network.  $ICs$  represents the loss at  $t = 0$  in the neural network output for the asymptomatic infectives  $J(0; \theta)$  and the actual asymptomatic infectives  $J(0)$  determined according to the formula  $J(0) = (1 - \xi)I(0)/\xi$ . And the loss at  $t = 0$  in the neural network output for the asymptomatic recovered  $U(0; \theta)$  and the actual asymptomatic recovered  $U(0)$  determined according to the formula  $U(0) = (1 - \xi)R(0)/\xi$ .  $KPs$  represent the known dynamics in the infection rates pattern.  $M$  is the number of training points.  $M$  does not necessarily correspond to the number of available data.  $M$  is generated by fitting the data with an interpolation function. For instance  $\tilde{I}(t_j)$ ,  $j = 1, \dots, M$  is the training data for the infectives after fitting with an interpolation function.  $M_\beta$  and  $M_\gamma$  are the number of training points used to enforce the known dynamics of the infection rates pattern. On the other hand,  $M_\kappa$  is the number of vaccination days in the model,  $M_\kappa$  is the number of days  $\kappa$  is not zero.  $V(t_j)$ ,  $j = 1, \dots, M_\kappa$  is the daily vaccination data for days  $\kappa$  is not zero. The input to EINN is  $t_j$ ,  $j = 1, \dots, M$ . The epidemiology model is embedded in the residual eq. (8). EINN solutions are not unique. In order to achieve good accuracy, we tune the hyperparameters; such as the number of layers, number of training points, and the learning rate. In all the simulations presented in this paper, we used 4 hidden layers, 64 neurons per layer, and the training loss was minimized in 40000 iterations. An interpolation function is used to generate 3000 training points from the original data-set. The loss function is minimized by a gradient-based optimizer such as the  $L - BFGS - B$  [26].

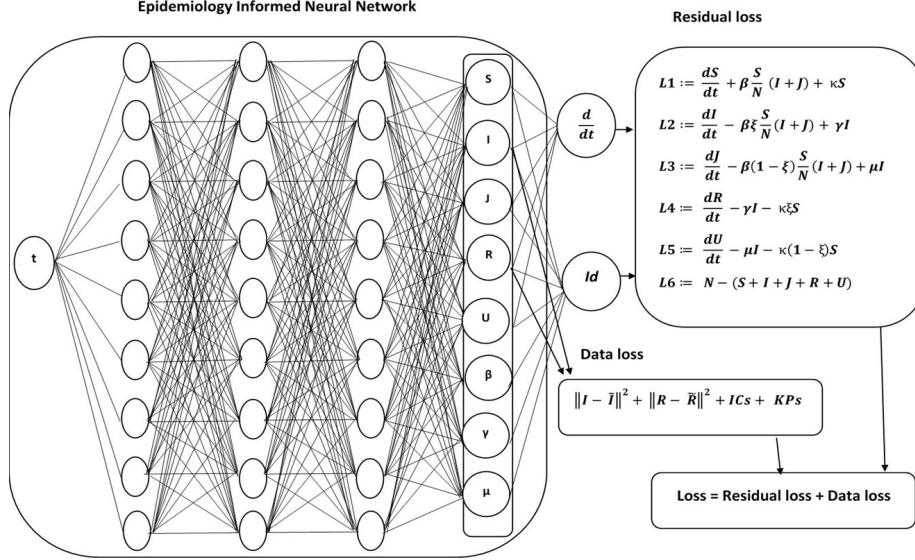


Figure (2) Schematic of the Epidemiology Informed Neural Network for nonlinear time-varying infection and recovery rates

#### 4. EINN algorithm for constant infection rates

In this section, we present the EINN algorithm 1 for the asymptomatic-SIR model with constant parameters. That is, in eq. (1), we set  $[\beta(t), \gamma(t), \mu(t)]$  to  $[\beta, \gamma, \mu]$ . The learned cumulative infectives and the recovered solution is matched against the training data. In this algorithm, the parameters represent average rates. We implement algorithm 1 using publicly available COVID-19 data [24].

Using data from Italy, South Korea, and the United States starting from the date of the first reported cases in the respective countries to the day before vaccination data was reported. The cumulative infective and recovered population data are observed to be non-exponential whenever a mitigation measure such as a comprehensive lockdown is detected in the data. We take the total population  $N$  to be  $60.36 \times 10^6$ ,  $51.64 \times 10^6$ , and  $328.2 \times 10^6$  in Italy, South Korea and USA respectively. In Figure 3a, Figure 4a, and Figure 5a,  $M_\kappa$  is zero and so  $\kappa = 0$  for all the period from the first reported cases to the day before vaccination data is reported. In addition to learning the parameters, EINN learns  $\xi$ , the probability that an infective is reported. Large  $\xi$  indicates high reported infectives.

As shown in Figures 3a, 4a, and 5a early in the pandemic, the cumulative infective and recovered data closely resemble an exponential function. In Figure 3, EINN algorithm 1 learns the constant model parameters as follows:  $\beta = 0.03$ ,  $\gamma = 0.0121$ ,  $\mu = 0.0128$ ,  $\xi = 0.37$ ,  $BRN = 2.3922$ . The mean squared error (MSE) is  $8.11 \times 10^{-4}$ . In Figure 4, EINN algorithm 1 learns the constant model parameters  $\beta = 0.0104$ ,  $\gamma = 0.0053$ ,  $\mu = 0.0046$ ,  $\xi = 0.22$ , and  $BRN = 2.1876$ . The mean squared error (MSE) is  $1.3 \times 10^{-5}$ . In Figures 5, EINN algorithm 1 learns the constant model parameters  $\beta = 0.0202$ ,  $\gamma = 0.0044$ ,  $\mu = 0.0089$  and  $\xi = 0.46$ ,  $BRN = 2.9575$ . The mean squared error (MSE) =  $1.99 \times 10^{-4}$ . The Centers for Disease Control and Prevention (CDC) estimate  $1 - \xi$  to be 0.40 in the USA [13]. A high population proportion of asymptomatic infectives was estimated in [12] for China and Singapore.

The COVID-19 infectives population surge witnessed in March and April 2020 around the world forced many countries to institute strict lockdown measures. This is largely successful in reducing the BRN in many countries, unfortunately, it also resulted in economic hardship, such that we seek other measures that also reduces the BRN to a number less than 1. In recent months, the measures that are promoted in most countries include contact tracing, social distancing, and facial covering. The epidemiological meaning of each of the model parameters in eq. (1) including  $\xi$  are presented in Sections 5.1, 5.3, and 5.2.

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**Algorithm 1** EINN algorithm for Asymptomatic-SIR model with constant parameters
 

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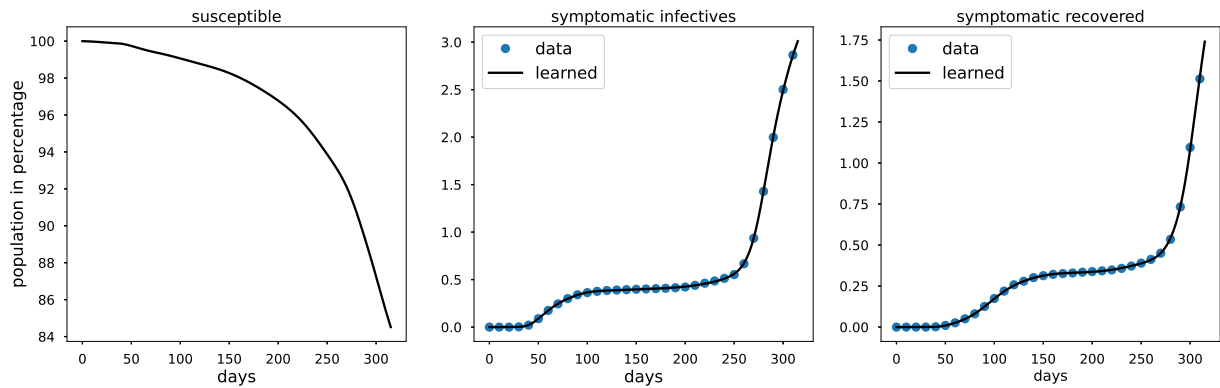
- 1: Construct EINN
  - specify the input:  $t_j, j = 1, \dots, M$
  - Initialize EINN parameter:  $\theta$
  - Initialize the epidemiology and vaccination parameters:  $\lambda = [\beta, \gamma, \mu, \xi, \kappa]$
- 2: Specify the training set
  - Training data:  $\tilde{I}(t_j), \tilde{R}(t_j), j = 1, \dots, M$  and  $\tilde{V}(t_j), j = 1, \dots, M_\kappa$ .
  - Asymptomatic population initialization:  $\tilde{J}(0) = (1 - \xi)\tilde{I}(0)/\xi$  and  $\tilde{U}(0) = (1 - \xi)\tilde{R}(0)/\xi$ .
- 3: Train the neural network
  - Specify an *MSE* loss function:

$$\begin{aligned}
 MSE &= \frac{1}{M} \sum_{j=1}^M \|I(t_j; \theta; \lambda) - \tilde{I}(t_j)\|_2^2 + \frac{1}{M} \sum_{j=1}^M \|R(t_j; \theta; \lambda) - \tilde{R}(t_j)\|_2^2 \\
 &+ \frac{1}{M_\kappa} \sum_{j=1}^{M_\kappa} \|\kappa S(t_j; \theta) - \tilde{V}(t_j)\|_2^2 \\
 &+ \|J(0; \theta; \lambda) - \tilde{J}(0)\|_2^2 + \|U(0; \theta; \lambda) - \tilde{U}(0)\|_2^2 \\
 &+ \frac{1}{M} \sum_{i=1}^6 \sum_{j=1}^M \|L_i(t_j; \theta; \lambda)\|_2^2.
 \end{aligned} \tag{9}$$

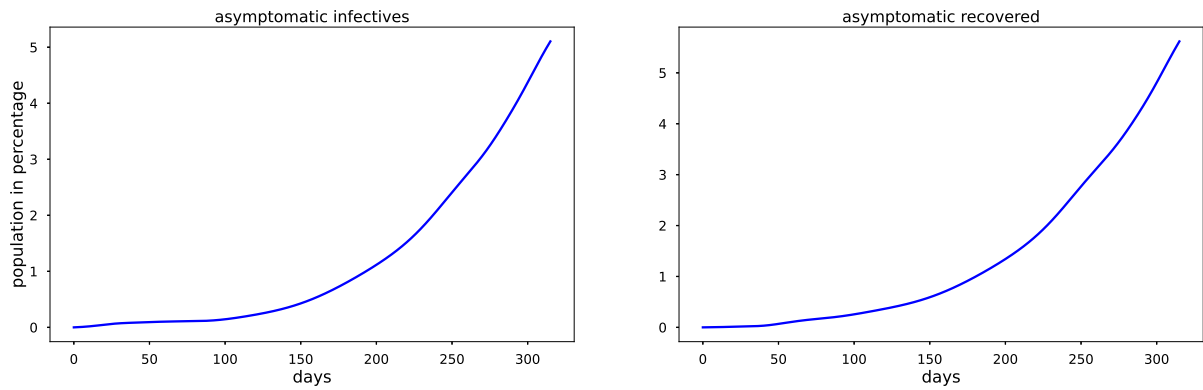
Minimize the *MSE* loss function eq. (7): compute  $\arg \min_{\{\theta; \lambda\}}(MSE)$  using an optimizer such as the

*L - BFGS - B*.

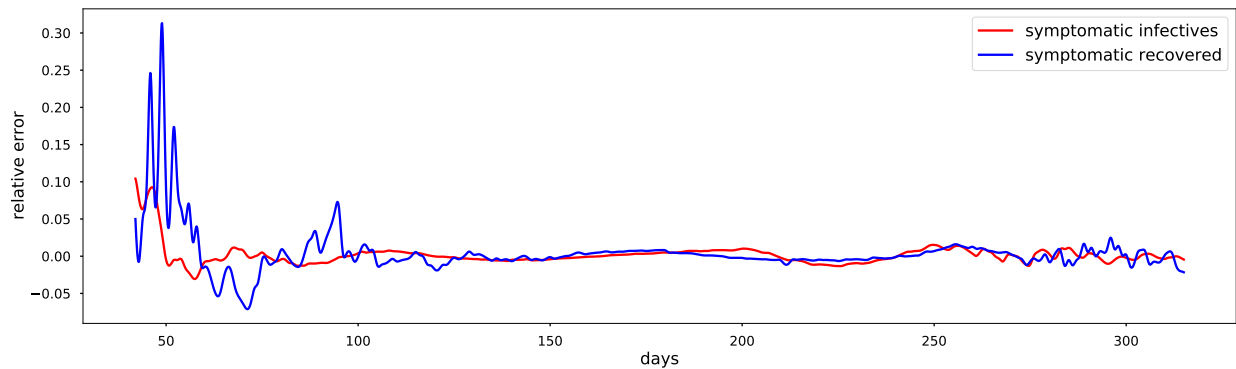
- 4: return EINN solution
    - $S(t_j; \theta; \lambda), I(t_j; \theta; \lambda), J(t_j; \theta; \lambda), R(t_j; \theta; \lambda), U(t_j; \theta; \lambda), j = 1, \dots, M$ .
    - parameters:  $\beta, \gamma, \mu, \xi, \kappa$ .
-



(a) The learned symptomatic infectives and recovered population by the EINN algorithm 1. An interpolation function is used to generate 3000 training points from the cumulative Italy symptomatic infective and recovered data. The learned plots are a near-perfect match of the original data. The relative  $L^2$ -error of the symptomatic infectives and symptomatic recovered are  $5.12 \times 10^{-3}$  and  $8.9 \times 10^{-3}$  respectively.



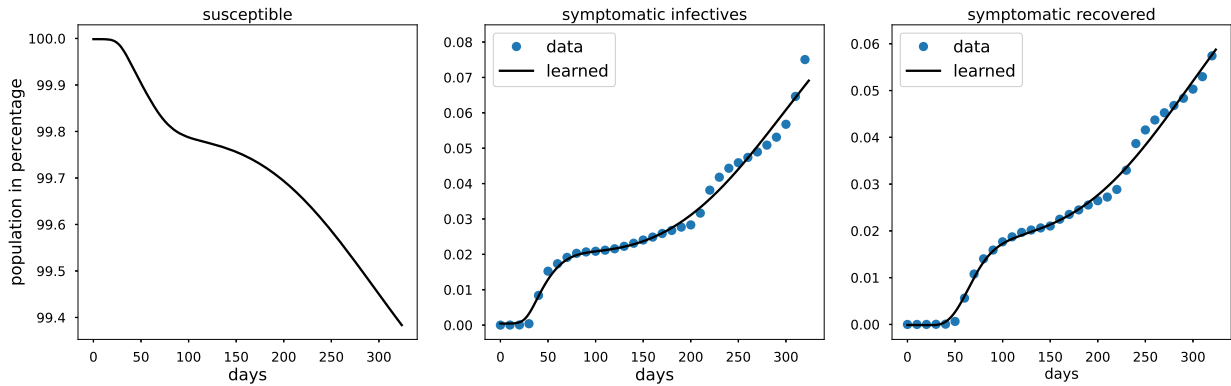
(b) EINN algorithm 1 learns the cumulative population of Italy that are asymptomatic infectives and asymptomatic recovered from January 31 to December 11. This represents the infective population that is unreported in the publicly available data [24].



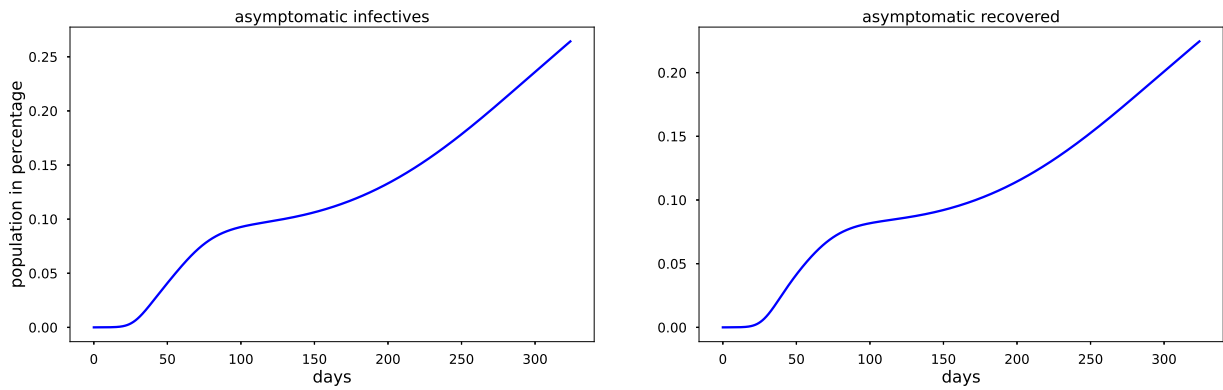
(c) relative error of the learned symptomatic infectives and recovered obtained by EINN, using the actual cumulative symptomatic infectives and recovered data.

Figure (3) Simulation of Italy COVID-19 data

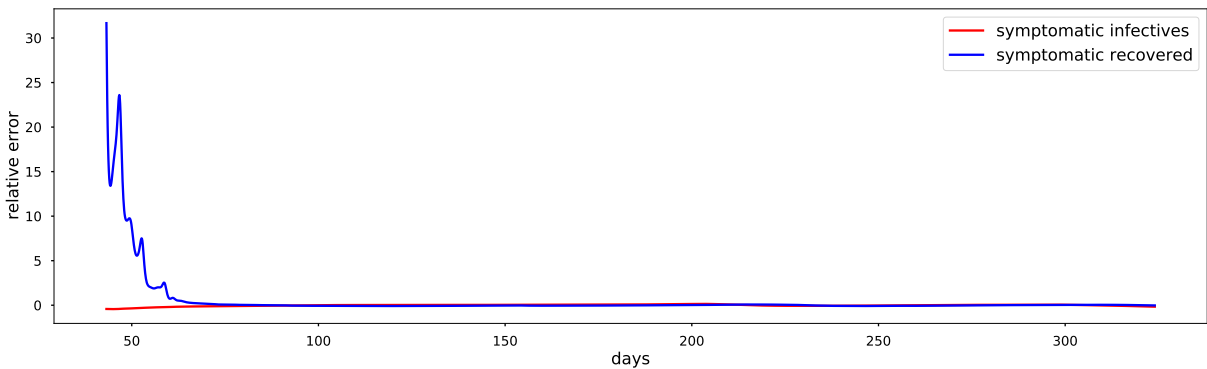




(a) The learned symptomatic infectives and recovered population by the EINN algorithm 1. An interpolation function is used to generate 3000 training points from the cumulative South Korea symptomatic infective and recovered data. The learned plots are a near-perfect match of the original data. The relative  $L^2$ -error of the symptomatic infectives and symptomatic recovered are  $7.73 \times 10^{-2}$  and  $7.01 \times 10^{-2}$ .

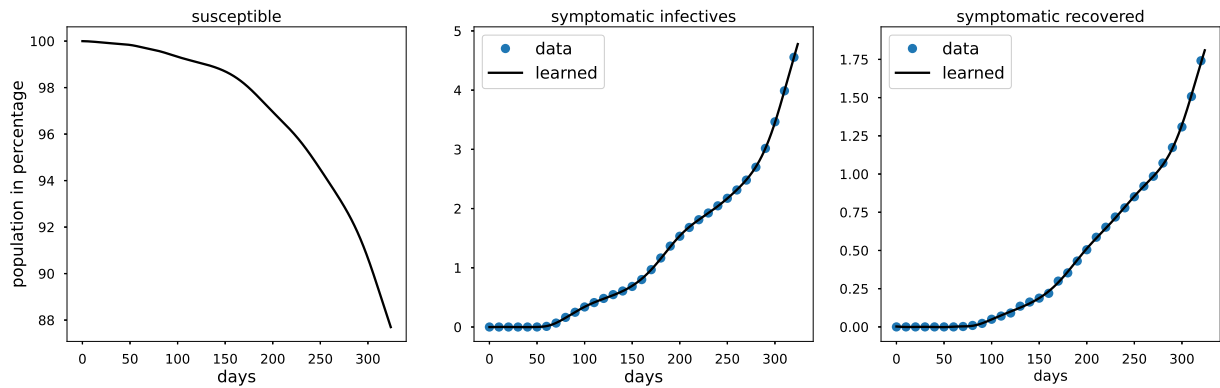


(b) EINN algorithm 1 learns the cumulative population of South Korea that are asymptomatic infectives and asymptomatic recovered from January 22 to December 11. This represent the infective population that are unreported in the publicly available data [24].

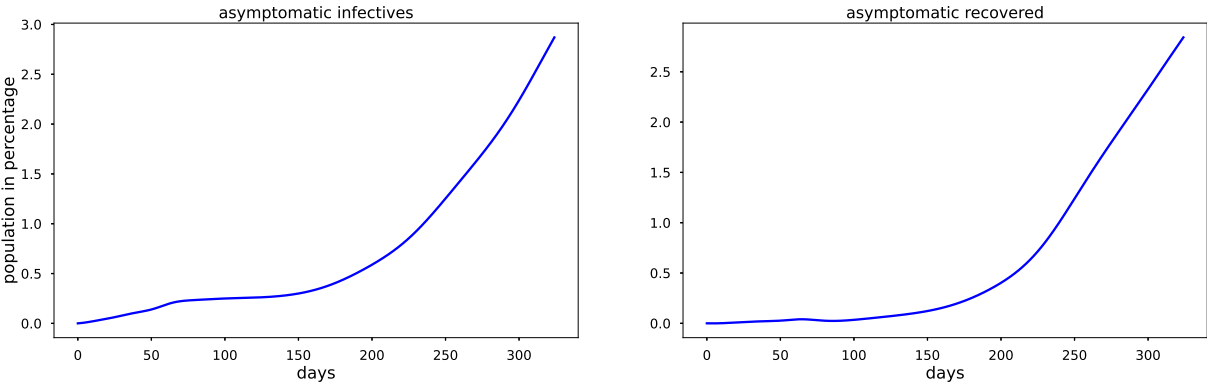


(c) relative error of the learned symptomatic infectives and recovered obtained by EINN, using the actual cumulative symptomatic infectives and recovered data.

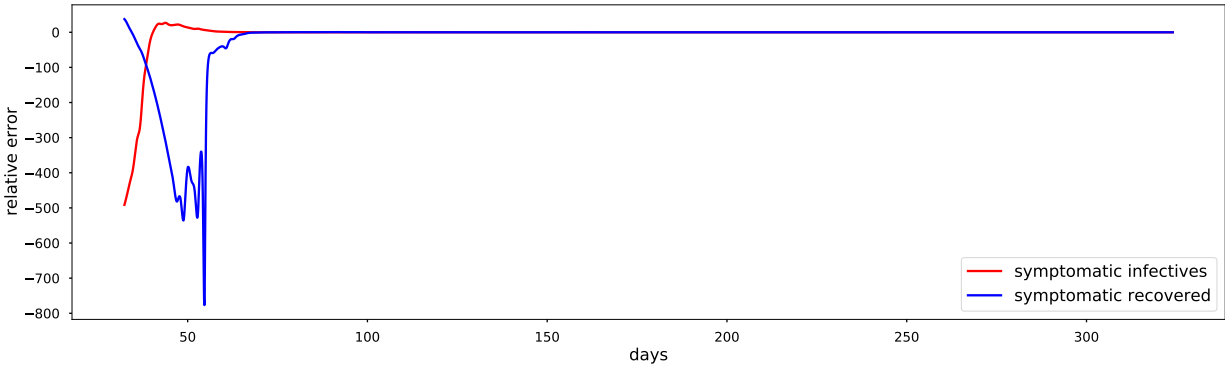
Figure (4) Simulation of South Korea COVID-19 data



(a) The learned symptomatic infectives and recovered population were obtained by the EINN algorithm 1. An interpolation function is used to generate 3000 training points from the cumulative USA symptomatic infective and recovered data. The learned plots are a near-perfect match of the original data. The relative  $L^2$ -error of the symptomatic infectives and symptomatic recovered are  $4.03 \times 10^{-3}$  and  $9.77 \times 10^{-3}$ .



(b) EINN algorithm 1 learns the cumulative population of USA that are asymptomatic infectives and asymptomatic recovered from January 22 to December 11. This represent the infective population that are unreported in the publicly available data [24].



(c) relative error of the learned symptomatic infectives and recovered obtained by EINN, using the actual cumulative symptomatic infectives and recovered data.

Figure (5) Simulation of USA COVID-19 data

## 5. Non-pharmaceutical mitigation measures

The model parameters in an epidemiology model are influenced by mitigation measures. For instance, social distancing correspond to reducing the infection rate by reducing human contact. In this Section, we simulate different levels of various non-pharmaceutical mitigation measure and we demonstrate its impact on BRN and the spread of COVID-19.

### 5.1. Early detection of infectives

Early detection of infectives population leads to higher reported infectives. This results in an early isolation of individuals who have had contact with infective individuals. There are no reported data for the asymptomatic infectives populations. Simulating with higher  $\xi$  increases the symptomatic infectives population. This corresponds to higher reported cases. Simulation is presented for Italy, South Korea, and the U.S.A see tables 1 and 2.

$\xi$	$\beta$	$\gamma$	$\mu$	$\beta \xi$	$\beta (1 - \xi)$	BRN
0.10	0.0081	0.0013	0.0081	0.00081	0.00729	1.0916
0.25	0.0099	0.0023	0.0161	0.00248	0.00743	0.7826
0.50	0.0140	0.0045	0.0496	0.00700	0.00700	0.5176
0.75	0.0109	0.0049	0.0144	0.00818	0.00273	1.4983

Table (1) We set the  $\xi$  parameters and use the EINN algorithm 1 to learn  $\beta$ ,  $\gamma$ , and  $\mu$  using cumulative Italy COVID-19 symptomatic infective and symptomatic recovered data from 01/31/2020 to 09/05/2020.

$\xi$	$\beta$	$\gamma$	$\mu$	$\beta \xi$	$\beta (1 - \xi)$	BRN
0.10	0.0090	0.0016	0.0044	0.0009	0.0081	2.1845
0.25	0.0150	0.0068	0.0095	0.0038	0.0113	1.6997
0.50	0.0153	0.0083	0.0121	0.0077	0.0077	1.5000
0.75	0.0116	0.0145	0.0033	0.0087	0.0029	0.9915

Table (2) Fixing  $\xi$ , we simulate predictions for  $\beta$ ,  $\gamma$ , and  $\mu$  using EINN algorithm 1 and cumulative South Korea COVID-19 symptomatic infective and symptomatic recovered data from 01/22/2020 to 09/05/2020.

Higher  $\xi$  values in tables 1 and 2, increases the symptomatic infectives population and reduces the asymptomatic population in general. This is reflected in increase in the  $\beta\xi$  column and decrease in the  $\beta(1 - \xi)$  column. It is assumed that the total infective population is unchanged as we raise  $\xi$ . This means that more people will be in hospitalization/isolation. This translates to more recovery in the symptomatic compartment. BRN decreases in general.

If the asymptomatic infectives population is much higher than the symptomatic population, this results in higher BRN. There is a right balance between  $I$  and  $J$  population. We also see that the detection of early infectives alone is not enough to mitigate an infectious disease like COVID-19. It should be combined with other measures such as contact tracing of infectives.

### 5.2. Social distancing

It is widely understood that measures such as a lockdown, social distancing, and widespread adoption of facial covering result in the mitigation of COVID-19. Social distancing is often the most sought after measure at reducing the BRN. The goal of social distancing is to reduce the average number of human contacts. This is demonstrated by reducing  $\beta$ , the infection rate. The impact of social distancing on the BRN is presented in the following tables 3 and 4 .

$\beta$	$\gamma$	$\xi$	$\mu$	$\beta \xi$	$\beta (1 - \xi)$	BRN
0.020	0.0071	0.6667	0.2234	0.0133	0.0067	0.2525
0.015	0.0042	0.4957	0.0653	0.0074	0.0076	0.4284
0.010	0.0048	0.6375	0.0060	0.0064	0.0036	1.9102
0.005	0.0031	0.7195	0.0037	0.0036	0.0014	1.5298

Table (3) Fixing  $\beta$ , we simulate predictions for  $\gamma$ ,  $\xi$ , and  $\mu$  using EINN algorithm 1 and cumulative Italy COVID-19 symptomatic infective and symptomatic recovered data 01/31/2020 to 09/05/2020.

$\beta$	$\gamma$	$\xi$	$\mu$	$\beta \xi$	$\beta (1 - \xi)$	BRN
0.020	0.0156	0.3359	0.0076	0.0067	0.0133	1.9442
0.015	0.0077	0.2839	0.0078	0.0043	0.0107	1.9301
0.010	0.0057	0.2454	0.0041	0.0025	0.0075	2.2259
0.005	0.0017	0.1369	0.0021	0.0007	0.0043	2.4447

Table (4) Fixing  $\beta$ , we simulate predictions for  $\gamma$ ,  $\xi$ , and  $\mu$  using EINN algorithm 1 and cumulative South Korea COVID-19 symptomatic infective and symptomatic recovered data 01/22/2020 to 09/05/2020.

Reducing  $\beta$  in tables 3 and 4 correspond to a reduced symptomatic infectives population  $I$ . However, the total infectives are unchanged as we reduce  $\beta$ . There is an increase in asymptomatic infectives population  $J$ . BRN does not decrease in general. Social distancing is effective when the asymptomatic infective population  $J$  diminishes.  $\beta\xi$  and  $\beta(1 - \xi)$  both decreases. Social distancing should be combined with contact tracing and early detection of infectives population.

### 5.3. Contact tracing of infectives

Contact tracing is equivalent to increasing the symptomatic recovery and asymptomatic recovery rates. However, since we do not have reported data for the asymptomatic population, in this paper, we pursue contact tracing as an increase in the symptomatic recovery rate. This is equivalent to reducing the number of days an infective individual stays infective. In tables 5, 6 and 7, the impact of contact tracing is demonstrated by increasing the symptomatic recovery rate.

$\gamma$	$\beta$	$\xi$	$\mu$	$\beta \xi$	$\beta (1 - \xi)$	BRN
0.0005	0.0095	0.3345	0.0093	0.0032	0.0063	1.4946
0.0010	0.0093	0.4727	0.0116	0.0044	0.0049	1.4114
0.0050	0.0118	0.5633	0.0133	0.0066	0.0052	1.3682
0.0100	0.0104	0.7669	0.0036	0.0080	0.0024	1.2224

Table (5) Fixing  $\gamma$ , we simulate predictions for  $\beta$ ,  $\xi$ , and  $\mu$  using EINN algorithm 1 and cumulative Italy COVID-19 symptomatic infective and symptomatic recovered data 01/31/2020 to 09/05/2020.

$\gamma$	$\beta$	$\xi$	$\mu$	$\beta \xi$	$\beta (1 - \xi)$	BRN
0.0010	0.0124	0.1556	0.0071	0.0019	0.0105	2.0160
0.0050	0.0109	0.2582	0.0043	0.0028	0.0081	2.4326
0.0100	0.0147	0.2690	0.0086	0.0040	0.0107	1.6376
0.0150	0.0145	0.3679	0.0079	0.0053	0.0092	1.3794

Table (6) Fixing  $\gamma$ , we simulate predictions for  $\beta$ ,  $\xi$ , and  $\mu$  using EINN algorithm 1 and cumulative South Korea COVID-19 symptomatic infective and symptomatic recovered data 01/22/2020 to 09/05/2020.

$\gamma$	$\beta$	$\xi$	$\mu$	$\beta \xi$	$\beta (1 - \xi)$	BRN
0.0005	0.0198	0.6240	0.0566	0.0124	0.0074	0.9169
0.0010	0.0227	0.4152	0.0222	0.0094	0.0133	1.6943
0.0050	0.0228	0.5783	0.0261	0.0132	0.0096	1.6405
0.0100	0.0295	0.5253	0.0345	0.0155	0.0140	1.3638

Table (7) Fixing  $\gamma$ , we simulate predictions for  $\beta$ ,  $\xi$ , and  $\mu$  using EINN algorithm 1 and cumulative USA COVID-19 symptomatic infective and symptomatic recovered data 01/22/2020 to 09/05/2020.

The raising of  $\gamma$  in tables 5 to 7, increases the symptomatic infectives population  $I$  which is demonstrated in increased  $\xi$  and increased  $\beta$ .  $\beta(1 - \xi)$  decreases somewhat while  $\beta\xi$  increases. This also results in a reduced BRN. Contact tracing is an efficient mitigation measure in lowering the spread of COVID-19.

## 6. Pharmaceutical mitigation measure

The mitigation measures described in Section 5 are non-pharmaceutical measures. In this Section, we discuss vaccination. In the fight against COVID-19, countries such as USA and United Kingdom began to vaccinate in December, 2020. A major goal of vaccination is to reduce the susceptible population, that is people recover without getting infected. This constitute a pharmaceutical mitigation measure. We considered the vaccination data for the USA and United Kingdom, and simulate the effectiveness of vaccination on the daily reported infectives. Algorithm 1 is implemented for the asymptomatic-SIR model given as follows.

In Figure 6a, using USA data, the mitigation effect of vaccination on the daily infectives is demonstrated. Implementing algorithm 1, we obtained  $\kappa = 0.00184$ , which is slightly different from the projection of  $\kappa = 0.00305$ , corresponding to 1 million people vaccination per day. In Figure 6b, using United Kingdom data, we simulate the impact of vaccination on the daily reported infectives, using a smoothed daily vaccination data from 12/13/2020 to 02/05/2020 and smoothed daily reported infectives data. We implement algorithm 1 and we obtained  $\kappa = 0.00305$ . Our model predicts that the daily infectives will be close to zero by April 2021. We demonstrate the impact of increased social distancing together with the vaccination effort. Social distancing corresponds to decreasing the infection rate  $\beta$ . Increased social distancing reduces the daily reported infectives but it extends the number of days daily infectives data is significant.

## 7. EINN algorithms for time-dependent infection rates

The model parameters in the previous section should be interpreted as average values throughout the pandemic. We cannot assume that these parameters stay constant because of the many government interventions and public perceptions and responses. In this section, we compare a delayed-mitigation time-series infection rate approach, a delayed-mitigation exponential time-dependent infection rate approach [5, 11, 3] and a piecewise time-dependent infection rate approach. In the case of the delayed-mitigation approach, Italy and the USA instituted the first COVID-19 lockdown on March 9th and March 18th respectively. That is 38 days after the first case in Italy and 57 days in the USA. In 7.1 and 7.2, we simulated the time-varying infection rate assuming there was no mitigation measure influencing the data pre-lockdown. We run simulations for Italy, South Korea, and the United States from the date of the first reported cases in the respective countries to the day before vaccination data was reported,  $\kappa = 0$ .

**Remark 1.** *The time-dependent infection rate is non-constant in the presence of mitigation measures in the cumulative infective data.*

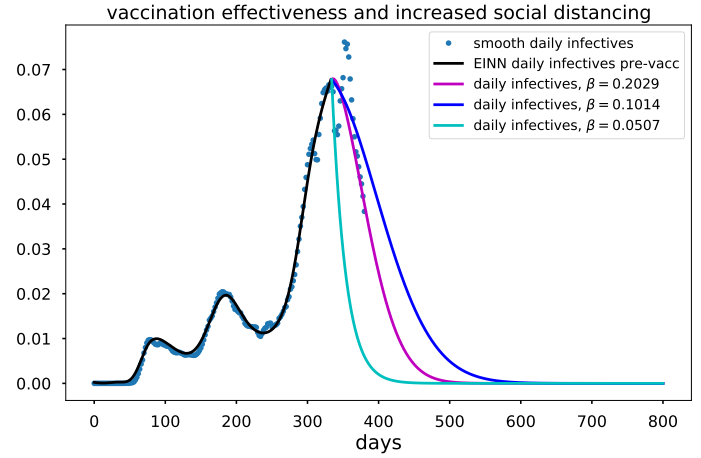
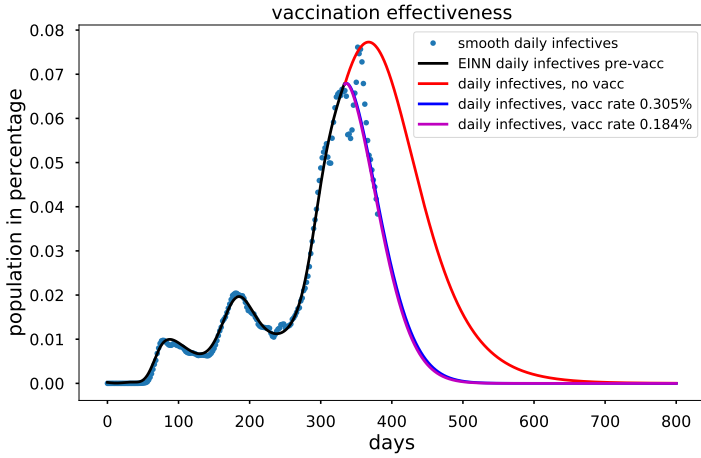
### 7.1. Delayed-mitigation time-series infection rate

Here, the infection rate  $\beta(t)$  in eq. (1) is nonlinear and time-dependent. The early COVID-19 lockdown that was instituted by several countries was widespread in reach and effective in mitigating the spread of the disease. The impact of lockdown was also immediately reflected in the cumulative infective data. As a result, we consider the infection rate to be constant pre-lockdown and a nonlinear time-series during and post-lockdown. Accordingly we write  $\beta(t)$

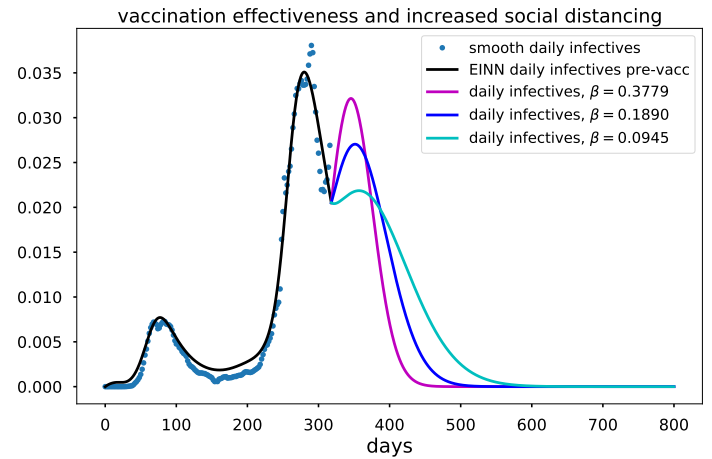
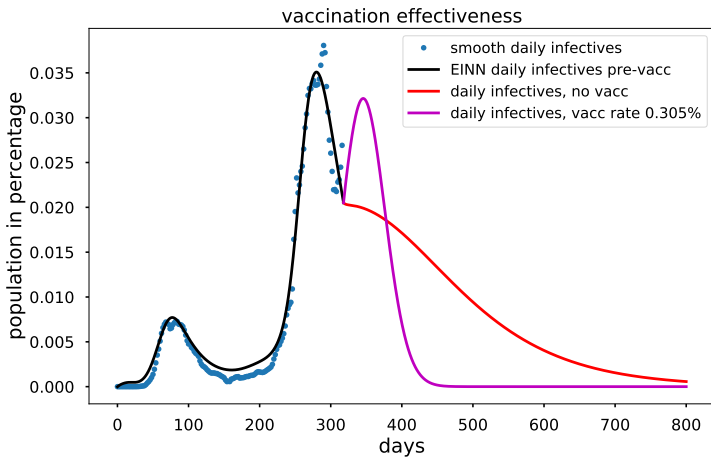
$$\beta(t) = \begin{cases} \beta_0 & 0 \leq t < K \\ \beta_0 q(t) & K < t, \end{cases} \quad (10)$$

In eq. (10),  $q(t) \in (0, 1)$  during and post-lockdown for each value of  $t$  [27].  $\beta_0$  correspond to the constant infection rate obtained from early data using nonlinear regression. It is the infection rate before any mitigation is enforced.  $q(t)$  represents the presence of some mitigation in the data. When  $q(t) = 1$ , this corresponds to an absence of mitigation. Using the reported cumulative data, the EINN algorithm 2 discovers the mitigation in the cumulative symptomatic infectives and recovered data and its impact on the infection rate. It is assumed that there is no mitigation very early in the data, and so the infection and recovery rates are constants during the early days of the pandemic.

In the EINN algorithm 2,  $M_\beta$  correspond to the number of days number of days mitigation is delayed in the data, which is equal to  $K$  in eq (10).



(a) The effectiveness of vaccination is demonstrated by learning the pre-vaccination and post-vaccination epidemiology parameters using smooth daily reported infectives data from the USA. The model is extrapolated for 3 cases. The red curve is the case of no vaccination, here  $\kappa = 0$ . For the blue curve, we used the USA projection of 1000000 daily vaccination. In the case of the magenta curve, we learned  $\kappa$  using the daily vaccination data. The first reported case was 01/22/2020, Vaccination data were first reported 12/19/2020. Our model suggests that daily reported infectives will be asymptotically zero by May - June, 2021 if the current vaccination rate is sustained together with the non-pharmaceutical mitigation measures.



(b) The effectiveness of vaccination is demonstrated by learning the pre-vaccination and post-vaccination epidemiology parameters using smooth daily reported infectives data from the United Kingdom. The model is extrapolated for 2 cases. The red curve is the case of no vaccination, here  $\kappa = 0$ . The magenta curve, we learned  $\kappa$  using the daily vaccination data. The first reported case was 01/31/2020, Vaccination data were first reported 12/13/2020. Our model suggests that daily reported infectives will be asymptotically zero by April, 2021 if the current vaccination rate is sustained together with the non-pharmaceutical mitigation measures.

Figure (6) Vaccination effectiveness for USA and United Kingdom

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**Algorithm 2** EINN algorithm for Asymptomatic-SIR model with delayed-mitigation time-series infection rate

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- 1: Construct EINN
  - specify the input:  $t_j, j = 1, \dots, M$
  - Initialize EINN parameter:  $\theta$
  - Initialize the epidemiology and vaccination parameters:  $\lambda = [\gamma, \mu, \kappa]$
- 2: Construct neural network:  $\beta$ 
  - specify the input:  $t_j, j = 1, \dots, M_\beta$
  - Initialize the neural network parameter:  $\phi$
- 3: Specify  $\beta_0$  obtained by nonlinear regression of early cumulative infective population data
- 4: Specify EINN training set
  - Training data:  $\tilde{I}(t_j)$  and  $\tilde{R}(t_j), j = 1, \dots, M$ .
  - Set  $\xi$  according to the value obtained from EINN algorithm 1.
  - Asymptomatic population initialization:  $\tilde{J}(0) = (1 - \xi)\tilde{I}(0)/\xi$  and  $\tilde{U}(0) = (1 - \xi)\tilde{R}(0)/\xi$ .
- 5: Train the neural networks
  - Specify an *MSE* loss function:

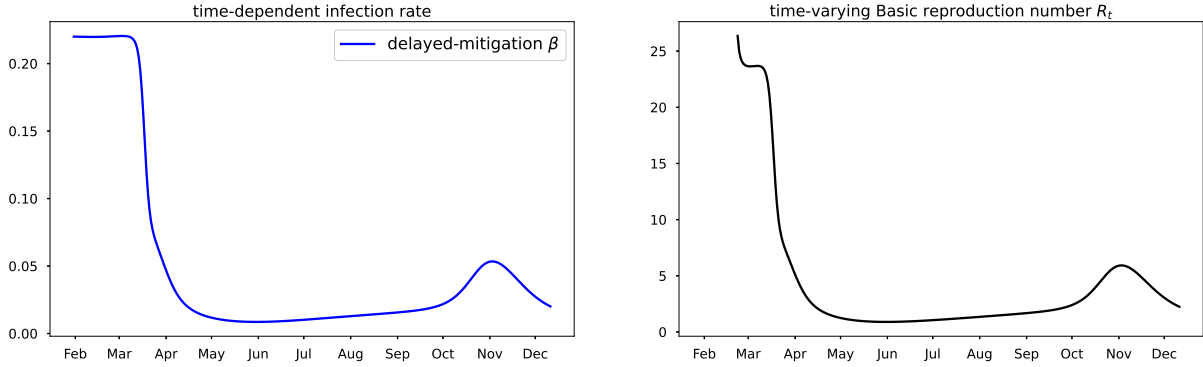
$$\begin{aligned}
 MSE = & \frac{1}{M} \sum_{j=1}^M \|I(t_j; \theta; \lambda) - \tilde{I}(t_j)\|_2^2 + \frac{1}{M} \sum_{j=1}^M \|R(t_j; \theta; \lambda) - \tilde{R}(t_j)\|_2^2 \\
 & + \frac{1}{M_\beta} \sum_{j=1}^{M_\beta} \|\beta(t_j; \phi) - \beta_0\|_2^2 \\
 & + \frac{1}{M_\kappa} \sum_{j=1}^{M_\kappa} \|\kappa S(t_j; \theta) - \tilde{V}(t_j)\|_2^2 \\
 & + \|J(0; \theta; \lambda) - \tilde{J}(0)\|_2^2 + \|U(0; \theta; \lambda) - \tilde{U}(0)\|_2^2 \\
 & + \frac{1}{M} \sum_{i=1}^6 \sum_{j=1}^M \|L_i(t_j; \theta; \phi; \lambda)\|_2^2.
 \end{aligned} \tag{11}$$

Set  $\beta(t_j; \phi)$  in the residual

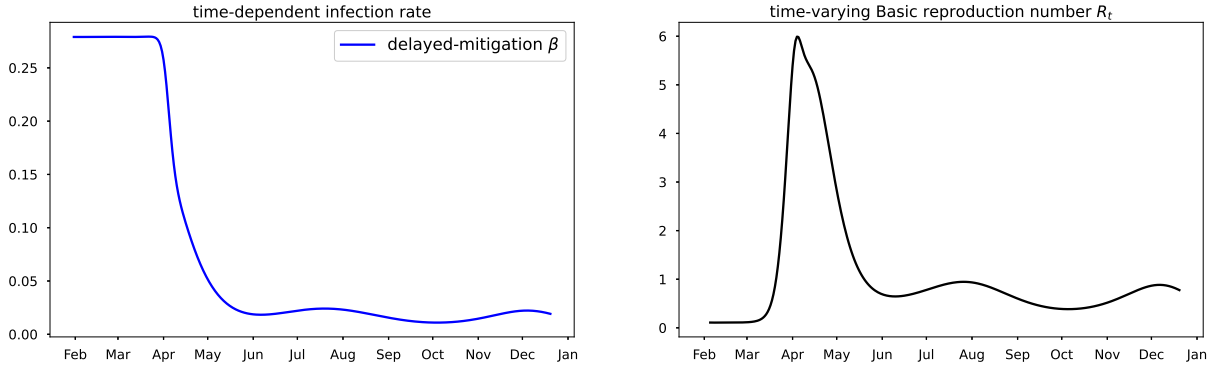
$$\beta(t_j; \phi) = \begin{cases} \beta_0 & 0 \leq t_j \leq M_\beta \\ \beta_0 q(t_j; \phi) & M_\beta < t_j, \end{cases} \tag{12}$$

Minimize the *MSE* loss function: compute  $\arg \min_{\{\theta; \phi; \lambda\}}(MSE)$  using an optimizer such as the *L - BFGS - B*.

- 6: return EINN solution
    - $S(t_j; \theta; \lambda), I(t_j; \theta; \lambda), J(t_j; \theta; \lambda), R(t_j; \theta; \lambda), U(t_j; \theta; \lambda), j = 1, \dots, M$ .
    - epidemiology parameters:  $\gamma$  and  $\mu$
    - vaccination parameter:  $\kappa$
  - 7: return time-series epidemiology parameter:
    - $\beta(t_j; \phi), j = 1, \dots, M_\beta$ .
-



(a) Using the infection rate eq. (10) in eq. (1), we fix  $\xi = 0.37$  and using early data and nonlinear regression to obtain  $\beta_0 = 0.22$ . A time-dependent infection rate  $\beta$  is learned and plotted for cumulative Italy COVID-19 data from January 31, 2020 to December 11, 2020.  $N = 60.36 \times 10^6$ . The first COVID-19 lockdown in Italy began March 9th. We set  $K = 40$  days. The time-varying basic reproduction rate  $R_t$  is obtained using the estimate eq. (4). The EINN algorithm 2 also learns  $\gamma = 0.0102$  and  $\mu = 0.0091$ . The MSE is  $9.4 \times 10^{-5}$ . Post lockdown, other mitigation measures including social distancing, widespread masking kept the infection rate down. This is followed by a relaxation period that results in COVID-19 surge that was reported during the holiday.



(b) Using the infection rate eq. (10) in eq. (1), we fix  $\xi = 0.46$  and using early data and nonlinear regression to obtain  $\beta_0 = 0.279$ . A time-dependent infection rate  $\beta$  is learned and plotted for cumulative U.S.A COVID-19 data from January 22, 2020 to December 11, 2020.  $N = 328.2 \times 10^6$ . The Nationwide Lockdown that was put in place in the USA in 2020 began March 19th. We set  $K = 57$  days. The time-varying basic reproduction rate  $R_t$  is obtained using the estimate eq. (4). The EINN algorithm 2 learns  $\gamma = 0.0014$  and  $\mu = 0.0261$ . The MSE is  $3.35 \times 10^{-4}$ . This approach underestimate  $R_t$  during the pre-lockdown period.

Figure (7) Delayed-mitigation time-series rates



In [11, 3], it was shown that during the early phase of the COVID-19 pandemic when the cumulative infection population grew exponentially, the infection rate was constant. This is demonstrated in Figures 7a and 7b. This coincide with the period before any mitigation measure. Incorporating measures such as social distancing, lock-down and widespread adoption of facial covering in an epidemiology model is complex. In [15] used a sigmoid function to model a time dependent decrease in the transmission of COVID-19. In [11], an exponentially decreasing function eq. (13) was used to represent the infection rate  $\beta(t)$  in eq. (1) to mimic the impact of lockdown. In Section 7.2, we learn an exponentially decreasing infection rate assuming that it takes the form of eq. (13). We demonstrate that this approach detects various other mitigation measures post-lockdown.

### 7.2. Delayed-mitigation exponential time-dependent infection rate

The infection rate  $\beta(t)$  in eq. (1) is defined as

$$\beta(t) = \begin{cases} \beta_0 & 0 \leq t \leq K \\ \beta_0 \exp(-\eta(t - K)) & K < t. \end{cases} \quad (13)$$

Where  $K$  signifies the onset of government intervention including isolation, quarantine and lock-down.  $\eta$  is the rate at which human contact decreases. We use EINN algorithm 3 to learn  $\beta(t)$  assuming that it takes the form of eq. (13) in eq. (1). This approach also learns  $\eta$ . We denote  $K$  to be the number of days between the date of the first reported case of COVID-19 and the date lockdown was instituted.

In the EINN algorithm 3,  $M_\beta$  corresponds to the number of days mitigation is delayed in the data, which is equal to  $K$  in eq (13).  $M_\kappa$  is the number of vaccination days.

In Section 7.2, the delayed-mitigation exponential time-dependent infection rate detects the impact of the year 2020 COVID-19 lockdown, as well as the other mitigation measures post-lockdown using the parameter  $\eta$ . It is however difficult to know if  $\eta$  captures all the pattern in the time-dependent infection rate as demonstrated in Figures 8a and 8b, that is if eq. (13) helps us to learn the most accurate form of  $\beta$ . For instance, the time-varying basic reproduction rate  $R_t$  is underestimated pre-lockdown in the USA data and overestimated pre-lockdown in Italy data.  $R_t$  is overestimated post-lockdown. One limitation in Section 7.1 and Section 7.2 is that the delayed-mitigation approach is not very useful when there is no big impacting mitigation measure such as lockdown. For example, In South Korea, mitigation measures such as social distancing, contact tracing, and quarantine were used to effectively combat the spread of COVID-19 early on without needing to lockdown entire cities [28].

### 7.3. Piecewise time-dependent infection rate

A piecewise time-dependent infection rate (16) is used to learn a time-dependent infection rate  $\beta$  in eq. (1). The piecewise-constant  $\beta(t)$  is defined as follows,

$$\beta(t) = \begin{cases} \beta_0 & t \leq M_1 \\ \beta_0 q_1 & M_1 < t \leq M_2 \\ \beta_0 q_2 & M_2 < t \leq M_3 \\ \beta_0 q_3 & M_3 < t \leq M_4 \\ \vdots & \\ \beta_0 q_n & M_n < t. \end{cases} \quad (16)$$

The goal of the parameters  $q_1, \dots, q_n$  in eq. (16) is to capture the decrease/increase that occur in the infection rate  $\beta(t)$ . We choose  $M_1, \dots, M_n$  and we use EINN algorithm 4 to learn  $q_1, \dots, q_n$ .

The learned piecewise time-dependent infection rate obtained in Section 7.3 captures the mitigation measures that occur early in the COVID-19 data. It is non-smooth as shown in Figures 9a, 9b, and 9c because we used a discrete piecewise function in the training. However, the COVID-19 surge that was reported in most publicly available data in tail end of 2020 are not detected in the time-varying basic reproduction number  $R_t$ . We see that it is difficult to know the most accurate form of a time-dependent infection rate that captures all the pre-lockdown mitigation measures as well as the post-lockdown ones. We seek a time-varying infection rate that also captures the COVID-19 surge that is reflected in the cumulative infective population data after low  $R_t$  for most of summer 2020.

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**Algorithm 3** EINN algorithm for Asymptomatic-SIR model with delayed-mitigation exponential time-dependent infection rate

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- 1: Construct EINN
  - specify the input:  $t_j, j = 1, \dots, M$
  - Initialize EINN parameter:  $\theta$
  - Initialize the epidemiology and vaccination parameters:  $\lambda = [\gamma, \mu, \kappa]$
- 2: construct neural network:  $\beta$ 
  - specify the input:  $t_j, j = 1, \dots, M_\beta$
  - Initialize the neural network parameter:  $\phi$
  - Initialize the exponential decay parameter:  $\eta$
- 3: Specify  $\beta_0$  obtained by nonlinear regression of early cumulative infective population data
- 4: Specify EINN training set
  - Training data:  $\tilde{I}(t_j)$  and  $\tilde{R}(t_j), j = 1, \dots, M$ .
  - Set  $\xi$  according to the value obtained from EINN algorithm 1
  - Asymptomatic population initialization:  $\tilde{J}(0) = (1 - \xi)\tilde{I}(0)/\xi$  and  $\tilde{U}(0) = (1 - \xi)\tilde{R}(0)/\xi$ .
- 5: Train the neural networks
  - Specify an *MSE* loss function:

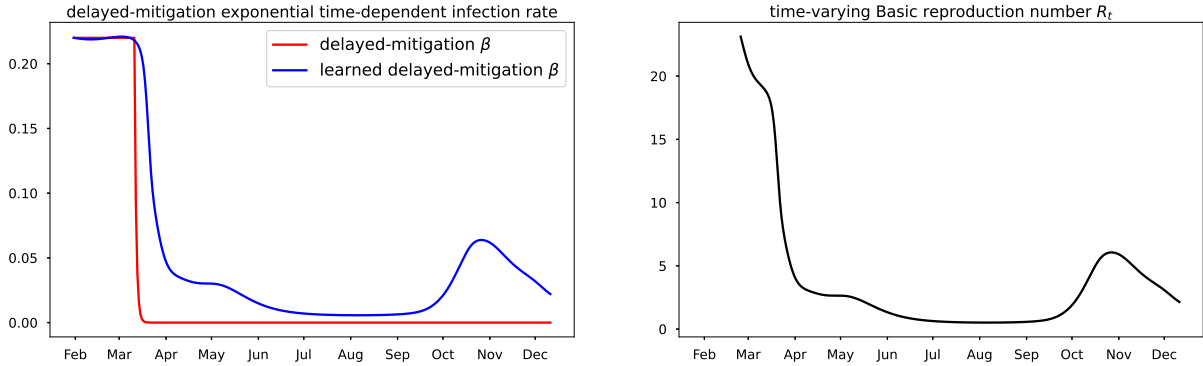
$$\begin{aligned}
 MSE = & \frac{1}{M} \sum_{j=1}^M \|I(t_j; \theta; \lambda) - \tilde{I}(t_j)\|_2^2 + \frac{1}{M} \sum_{j=1}^M \|R(t_j; \theta; \lambda) - \tilde{R}(t_j)\|_2^2 \\
 & + \frac{1}{M_\beta} \sum_{j=1}^{M_\beta} \|\beta(t_j; \phi; \eta) - \beta_0\|_2^2 \\
 & + \frac{1}{M_\kappa} \sum_{j=1}^{M_\kappa} \|\kappa S(t_j; \theta) - \tilde{V}(t_j)\|_2^2 \\
 & + \|J(0; \theta; \lambda) - \tilde{J}(0)\|_2^2 + \|U(0; \theta; \lambda) - \tilde{U}(0)\|_2^2 \\
 & + \frac{1}{M} \sum_{i=1}^6 \sum_{j=1}^M \|L_i(t_j; \theta; \phi; \lambda; \eta)\|_2^2.
 \end{aligned} \tag{14}$$

Set  $\beta(t_j; \phi; \eta)$  in the residual

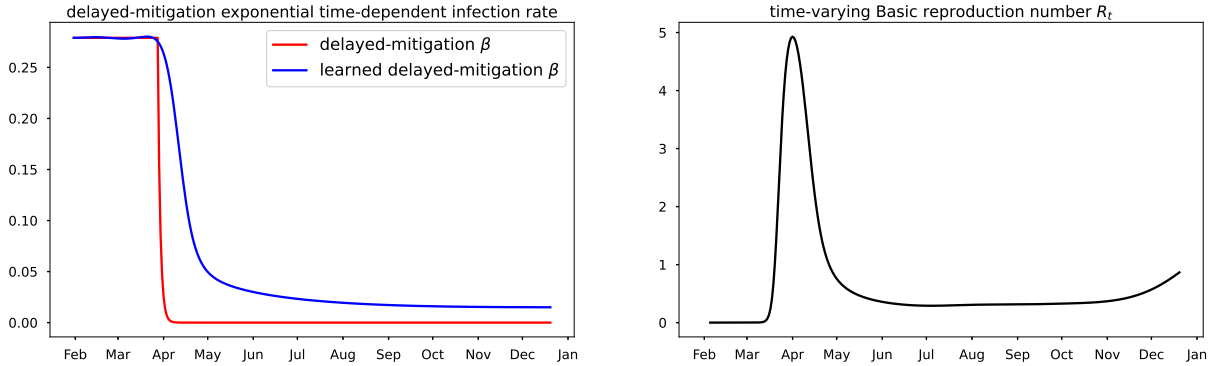
$$\beta(t_j; \phi; \eta) = \begin{cases} \beta_0 & 0 \leq t_j \leq M_\beta \\ \beta_0 \exp(-\eta(t_j - M_\beta)) & M_\beta < t_j, \end{cases} \tag{15}$$

Minimize the *MSE* loss function: compute  $\arg \min_{\{\theta; \phi; \lambda; \eta\}}(MSE)$  using an optimizer such as the *L - BFGS - B*.

- 6: return EINN solution
    - $S(t_j; \theta; \lambda), I(t_j; \theta; \lambda), J(t_j; \theta; \lambda), R(t_j; \theta; \lambda), U(t_j; \theta; \lambda), j = 1, \dots, M$ .
    - epidemiology parameters:  $\gamma$  and  $\mu$
    - vaccination parameter:  $\kappa$
  - 7: return time-dependent epidemiology parameter:
    - $\beta(t_j; \phi; \eta), j = 1, \dots, M_\beta$ .
    - Rate of human contact decrease:  $\eta$ .
-



(a) The delayed-mitigation exponential infection rate is learned using eq. (13) in eq. (1). We set  $K = 40$  and we fix  $\xi = 0.37$  in EINN algorithm 3. We take  $\beta_0 = 0.22$ , obtained using early data and nonlinear regression. EINN algorithm 3 learns  $\eta = 0.87$ , the rate at which human contact decreases. A learned delayed-mitigation exponential time-dependent infection rate  $\beta$  is plotted for cumulative Italy COVID-19 data from January 31, 2020 to December 11, 2020.  $N = 60.36 \times 10^6$ . The plotted time-varying basic reproduction rate  $R_t$  shows the impact of lockdown and the mitigation measures post lockdown. The relaxation that followed resulted in the COVID-19 surge and is detected in the learned  $\beta$  and  $R_t$ . The EINN algorithm 3 also learns  $\gamma = 0.0121$  and  $\mu = 0.0106$ . The MSE is  $7.5 \times 10^{-5}$ .



(b) The delayed-mitigation exponential infection rate is learned using eq. (13) in eq. (1). We set  $K = 57$  and we fix  $\xi = 0.46$  in EINN algorithm 3. We take  $\beta_0 = 0.279$ , obtained using early data and nonlinear regression. EINN algorithm 3 learns  $\eta = 0.60$ , the rate at which human contact decreases. A learned delayed-mitigation exponential time-dependent infection rate  $\beta$  is plotted for cumulative U.S.A COVID-19 data from January 22, 2020 to December 11, 2020.  $N = 328.2 \times 10^6$ . The plotted time-varying basic reproduction rate  $R_t$  is underestimated in the pre-lockdown stage. The EINN algorithm 3 also learns  $\gamma = 0.001$  and  $\mu = 0.0224$ . The MSE is  $3.88 \times 10^{-4}$ .

Figure (8) Delayed-mitigation exponential time-dependent rates

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**Algorithm 4** EINN algorithm for Asymptomatic-SIR model with piecewise time-dependent infection rate
 

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- 1: Construct EINN
  - specify the input:  $t_j, j = 1, \dots, M$
  - Initialize EINN parameter:  $\theta$
  - Initialize the epidemiology and vaccination parameter:  $\lambda = [\gamma, \mu, \kappa]$
- 2: construct neural network:  $\beta$ 
  - specify the input:  $t_j, j = 1, \dots, M_\beta$
  - Initialize the neural network parameter:  $\phi$
  - Initialize the exponential decay parameters:  $\omega = [q_1, \dots, q_n]$
- 3: Specify  $\beta_0$  obtained by nonlinear regression of early cumulative infective population data
- 4: Specify EINN training set
  - Training data:  $\tilde{I}(t_j)$  and  $\tilde{R}(t_j), j = 1, \dots, M$ .
  - Set  $\xi$  according to the value obtained from EINN algorithm 1
  - Asymptomatic population initialization:  $\tilde{J}(0) = (1 - \xi)\tilde{I}(0)/\xi$  and  $\tilde{U}(0) = (1 - \xi)\tilde{R}(0)/\xi$ .
- 5: Train the neural networks
  - Specify an *MSE* loss function:

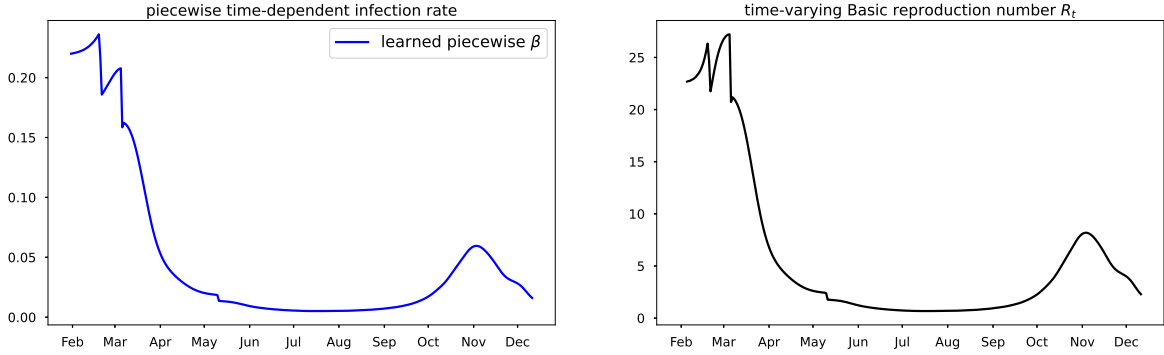
$$\begin{aligned}
 MSE &= \frac{1}{M} \sum_{j=1}^M \|I(t_j; \theta; \lambda) - \tilde{I}(t_j)\|_2^2 + \frac{1}{M} \sum_{j=1}^M \|R(t_j; \theta; \lambda) - \tilde{R}(t_j)\|_2^2 \\
 &+ \|\beta(0; \phi; \omega) - \beta_0\|_2^2 \\
 &+ \frac{1}{M_\kappa} \sum_{j=1}^{M_\kappa} \|\kappa S(t_j; \theta) - \tilde{V}(t_j)\|_2^2 \\
 &+ \|J(0; \theta; \lambda) - \tilde{J}(0)\|_2^2 + \|U(0; \theta; \lambda) - \tilde{U}(0)\|_2^2 \\
 &+ \frac{1}{M} \sum_{i=1}^6 \sum_{j=1}^M \|L_i(t_j; \theta; \phi; \lambda; \omega)\|_2^2.
 \end{aligned} \tag{17}$$

Set  $\beta(t_j; \phi; \omega)$  in the residual

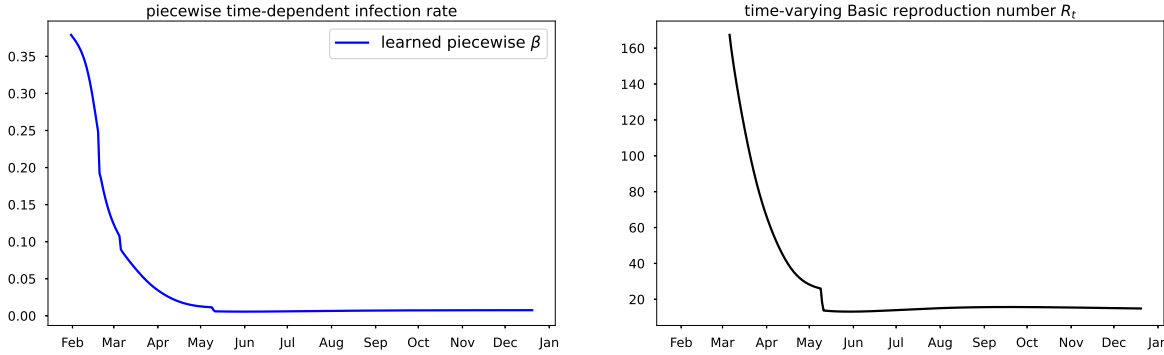
$$\beta(t_j; \phi; \omega) = \begin{cases} \beta_0 & 0 \leq t_j \leq M_1 \\ \beta_0 q_1 & M_1 < t_j \leq M_2 \\ \beta_0 q_2 & M_2 < t_j \leq M_3 \\ \beta_0 q_3 & M_3 < t_j \leq M_4 \\ \vdots & \\ \beta_0 q_n & M_n < t_j. \end{cases} \tag{18}$$

Minimize the *MSE* loss function: compute  $\arg \min_{\{\theta; \phi; \lambda; \omega\}} (MSE)$  using an optimizer such as the *L - BFGS - B*.

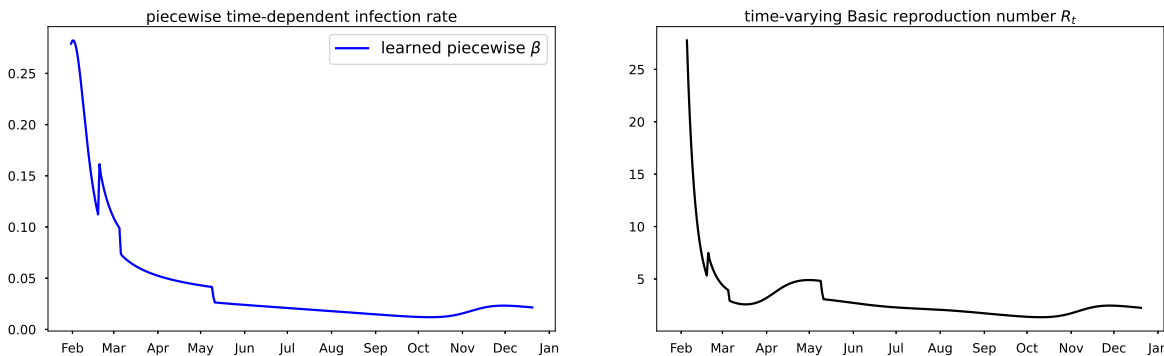
- 6: return EINN solution
    - $S(t_j; \theta; \lambda), I(t_j; \theta; \lambda), J(t_j; \theta; \lambda), R(t_j; \theta; \lambda), U(t_j; \theta; \lambda), j = 1, \dots, M$ .
    - epidemiology parameters:  $\gamma$  and  $\mu$
    - vaccination parameter:  $\kappa$
  - 7: return time-dependent epidemiology parameter:
    - $\beta(t_j; \phi; \omega), j = 1, \dots, M_\beta$ .
    - Rate of human contact decrease:  $q_1, \dots, q_n$ .
-



(a) Using the piecewise-constant infection rate eq. (16) to learn a time-dependent infection rate  $\beta(t)$  in eq. (1). We choose  $M_1, \dots, M_n$  and we use EINN algorithm 4 to learn  $q_1, \dots, q_n$ . For the Italy COVID-19 data, we fix  $\xi = 0.37$  and set  $\beta_0 = 0.22$ . Choosing  $M_1 = \text{day } 20$ ,  $M_2 = \text{day } 35$ , and  $M_3 = \text{day } 100$ , we learn  $q_1 = 0.77$ ,  $q_2 = 0.60$ , and  $q_3 = 0.44$ . A learned piecewise time-dependent infection rate  $\beta$  is plotted for cumulative Italy COVID-19 data from January 31, 2020 to December 11, 2020.  $N = 60.36 \times 10^6$ . It is not smooth because it is learned using a discrete piecewise function. The plot of the time-varying basic reproduction rate  $R_t$  detects the lockdown, the mitigation measures post-lockdown and the relaxation period that followed. The algorithm also learns  $\gamma = 0.0112$  and  $\mu = 0.0075$ . The MSE is  $9.7 \times 10^{-5}$ .



(b) Using the piecewise-constant infection rate eq. (16) to learn a time-dependent infection rate  $\beta(t)$  in eq. (1). We choose  $M_1, \dots, M_n$  and we use EINN algorithm 4 to learn  $q_1, \dots, q_n$ . For the Italy COVID-19 data, we fix  $\xi = 0.22$  and set  $\beta_0 = 0.379$ . Choosing  $M_1 = \text{day } 20$ ,  $M_2 = \text{day } 35$ , and  $M_3 = \text{day } 100$ , we learn  $q_1 = 0.83$ ,  $q_2 = 0.71$ , and  $q_3 = 0.38$ . A learned piecewise time-dependent infection rate  $\beta$  is plotted for cumulative South Korea COVID-19 data from January 22, 2020 to December 11, 2020.  $N = 51.64 \times 10^6$ . It is not smooth because it is learned using a discrete piecewise function. The plot of the time-varying basic reproduction rate  $R_t$  detects the impact of the various mitigation measures. The algorithm also learns  $\gamma = 0.0173$  and  $\mu = 0.0021$ . The MSE is  $9.0 \times 10^{-6}$ .



(c) Using the piecewise-constant infection rate eq. (16) to learn a time-dependent infection rate  $\beta(t)$  in eq. (1). We choose  $M_1, \dots, M_n$  and we use EINN algorithm 4 to learn  $q_1, \dots, q_n$ . For the Italy COVID-19 data, we fix  $\xi = 0.46$  and set  $\beta_0 = 0.279$ . Choosing  $M_1 = \text{day } 20$ ,  $M_2 = \text{day } 35$ , and  $M_3 = \text{day } 100$ , we learn  $q_1 = 1.48$ ,  $q_2 = 1.12$ , and  $q_3 = 0.72$ . A learned piecewise time-dependent infection rate  $\beta$  is plotted for cumulative USA COVID-19 data from January 22, 2020 to December 11, 2020.  $N = 328.2 \times 10^6$ . The plot of the time-varying basic reproduction rate  $R_t$  detects the lockdown, mitigation measures post-lockdown and the relaxation period that followed. It is not smooth because it is learned using a discrete piecewise function. The algorithm also learns  $\gamma = 0.0035$  and  $\mu = 0.0088$ . The MSE is  $1.7 \times 10^{-4}$ .

Figure (9) piecewise time-dependent rates

## 8. EINN algorithm for time-series infection and recovery rates

At this point, we are convinced that the infection rate  $\beta(t)$  is nonlinear time-varying. It is non-constant in the presence of mitigation measures in the cumulative data, and it closely resembles an exponentially decreasing function immediately after a wide reaching mitigation measure such as lockdown. The impact of various combination of mitigation measures post-lockdown including social distancing, contact tracing, early detection of infectives and strict adoption of facial covering are detected in the cumulative infective and recovered data. We show in this section, that we can learn the infection rate  $\beta(t)$  from data without any prior knowledge on how the aforementioned mitigation measures influence the interactions between the compartments of an epidemiological model such as in the asymptomatic-SIR. This approach learns the infection rate  $\beta(t)$  in eq. (1) as a nonlinear time-series without knowing an explicit form for the infection rate. This is called equation learning [23].

We showed earlier in section (4) that increasing  $\gamma$ , and  $\mu$  correspond to promoting measures such as contact tracing. In this section, it is shown that the symptomatic recovery rate  $\gamma(t)$  and the asymptomatic recovery rate  $\mu(t)$  are also nonlinear time-varying. Observe that in Section 7.1, we used the early infection rate  $\tilde{\beta}$  in the loss function eq. (7), this is an example of using data and some properties of the epidemiology model in the loss function. In Section 7.2, we used an explicit formula for  $\beta(t)$  given by equation (13) in the loss function eq. (7), this corresponds to using data and some known dynamics of the epidemiology model in the loss function and similarly, Section 7.3 is also an example of using data and some known dynamics of the epidemiology model in the loss function eq. (7). In all of these approaches, we learn nonlinear time-dependent infection rates. In this Section, we learn the infection and recovery rates as nonlinear time-series without knowing their explicit form. We already established that the infection and recovery rates are non-constant in the presence of mitigation measures. Using early data and nonlinear regression,  $\tilde{\beta}$  and  $\tilde{\gamma}$  are obtained. This approach involves using a feed forward neural network to learn the time-series  $S(t)$ ,  $I(t)$ ,  $J(t)$ ,  $R(t)$ , and  $U(t)$ , which are the solutions to the Asymptomatic-SIR model, and using a feed forward neural network to learn the time-series  $\beta(t)$ ,  $\gamma(t)$ , and  $\mu(t)$ . We run simulations for Italy, South Korea, and the United States from the date of the first reported cases in the respective countries to the day before vaccination data was reported,  $\kappa = 0$ . See Figures 10a, 10b, 11a, 11b, 12a, and 12b.

In this section, we estimate the time-series basic reproduction rate  $R_t$  by

$$R_t = \frac{\gamma(t)\beta(t)}{\left[\gamma(t)\left(\frac{I(t)}{I(t)+J(t)}\right) + \mu(t)\left(1 - \frac{I(t)}{I(t)+J(t)}\right)\right]^2}.$$

## 9. Conclusion

We have presented data-driven deep learning algorithms that discovers infection and recovery rate patterns in an epidemiology model using cumulative symptomatic infective and recovered data. These algorithms predict asymptomatic infectives and asymptomatic recovered populations. The asymptomatic population is usually unreported in the publicly available data. We learn this population group from symptomatic population data. It is demonstrated that time-dependent functions model the nonlinear infection rates, however, it is difficult to know the most accurate form of these time-dependent functions. The EINN algorithms presented, learns the nonlinear time-varying infection and recovery rates without a pre-assumed pattern. This approach is useful when the dynamics of an epidemiological model such as an asymptomatic-SIR model is impacted by various mitigation measures. These algorithms can be adapted to most epidemiology models. Non-pharmaceutical mitigation measures such as early detection of symptomatic infectives population, contact tracing, and social distancing are promoted by showing their impact on the spread of COVID-19. The pharmaceutical mitigation measure such as the effect of vaccination in reducing the daily reported infectives is demonstrated. This study is useful in the event of a pandemic such as COVID-19, where governmental interventions and public response and perceptions interfere in the interaction of the compartments in an epidemiology model. In future work, we will study Spatio-temporal differential equation models with nonlocal diffusion term that arise naturally in migratory patterns of infectives and non-infectives during a pandemic such as COVID-19.

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**Algorithm 5** EINN algorithm for Asymptomatic-SIR model with time-series infection and recovery rates

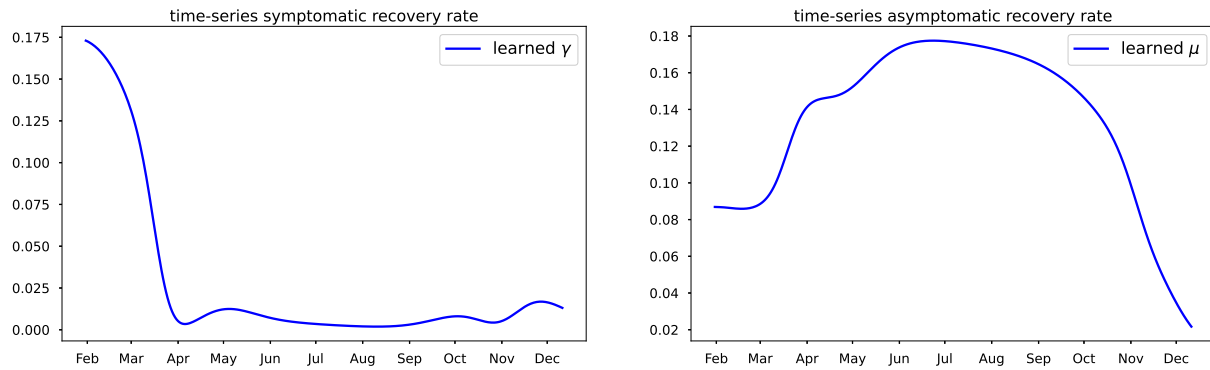
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- 1: Construct EINN
  - specify the input:  $t_j, j = 1, \dots, M$
  - Initialize the neural network parameter:  $\theta$
  - Initialize the vaccination parameter:  $\kappa$
  - Specify  $\beta_0$  obtained by nonlinear regression of early cumulative infective population data
  - Specify  $\gamma_0$  obtained by nonlinear regression of early cumulative recovered population data
- 2: Specify EINN training set
  - Training data:  $\tilde{I}(t_j)$  and  $\tilde{R}(t_j), j = 1, \dots, M$ .
  - Set  $\xi$  according to the value obtained from EINN algorithm 1
  - Asymptomatic population initialization:  $\tilde{J}(0) = (1 - \xi)\tilde{I}(0)/\xi$  and  $\tilde{U}(0) = (1 - \xi)\tilde{R}(0)/\xi$ .
- 3: Train the neural network
  - Specify an *MSE* loss function:

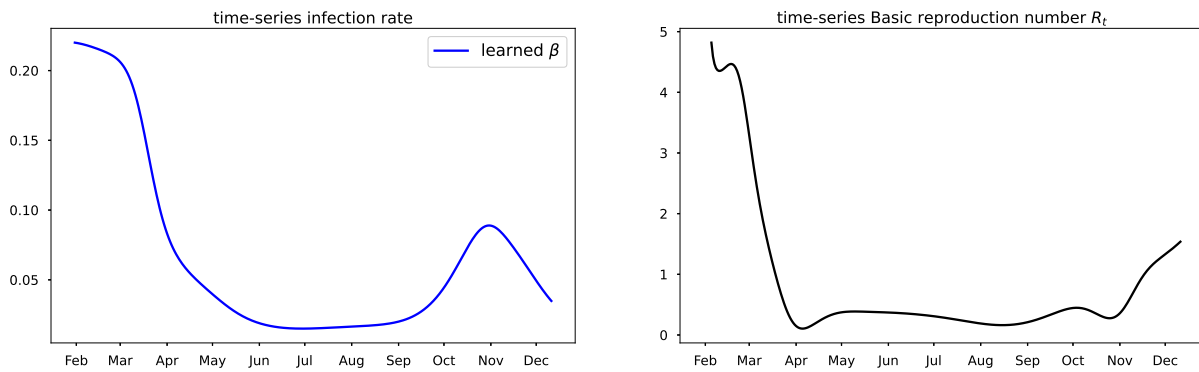
$$\begin{aligned}
 MSE = & \frac{1}{M} \sum_{j=1}^M \|I(t_j; \theta; \kappa) - \tilde{I}(t_j)\|_2^2 + \frac{1}{M} \sum_{j=1}^M \|R(t_j; \theta; \kappa) - \tilde{R}(t_j)\|_2^2 \\
 & + \|\beta(0; \theta; \kappa) - \beta_0\|_2^2 + \|\gamma(0; \theta; \kappa) - \gamma_0\|_2^2 \\
 & + \frac{1}{M_\kappa} \sum_{j=1}^{M_\kappa} \|\kappa S(t_j; \theta; \kappa) - \tilde{V}(t_j)\|_2^2 \\
 & + \|J(0; \theta) - \tilde{J}(0)\|_2^2 + \|U(0; \theta; \kappa) - \tilde{U}(0)\|_2^2 \\
 & + \frac{1}{M} \sum_{i=1}^6 \sum_{j=1}^M \|L_i(t_j; \theta; \kappa)\|_2^2.
 \end{aligned} \tag{19}$$

Minimize the *MSE* loss function: compute  $\arg \min_{\theta} (MSE)$  using an optimizer such as the *L - BFGS - B*.

- 4: return EINN solution
    - $S(t_j; \theta; \kappa), I(t_j; \theta; \kappa), J(t_j; \theta; \kappa), R(t_j; \theta; \kappa), U(t_j; \theta; \kappa), j = 1, \dots, M$ .
    - Time-series epidemiology parameters:  $\beta(t_j; \theta; \kappa), \gamma(t_j; \theta; \kappa), \mu(t_j; \theta), j = 1, \dots, M$ .
    - vaccination parameter:  $\kappa$ .
-



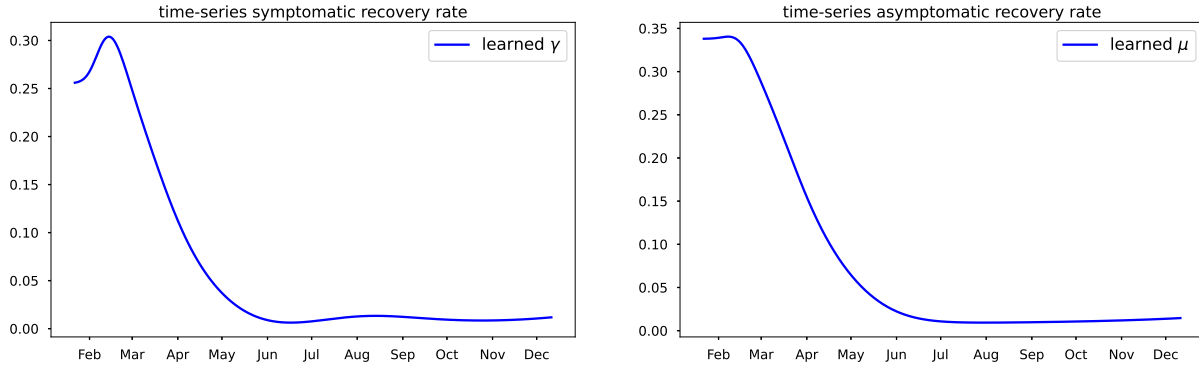
(a) Using EINN algorithm 5 and fixing  $\xi = 0.37$ , graph of the time-series symptomatic recovery rate  $\gamma(t)$  and time-series asymptomatic recovery rate  $\mu(t)$  for cumulative Italy COVID-19 data from January 31, 2020 to December 11, 2020 are obtained by initializing to  $\gamma(0)$  to  $\tilde{\gamma} = 0.173$  obtained from nonlinear regression of early data.



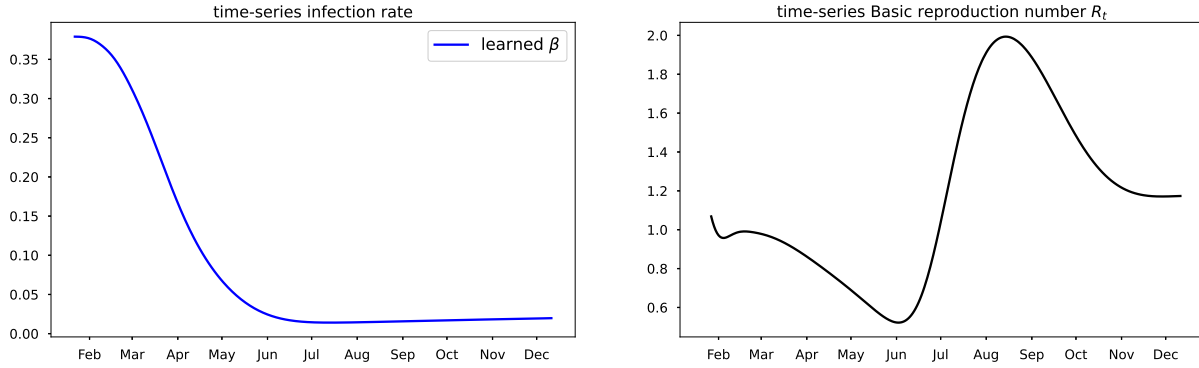
(b) Time-series infection rate  $\beta(t)$  is learned from data using EINN algorithm 5 for cumulative Italy COVID-19 data from January 31, 2020 to December 11, 2020.  $\beta(0)$  is initialized to  $\tilde{\beta} = 0.22$ , where  $\tilde{\beta}$  is obtained by nonlinear regression on early cumulative infective data. The plot of the time-series basic reproduction rate  $R_t$  detects the lockdown in March, the impact of the mitigation measures post-lockdown and the COVID-19 surge following the end of the year holidays. The MSE is  $3.0 \times 10^{-5}$ .

Figure (10) Time-series infection and recovery rates for Italy data



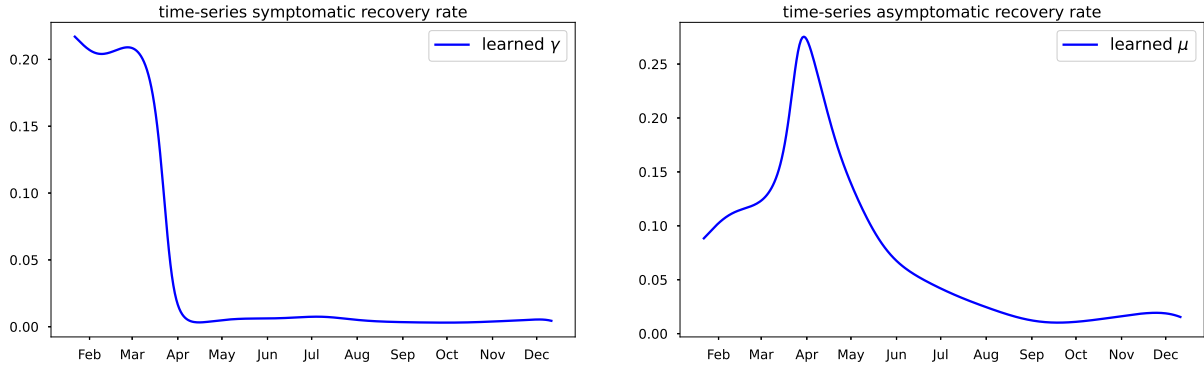


(a) Using EINN algorithm 5 and fixing  $\xi = 0.22$ , graph of the time-series symptomatic recovery rate  $\gamma(t)$  and time-series asymptomatic recovery rate  $\mu(t)$  for cumulative South Korea COVID-19 data from January 22, 2020 to December 11, 2020 are obtained by initializing to  $\gamma(0)$  to  $\tilde{\gamma} = 0.256$  obtained from nonlinear regression of early data.

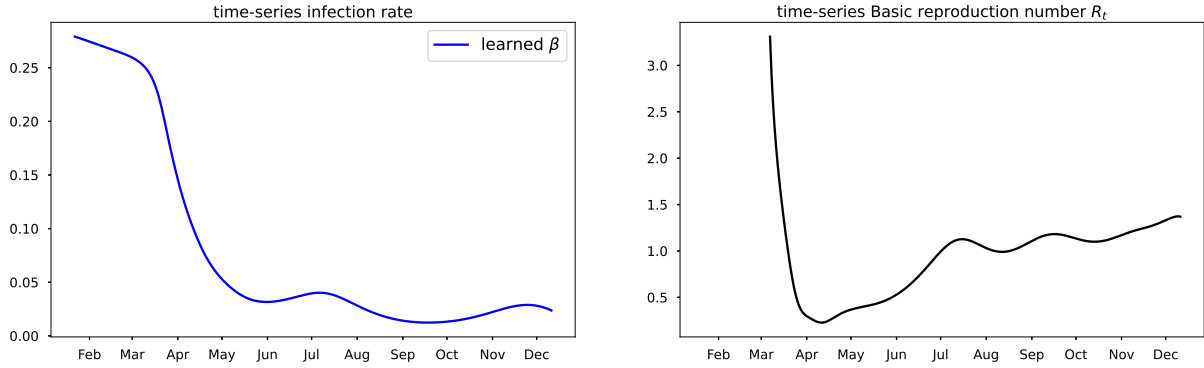


(b) Time-series infection rate  $\beta(t)$  is learned from data using EINN algorithm 5 for cumulative South Korea COVID-19 data from January 22, 2020 to December 11, 2020.  $\beta(0)$  is initialized to  $\tilde{\beta} = 0.379$ , where  $\tilde{\beta}$  is obtained by nonlinear regression on early cumulative infective data. The plot of the time-series basic reproduction rate  $R_t$  shows the early success in reducing the spread of COVID-19. However, it also shows the COVID-19 surge that happened in South Korea in August, 2020. The MSE is  $1.2 \times 10^{-5}$ .

Figure (11) Time-series infection and recovery rates for South Korea data



(a) Using EINN algorithm 5 and fixing  $\xi = 0.46$ , graph of the time-series symptomatic recovery rate  $\gamma(t)$  and time-series asymptomatic recovery rate  $\mu(t)$  for cumulative USA COVID-19 data from January 22, 2020 to December 11, 2020 are obtained by initializing to  $\gamma(0)$  to  $\tilde{\gamma} = 0.217$  obtained from nonlinear regression of early data.



(b) Time-series infection rate  $\beta(t)$  is learned from data using EINN algorithm 5 for cumulative USA COVID-19 data from January 22, 2020 to December 11, 2020.  $\beta(0)$  is initialized to  $\tilde{\beta} = 0.279$ , where  $\tilde{\beta}$  is obtained by nonlinear regression on early cumulative infective data. The plot of the time-series basic reproduction rate  $R_t$  shows the impact of the lockdown on the spread of COVID-19, it also shows that there is a surge in COVID-19. The MSE is  $3.8 \times 10^{-5}$ .

Figure (12) Time-series infection and recovery rates for USA data

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